PERPETUAL AMERICAN OPTIONS UNDER LÉVY PROCESSES

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Abstract. We consider perpetual American options assuming that under a chosen equivalent martingale measure the stock returns follow a Lévy process. For put and call options, their analogs for more general payoffs, and a wide class of Lévy processes, which contains Brownian Motion, Normal Inverse Gaussian Processes, Hyperbolic Processes, Truncated Lévy Processes and their mixtures, we obtain formulas for the optimal exercise price and the fair price of the option in terms of the factors in the Wiener-Hopf factorization formula, i.e. in terms of the resolvents of the supremum and infimum processes, and derive explicit formulas for these factors. For calls, puts and some other options, the results are valid for any Lévy process.

We use the Dynkin's formula and the Wiener-Hopf factorization to find the explicit formula for the price of the option for any candidate for the exercise boundary, and by using this explicit representation, we select the optimal solution.

We show that in some cases the principle of the smooth fit fails, and suggest a generalization of this principle.

Key words. Lévy processes, perpetual American options, Wiener-Hopf factorization

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1. Introduction. Consider a market of a riskless bond and a stock whose returns follow a Lévy process. If the Lévy process is neither a Brownian motion nor a Poisson process, the market is incomplete. According to the modern martingale approach to option pricing [16], arbitrage-free prices can be obtained as expectations under any equivalent martingale measure (EMM), which is absolutely continuous w.r.t. the historic measure.

Let the riskless rate $r > 0$ and the dividend rate $\lambda \geq 0$ be fixed, let $S = \{S_t\}_{t\geq 0}$, $S_t = \exp X_t$ be the price process of the stock, and let $\mathbf{Q}$ be an EMM chosen by the market. Let $\{X_t\}$ be a Lévy process under $\mathbf{Q}$, and $(\Omega, \mathcal{F}, \mathbf{Q})$ the corresponding probability space (for general definitions of the theory of Lévy processes, see e.g. [33], [5] and [34]). Let $g(X_t)$ be the payoff function for a perpetual American option on the stock (e.g. for a put, $g(x) = K - e^x$, and for a call, $g(x) = e^x - K$, where K is the strike price; for the formulation of our results, it is more convenient to use $g(X_t)$ rather than $\max\{g(X_t), 0\}$). Set $q = r + \lambda$, and denote by $V_*(x)$, where $x = \ln S$, the rational price of the perpetual American option. It is given by

$$V_*(x) = \sup E^x[e^{-q\tau} g(X_\tau)], \tag{1.1}$$

where E^x denotes the expectation under $\mathbf{Q}$, and the supremum is taken over a set $\mathcal{M}$ of all stopping times $\tau = \tau(\omega)$ satisfying $0 \leq \tau(\omega) < \infty$, $\omega \in \Omega$ (see e.g. [35], XVIII, 2).

Suppose that the optimal stopping time is the hitting time of the exterior of an open set $\mathcal{C} \subset \mathbf{R}$:

$$\tau_* = \inf\{t \geq 0 \mid X_t \notin \mathcal{C}\}. \tag{1.2}$$

In the pure diffusion case, one finds a candidate for the optimal stopping time (1.2) or equivalently a boundary of $\mathcal{C}$ by using the smooth fit principle, as in [21] and in [28]; see also [35]. When jumps are present, this principle may fail. This effect was demonstrated in [31] for sequential testing problems for the Poisson process, and in [7], [8] and [10]–[12] for a discrete-time model of the investment under uncertainty, the perpetual American put in discrete time, and the perpetual American put in continuous time, respectively (in [7], [8], and [10]–[12], only the free boundary value problem was considered).

We use a direct reduction of the problems for puts and calls to the free boundary problem based on the Dynkin's formula, and we solve this problem directly by using the Wiener-Hopf factorization

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method in the form which is standard in the theory of pseudo-differential operators (PDO) (see e.g. [20]).[1] Similar but less direct approach was used in [29] for pure jumps and jump-diffusion mixtures of special forms; only puts and calls were considered. In [15], the perpetual call for random walks was considered, and the answer in terms of the supremum process was obtained. In [30], the paper [15] was used to derive results in the same non-explicit form for calls and puts and any Lévy process.

We do not use the smooth pasting principle but make the direct comparison of expected payoffs for different choices of candidates for the exercise price. We formulate the optimality conditions for a relatively general payoff, and verify them for puts, calls and other options with payoffs of the form

$$g(x) = \sum_{j=1}^{m} c_j \exp(\gamma_j x); \tag{1.3}$$

the list of examples can be extended. An example of (1.3) is an option which gives its owner the right to sell the stock for $K + a\sqrt{S_t}$.

We obtain the optimal solution in the class $\mathcal{M}_0$ of hitting times of semi-infinite intervals; the verification in the class $\mathcal{M}$ is made for Brownian Motions (BM), Normal Inverse Gaussian processes (NIG) and their generalizations, Hyperbolic Processes (HP), Truncated Lévy Processes (TLP) and any finite mixture of independent BM, NIG, HP and TLP.

The results are formulated in terms of the infinitesimal generator and the factors in the Wiener-Hopf factorization formula (equivalently, in terms of the resolvents of the supremum and infimum processes); in this form, they make sense for any Lévy process. We prove the results by using explicit analytic expressions for the factors, obtained in the paper for a wide class of Lévy processes. This class can be loosely characterized as a class of Lévy processes with the Lévy measures exponentially decaying at infinity and having polynomial singularity at the origin; we call these processes Regular Lévy Processes of Exponential type (RLPE)[2]. Notice that BM, NIG, HP and TLP, and any finite mixture of independent BM, NIG, HP and TLP are RLPE.

We exclude Variance Gamma Processes (VGP), since they need special treatment at many places; in particular, the explicit formulas for the factors in the Wiener-Hopf factorization formula, which we use, need regularization in the case of VGP.

Not only BM, but the other mentioned processes as well have been widely used to describe the behavior of stock prices in real financial markets:

Variance Gamma Processes have been used by Madan and co-authors in a series of papers during 90th (see [23], [24] and the bibliography there);

NIG were constructed in [2] and used to model German stocks in [3];

HP were constructed and used by Eberlein and co-authors [17], [18], [19]; hyperbolic distributions were constructed in [1];

TLP constructed in [22] were used for modeling in real financial markets in [6], [14] and [27]; a generalization of this family was constructed in [9], [11], [12]. As A.N. Shiryaev and O.E. Barndorff-Nielsen remarked, the name TLP was misleading, and so from now on we will call this family of processes KoBoL family.

Earlier, non-infinitely divisible truncations of stable Lévy distributions had been constructed and used to model the behavior of the Standard & Poor 500 Index by Mantegna and Stanley [25], [26].

Notice that the Lévy measure of any Lévy process can be approximated by a sequence of Lévy measures of RLPE so that the factors in the Wiener-Hopf factorization formula also converge, and in

[1] Notice that the stochastic version of the Wiener-Hopf method was used to solve boundary value problems for stochastic processes in Queuing Theory and Insurance (see [13] and [32]).

[2] In [9]-[12], a misleading name Generalized Truncated Lévy Processes was used. Here we use the name suggested in [4].

the case of payoffs of the form (1.3), the answers and conditions are formulated in terms of these factors. Hence, for these payoffs, our results are valid for any Lévy process. Whether they are valid for any Lévy process, when payoffs are more general than (1.3), remains an open question.

As it is well-known, simple formulas for the factors in the Wiener-Hopf factorization formula can be obtained in few cases only. Here we obtain them (in two versions) by using only one integration, and for model classes HP, NIG, and KoBoL family, we derive really simple approximate formulas, with small errors if the rate of decay of the tails is large. As empirical studies in [3] and [27] suggest, usually this is the case, and so these approximate formulas may be of some interest.

The plan of the paper is as follows. In Section 2, we introduce the class RLPE, give examples, and prove several properties of the characteristic exponents of RLPE. In Section 3, we derive two sets of explicit formulas for factors in the Wiener-Hopf factorization formula, and necessary bounds for these factors; for model classes of RLPE, we also obtain approximate effective formulas for the factors.

In Section 4 (resp., Section 5), we solve the problem for the perpetual American put (resp., call) and similar more general payoffs, in the class $\mathcal{M}_0$. In Section 6, we formulate the free boundary value problem, prove that its solution solves the optimal stopping problem in the class $\mathcal{M}$, and for model classes of RLPE and mixtures of independent processes of the model classes, we verify that the explicit solutions found in Sections 4-5 for puts, calls and some other options with the payoffs of the form (1.3), solve the free boundary value problem, and hence, solve the problem (1.1).

In Section 7 we show that in some cases the smooth pasting condition fails, and offer its generalization, which is valid for RLPE; in Appendix, we prove the most technical statements of Sections 2-6.

2. Regular Lévy processes of exponential type.

2.1. Some basic facts about Lévy processes. (See e.g. [33] (1990), Section I.4, [5], p.p. 3, 13, and [34], p.3). We assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t<+\infty}, \mathbf{P})$ satisfying the usual hypotheses.

DEFINITION 2.1. *An adapted process $X = (X_t)_{0\le t<+\infty}$ with $X_0 = 0$ a.s. is a Lévy process if and only if*

(i) X has increments independent of the past, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s, 0 \le s < t < +\infty$;

(ii) X has stationary increments, i.e. $X_t - X_s$ has the same distribution as $X_{t-s}, 0 \le s < t < +\infty$;

(iii) X is continuous in probability.

There exists a nice formula (the Lévy-Khintchine formula) which explicitly describes a Lévy process in terms of its characteristic exponent, ψ, defined by $E[e^{i\xi X_t}] = e^{-t\psi(\xi)}$. Since we consider only 1-dimensional Lévy processes here, we formulate the corresponding theorem in the 1-dimensional case only.

THEOREM 2.2. *(i) Let X be a Lévy process on $\mathbf{R}$. Then its characteristic exponent admits the representation*

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\gamma\xi - \int_{\mathbf{R}^n} (e^{ix\xi} - 1 - ix\xi\mathbf{1}_{[-1,1]}(x))\Pi(dx), \tag{2.1}$$

where $\sigma \ge 0$, $\gamma \in \mathbf{R}$, and Π is a measure supported on $\mathbf{R}\setminus\{0\}$, which satisfies

$$\Pi(\{0\}) = 0, \quad \int_{-\infty}^{+\infty} (|x|^2 \wedge 1)\Pi(dx) < \infty. \tag{2.2}$$

(ii) The representation (2.1) is unique.

(iii) Conversely, if $\sigma \ge 0$, $\gamma \in \mathbf{R}$, and Π is a measure supported on $\mathbf{R}\setminus\{0\}$, which satisfies (2.2), then there exists a Lévy process X with the characteristic exponent defined by (2.1); X is uniquely defined in law.

The triple (σ^2, Π, γ) is called the *generating triplet* of X. The σ^2 and Π are called the *Gaussian coefficient* and *Lévy measure* of X. When $\Pi = 0$, X is Gaussian, and if $\sigma = 0$, X is called purely non-Gaussian.

The infinitesimal generator, L, of a Lévy process X acts as follows:

$$Lf(x) = \frac{\sigma^2}{2} f''(x) + \gamma f'(x) + \int_{-\infty}^{+\infty} (f(x+y) - f(x) - yf'(x)\mathbf{1}_{[-1,1]}(y))\Pi(dy). \tag{2.3}$$

Apply $-L$ to an oscillating exponent $f(x) = e^{ix\xi}$, and use (2.3):

$$(-L)e^{ix\xi} = \left[\frac{\sigma^2}{2}\xi^2 - i\gamma\xi - \int_{-\infty}^{+\infty} (e^{iy\xi} - 1 - iy\xi\mathbf{1}_{(-1,1)}(y))\Pi(dy)\right] e^{ix\xi} = \psi(\xi)e^{ix\xi}. \tag{2.4}$$

By decomposing a sufficiently regular function u into the Fourier integral

$$u(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi}\hat{u}(\xi)d\xi,$$

where

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(x)dx \tag{2.5}$$

is the Fourier transform of a function u (this is the standard definition in the literature on pseudodifferential operators (PDO)), and using (2.4), we conclude that $-L$ as a pseudo-differential operator with the symbol $\psi(\xi)$:

$$-L = \psi(D).$$

Recall that a pseudo-differential operator with the (constant) symbol a is defined by

$$a(D)u(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi}a(\xi)\hat{u}(\xi)d\xi.$$

For an introduction to the general theory of PDO, see [20].

2.2. Lévy processes of exponential type.

DEFINITION 2.3. *Let $\lambda_- < 0 < \lambda_+$. We call X a Lévy process of exponential type $[\lambda_-, \lambda_+]$ if its Lévy measure satisfies*

$$\int_{-\infty}^{-1} e^{-\lambda_+ x}\Pi(dx) + \int_{1}^{+\infty} e^{-\lambda_- x}\Pi(dx) < \infty. \tag{2.6}$$

LEMMA 2.4. *Let X be a Lévy process of exponential type $[\lambda_-, \lambda_+]$. Then*

(i) the characteristic exponent ψ is holomorphic in the strip $\Im\xi \in (\lambda_-, \lambda_+)$, and continuous up to the boundary of the strip;

(ii) there exist C and $\nu > 0$ such that for all ξ in the strip $\Im\xi \in [\lambda_-, \lambda_+]$,

$$|\psi(\xi)| \le C(1 + |\xi|)^\nu; \tag{2.7}$$

(iii) for any $q > 0$, there exist $\delta > 0$ and $\sigma_- < 0 < \sigma_+$ such that for any $[\omega_-, \omega_+] \subset (\sigma_-, \sigma_+)$ and all ξ in the strip $\Im\xi \in [\omega_-, \omega_+]$,

$$q + \Re\psi(\xi) \ge \delta, \tag{2.8}$$

where $\delta = \delta(\omega_-, \omega_+) > 0$*;*

(iv) if

$$q + \psi(i(\sigma_- + 0)) \geq 0, \quad \text{and} \quad q + \psi(i(\sigma_+ - 0)) \geq 0, \tag{2.9}$$

then (2.8) holds.

(v) for any $q > 0$*, the equation*

$$q + \psi(\xi) = 0 \tag{2.10}$$

has at most one purely imaginary root in the lower half-plane, call it $-i\beta_+$*, and at most one,* $-i\beta_-$*, in the upper one half-plane;*

(vi) the root $-i\beta_\mp$ *exists if and only if*

$$q + \psi(i\lambda_\pm \mp 0) < 0, \tag{2.11}$$

and if it exists, it is a simple root

Proof. (i) is immediate from (2.6), and (ii) can be easily deduced from the Lévy-Khintchine formula, by considering separately the integral over $|x| \leq |\xi|^{-1}$ and $|x| \geq |\xi|^{-1}$.

(iii)-(iv) Set $M_1(\sigma) = \int_{-\infty}^{+\infty} e^{-\sigma x} \mu^1(dx)$, where $\mu^1(dx)$ is the probability distribution of X_1. By differentiating twice, we conclude that M_1 is convex, and clearly, $M_1(0) = 1 < e^q$. Hence, there exist $\omega_- < 0 < \omega_+$ and $\delta > 0$ such that for all $\sigma \in [\omega_-, \omega_+]$, $M_1(\sigma) \leq e^{q-\delta}$.

Now, for any $\xi \in \mathbf{R}$, and these σ,

$$\exp(-\Re\psi(\xi + i\sigma)) = |\exp(-\psi(\xi + i\sigma))| =$$

$$= \left| \int_{-\infty}^{+\infty} e^{i\xi x - \sigma x} \mu^1(dx) \right| \leq \int_{-\infty}^{+\infty} e^{-\sigma x} \mu^1(dx),$$

therefore (2.8) holds with $\sigma_- = \inf \omega_-, \sigma_+ = \sup \omega_+$, and (2.9) implies (2.8).

(v)-(vi) Notice that by the proof of (iii), $\sigma \mapsto q + \psi(i\sigma)$ is concave and equals to $q > 0$ at 0. □

2.3. Two definitions of Regular Lévy Processes of Exponential type. For the sake of brevity, we consider processes with Lévy measures (almost) symmetric in a neighborhood of the origin.

DEFINITION 2.5. *Let* $\lambda_- < 0 < \lambda_+$ *and* $\nu \in [0, 2)$*. A purely non-Gaussian Lévy process is called a Regular Lévy Process of Exponential type* $[\lambda_-, \lambda_+]$ *and order* ν *if its Lévy measure satisfies (2.6) and, in a neighborhood of zero, admits a representation* $\Pi(dx) = f(x)dx$*, where* f *satisfies the following condition:*

there exist $\nu' < \nu$*,* $c > 0$*, and* $C > 0$ *such that*

$$\left| f(x) - c|x|^{-\nu-1} \right| \leq C|x|^{-\nu'-1}, \quad \forall\ |x| \leq 1. \tag{2.12}$$

If the sample paths of a Lévy process have bounded variation on every compact time interval a.s., one says that the Lévy process has bounded variation. A regular Lévy process of exponential type has bounded variation if and only if $\nu < 1$, since this is equivalent to $\int_{-\infty}^{+\infty} (|x| \wedge 1)\Pi(dx) < +\infty$ (see e.g. [5], p.15).

Straightforward calculation (see [9]) shows that an RLPE of order $\nu > 0$ in the sense of Definition 2.5 is an RLPE in the sense of the following definition.

DEFINITION 2.6. *Let* $\lambda_- < 0 < \lambda_+$ *and* $\nu \in (0, 2]$*. A Lévy process is called a Regular Lévy Process of exponential type* $[\lambda_-, \lambda_+]$ *and order* $\nu > 0$ *if the following two conditions are satisfied:*

a) the characteristic exponent admits a representation

$$\psi(\xi) = -i\mu\xi + \phi(\xi), \tag{2.13}$$

where ϕ is holomorphic in the strip $\Im\xi \in (\lambda_-, \lambda_+)$, is continuous up to the boundary of the strip, and admits a representation

$$\phi(\xi) = c|\xi|^\nu + O(|\xi|^{\nu_1}), \tag{2.14}$$

as $\xi \to \infty$ in the strip $\Im\xi \in [\lambda_-, \lambda_+]$, where $\nu_1 < \nu$;

b) there exist $\nu_2 < \nu$ and C such that the derivative of ϕ in (2.13) admits a bound

$$|\phi'(\xi)| \le C(1 + |\xi|)^{\nu_2}, \quad \Im\xi \in [\lambda_-, \lambda_+]. \tag{2.15}$$

One can easily generalize both definitions by using $c_\pm \ge 0$ in (2.12) on the half-axis $\pm x > 0$, and in (2.14)–(2.15), as $\Re\xi \to \pm\infty$.

2.4. Model classes of Regular Lévy Processes of Exponential type. All model classes listed in Introduction but VGP are RLPE:

- BM are RLPE of order 2, and any exponential type;
- a KoBoL process of order $\nu \in (0, 2)$, with steepness parameters $\lambda_- < 0$ and $\lambda_+ > 0$, is an RLPE of orger ν and exponential type $[\lambda_-, \lambda_+]$. An (asymmetric) version can be defined as a purely non-Gaussian Lévy process with the Lévy measure

  $$\Pi(dx) = c_+ x_+^{-\nu-1} e^{\lambda_- x} dx + c_- x_-^{-\nu-1} e^{\lambda_+ x} dx, \tag{2.16}$$

  where $x_+ = \max\{x, 0\}$, and $c_\pm > 0$. An RLPE in the sense of Definition 2.5–Definition 2.6 obtains with $c_+ = c_-$.
  Direct calculation show that if $\nu \in (0, 2), \nu \ne 1$, and $c_+ = c_- = c$, then the characteristic exponent of a KoBoL process is of the form

  $$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu]. \tag{2.17}$$

  In the case $\nu = 1$, the formula differs from (2.17) (see [9], [12]).
- a Normal Tempered Stable Lévy process of order $\nu \in (0, 2)$, with parameters $\delta > 0, \alpha > \beta > -\alpha$, is an RLPE of order ν and exponential type $[-\alpha + \beta, \alpha + \beta]$; in particular, NIG processes are RLPE of order 1. The characteristic exponent is of the form

  $$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}]; \tag{2.18}$$

- a HP with parameters $\delta > 0, \alpha > \beta > -\alpha$, is an RLPE of order 1 and exponential type $[\lambda_-, \lambda_+]$, for any $[\lambda_-, \lambda_+] \subset (-\alpha + \beta, \alpha + \beta)$. Its characteristic exponent is equal to

  $$\psi(\xi) = -i\mu\xi - \ln\left[\frac{\alpha\delta}{K_1(\alpha\delta)} \frac{K_1(\delta\sqrt{\alpha^2 - (\beta + i\xi)^2})}{\delta\sqrt{\alpha^2 - \beta^2}}\right]. \tag{2.19}$$

2.5. Properties of the characteristic exponent of RLPE.

2.5.1. General properties. Clearly, an RLPE is a Lévy process of exponential type, therefore the properties listed in Lemma 2.4 hold for any RLPE.

2.5.2. Additional properties of the characteristic exponents from model classes. In order to derive simple approximate formulas for the factors in the Wiener-Hopf factorization formula, we need the following lemma, which we managed to prove only for model classes, on the case-by-case basis. We conjecture that this lemma holds for a much wider variety of RLPE if not for all RLPE.

LEMMA 2.7. *Let X be one of the model processes, of order $\nu > 0$ and exponential type $[\lambda_-, \lambda_+]$.*

Then (i) the ϕ in (2.13) admits the analytic continuation into the complex plane with two cuts: $(-i\infty, i\lambda_-]$ and $[i\lambda_+, +i\infty)$, and outside any neighborhood of $i\lambda_-$ and $i\lambda_+$ satisfies the following estimate:

$$|\phi(\xi)| \le C(1+|\xi|)^\nu; \tag{2.20}$$

(ii) all the roots in the plane with the cuts are purely imaginary.

Proof. See Appendix. □

3. The Wiener-Hopf factorization.

3.1. General Lévy processes. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, on which a one-dimensional Lévy process X is defined, and let Ω_0 be a subset of Ω such that for each $\omega \in \Omega_0$, the trajectory $X_\cdot(\omega)$ is right-continuous with left limits. Define, on Ω_0, $M_t = \sup_{0\le s\le t} X_s$ and $N_t = \inf_{0\le s\le t} X_s$. On $\Omega \setminus \Omega_0$, both M_t and N_t are set to be 0. $M = \{M_t\}$ and $N = \{N_t\}$ are called the supremum process and the infimum process, respectively. The Laplace transform (in t) of the distribution of X_t, or more precisely,

$$qE^x\left[\int_0^\infty e^{-qt}e^{i\xi X_t}dt\right] = q(q+\psi(\xi))^{-1},$$

can be factorized by using the Laplace transforms (in t) of the distributions of the supremum and infimum processes. Among many factorization identities, we will use only the simplest one ([34], Theorems 45.2 and 45.5; for more detailed exposition, see [34], Section 45).

Let μ^t be the law of X.

THEOREM 3.1. *(i) Let $q > 0$. There exists a unique pair of infinitely divisible distributions μ_q^+ and μ_q^- supported on $(-\infty, 0]$ and $[0, +\infty)$, respectively, such that their Fourier transforms ϕ_q^+ and ϕ_q^- satisfy*

$$q(q+\psi(\xi))^{-1} = \phi_q^+(\xi)\phi_q^-(\xi), \quad \xi \in \mathbf{R}. \tag{3.1}$$

(ii) The functions ϕ_q^+ and ϕ_q^- admit the following representations

$$\phi_q^+(\xi) = q\int_0^{+\infty} e^{-qt}E[e^{i\xi M_t}]dt = q\int_0^{+\infty} e^{-qt}E[e^{i\xi(X_t-N_t)}]dt, \tag{3.2}$$

$$\phi_q^-(\xi) = q\int_0^{+\infty} e^{-qt}E[e^{i\xi N_t}]dt = q\int_0^{+\infty} e^{-qt}E[e^{i\xi(X_t-M_t)}]dt, \tag{3.3}$$

and

$$\phi_q^+(\xi) = \exp\left[\int_0^{+\infty} t^{-1}e^{-qt}dt\int_0^{+\infty}(e^{ix\xi}-1)\mu^t(dx)\right], \tag{3.4}$$

$$\phi_q^-(\xi) = \exp\left[\int_0^{+\infty} t^{-1}e^{-qt}dt\int_{-\infty}^{0}(e^{ix\xi}-1)\mu^t(dx)\right]. \tag{3.5}$$

Notice that $\phi_q^+(\xi)$ (resp., $\phi_q^-(\xi)$) admits the analytic continuation into the upper half-plane $\Im\xi > 0$ (resp., lower half-plane $\Im\xi < 0$) and does not vanish there. Thus, (3.1) is a special case of the *Wiener-Hopf factorization* introduced in solving integral equations by Wiener and Hopf in 1931, and widely used in the theory of boundary value problems for PDE and PDO.

The formulas (3.2)–(3.3) are by no means explicit though very convenient for theoretical considerations, and (3.4)–(3.5) are rather involved. Simple explicit analytical formulas can be obtained for special cases only.

Example 3.1. Let X be a Brownian motion with the drift γ and variance σ^2. Then the characteristic exponent is $\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\gamma\xi$. It is clear that for $q > 0$, the equation $q + \psi(\xi) = 0$ has two roots $-i\beta_-$ and $-i\beta_+$ in the upper and lower half-planes, respectively, and therefore, $q(q+\psi(\xi))^{-1}$ admits the factorization (3.1) with

$$\phi_q^+(\xi) = \frac{\beta_+}{\beta_+ - i\xi}, \quad \phi_q^-(\xi) = \frac{-\beta_-}{-\beta_- + i\xi}. \tag{3.6}$$

Clearly, ϕ_q^- is the Fourier transform of the exponential distribution with parameter $-\beta_-$, and ϕ_q^+ is the Fourier transform of the dual to the exponential distribution with parameter β_+.

3.2. Lévy processes of exponential type. We fix a branch of ln by the requirement $\ln a \in \mathbf{R}$ for $a > 0$. We also fix $\omega_- < 0 < \omega_+$, for which (2.8) hold.

THEOREM 3.2. *Let X be a Lévy process of exponential type, let there exist $C, c, \nu > 0$ such that*

$$\Re\psi(\xi) \ge c(1+|\xi|)^\nu, \quad \Im\xi \in [\omega_-, \omega_+], \tag{3.7}$$

and let there exist $q > 0$ such that for $\omega = \omega_\pm$,

$$\int_{-\infty+i\omega}^{+\infty+i\omega} \frac{|\psi'(\eta)|}{(1+|\eta|)(q+\Re\psi(\eta))} d\eta < +\infty. \tag{3.8}$$

Then a) $\phi_q^+(\xi)$ admits the analytic continuation into a half-plane $\Im\xi > \omega_-$ and can be calculated as follows:

$$\phi_q^+(\xi) = \exp\left[(2\pi i)^{-1}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-} \frac{\psi'(\eta)}{q+\psi(\eta)} \ln\frac{\eta-\xi}{\eta} d\eta\right] \tag{3.9}$$

$$= \exp\left[(2\pi i)^{-1}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-} \frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)} d\eta\right] \tag{3.10}$$

b) $\phi_q^-(\xi)$ admits the analytic continuation into a half-plane $\Im\xi < \omega_+$ and can be calculated as follows:

$$\phi_q^-(\xi) = \exp\left[-(2\pi i)^{-1}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+} \frac{\psi'(\eta)}{q+\psi(\eta)} \ln\frac{\eta-\xi}{\eta} d\eta\right] \tag{3.11}$$

$$= \exp\left[-(2\pi i)^{-1}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+} \frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)} d\eta\right]; \tag{3.12}$$

c) $\phi_q^+(\xi)^{-1}$ (resp., $\phi_q^-(\xi)^{-1}$) admits the analytic continuation into a wider half-plane $\Im\xi > \lambda_-$ (resp., $\Im\xi < \lambda_+$) by

$$\phi_q^+(\xi)^{-1} = q^{-1}(q+\psi(\xi))\phi_q^-(\xi), \quad \Im\xi \in (\lambda_-, \omega_-]; \tag{3.13}$$

$$\phi_q^-(\xi)^{-1} = q^{-1}(q+\psi(\xi))\phi_q^+(\xi), \quad \Im\xi \in [\omega_+, \lambda_+). \tag{3.14}$$

Proof. a) Consider the expression under the exponent sign in (3.4):

$$f(\xi) := \int_0^{+\infty} \frac{e^{-qt}}{t} \int_0^{+\infty} (e^{ix\xi} - 1)\mu^t(dx)dt$$
$$= \int_0^{+\infty} \frac{e^{-qt}}{t} \int_0^{+\infty} (e^{ix\xi} - 1)(2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\eta - t\psi(\eta)} d\eta dx dt$$

On the strength of (3.7), we may apply the Cauchy theorem and shift the line of integration:

$$f(\xi) = \int_0^{+\infty} \frac{e^{-qt}}{t} \int_0^{+\infty} (e^{ix\xi} - 1)(2\pi)^{-1} \int_{-\infty+i\omega_-}^{+\infty+i\omega_-} e^{-ix\eta - t\psi(\eta)} d\eta dx dt.$$

Now the inner double integral converges absolutely, hence we can apply the Fubini theorem and integrate w.r.t. x first:

$$= \int_0^{+\infty} \frac{e^{-qt}}{t} (2\pi i)^{-1} \int_{-\infty+i\omega_-}^{+\infty+i\omega_-} e^{-t\psi(\eta)}((\eta - \xi)^{-1} - \eta^{-1}) d\eta dt$$

Integrate by part:

$$= \int_0^{+\infty} \frac{e^{-qt}}{t} (2\pi i)^{-1} \int_{-\infty+i\omega_-}^{+\infty+i\omega_-} \ln \frac{\eta - \xi}{\eta} t\psi'(\eta) e^{-t\psi(\eta)} d\eta dt$$
$$= (2\pi i)^{-1} \int_0^{+\infty} \int_{-\infty+i\omega_-}^{+\infty+i\omega_-} \ln \frac{\eta - \xi}{\eta} \psi'(\eta) e^{-t(q+\psi(\eta))} d\eta dt.$$

From (3.8), the integral above calculated in the reverse order $dtd\eta$ converges absolutely. Hence we can apply the Fubini theorem once again and obtain (3.9); integrating in (3.9) by part, we arrive at (3.10).

b) The dual process $\tilde{X}$ is of exponential type $[-\lambda_+, -\lambda_-]$, its characteristic exponent is $\bar{\psi}$, and $[-\omega_+, -\omega_-]$ plays the part of $[\omega_-, \omega_+]$ in Lemma 2.4. Write down the Wiener-Hopf factorization for $\tilde{X}$ and apply the complex conjugation; then the "+"-factor for $\tilde{X}$ becomes the "-"-factor for X, and (3.9) for $\tilde{X}$ becomes (3.11) for X.

c) follows from (3.1) and Lemma 2.4, (i). □

Remark 3.1 If X is an RLPE in the sense of Definition 2.6, then (3.8) and (3.7) hold; hence, Theorem 3.2 holds as well.

LEMMA 3.3. *Let ω_- and ω_+ be as in Theorem 3.2.*

Then there exists $C > 0$ such that in the half-plane $\pm\Im\xi \geq \pm\omega_\mp$, $\phi_q^\pm$ admits estimates

$$(1 + |\xi|)^{-C} \leq |\phi_q^\pm(\xi)| \leq (1 + |\xi|)^C. \tag{3.15}$$

Proof. In (3.9) and (3.11), make change of variables $\eta \mapsto |\xi|\eta$ and use (2.7) to notice that the expressions under the exponential sign admit an estimate via $C \ln(2 + |\xi|)$. □

(3.15) is insufficient for the proofs in Sections 4-6. More information about properties of the factors is obtained below.

3.3. Regular Lévy Processes of Exponential type. Let $\sigma_- < 0 < \sigma_+$ be from (2.8). Fix $\lambda > \max\{-\sigma_-, \sigma_+\}$, and set $\Lambda_\pm(\xi)^s = (\lambda \mp i\xi)^s = \exp[s \ln(\lambda \mp i\xi)]$. Next, choose $d > 0$ and $\kappa_-, \kappa_+ \in \mathbf{R}$ so that

$$B(\xi) := d^{-1}\Lambda_+(\xi)^{-\kappa_+}\Lambda_-(\xi)^{-\kappa_-}(q + \psi(\xi)) \tag{3.16}$$

satisfies

$$\lim_{\xi\to\pm\infty} B(\xi) = 1. \tag{3.17}$$

Choices of d, κ_+ and κ_- depending on properties of ψ, hence on ν, μ and c in (2.13)–(2.14), we have to consider four cases.

1. If $\nu \in (1,2)$ or $\nu \in (0,1]$ and $\mu = 0$, we set $d = c, \kappa_+ = \kappa_- = \nu/2$.
2. If $\nu \in (0,1)$ and $\mu > 0$, we set $d = \mu$, $\kappa_+ = 1, \kappa_- = 0$.
3. If $\nu \in (0,1)$ and $\mu < 0$, we set $d = |\mu|$, $\kappa_+ = 0, \kappa_- = 1$.
4. If $\nu = 1$, we set $d = (c^2 + \mu^2)^{1/2}$, $\kappa_\pm = 1/2 \pm \pi^{-1} \arctan(\mu/c)$.

In all cases, (3.17) follows from (2.13)–(2.14). In the first three cases, (3.17) is immediate, and in the last case, the simplest way is to check that $\ln B(\xi) \to 0$ as $\xi \to \pm\infty$:

$$\begin{aligned}\lim_{\xi\to\pm\infty} \ln B(\xi) &= \pm\frac{\pi i}{2}\kappa_+ \mp \frac{\pi i}{2}\kappa_- + \ln\frac{c \mp i\mu}{(c^2+\mu^2)^{1/2}} + (-\kappa_+ - \kappa_- + 1)\ln|k| \\ &= \pm(\kappa_+ - \kappa_-)\frac{\pi i}{2} \mp i \arctan\frac{\mu}{c} = 0\end{aligned}$$

by our choice of κ_+ and κ_-.

The last factor in (3.16) assumes values in a half-plane $\Re z > 0$ by (2.8), and the same is true of the product of the first three factors, since the first one is positive, $\Lambda_-(\xi)$ and $\Lambda_+(\xi)$ assume values in the half-plane but in different quadrants, and $0 \le \kappa_\pm \le 1$. Hence, for all $\xi \in \mathbf{R}$, $-\pi < \arg B(\xi) < \pi$, and therefore, $b = \ln B$ is well-defined on $\mathbf{R}$. Fix $\omega_- < 0 < \omega_+$ such that $\sigma_- < \omega_-$, $\omega_+ < \sigma_+$, where $\sigma_\pm$ are from (2.8), and notice that all the arguments above are valid on any line $\Im\xi = \sigma \in [\omega_-, \omega_+]$.

Next, for $\tau > \omega_-$, $\tau_1 \in [\omega_-, \tau)$ and real ξ, set

$$b_+(\xi + i\tau) = \frac{i}{2\pi}\int_{-\infty+i\tau_1}^{+\infty+i\tau_1} \frac{b(\eta)}{\xi + i\tau - \eta} d\eta \tag{3.18}$$

(by the Cauchy theorem, $b_+(\eta + i\tau)$ is independent of a choice of τ_1), and for $\tau < \omega_+$, $\tau_2 \in (\tau, \omega_+]$ and real ξ, set

$$b_-(\xi + i\tau) = -\frac{i}{2\pi}\int_{-\infty+i\tau_2}^{+\infty+i\tau_2} \frac{b(\eta)}{\xi + i\tau - \eta} d\eta. \tag{3.19}$$

It follows from (2.13), (2.14), (3.16) and (3.17) that there exist $C_1, \rho > 0$ such that for any η in a strip $\Im\eta \in [\omega_-, \omega_+]$,

$$|b(\eta)| \le C_1(1 + |\eta|)^{-\rho}. \tag{3.20}$$

Hence, the integrals in (3.18)–(3.19) converge, and $b_+(\xi)$ (resp., $b_-(\xi)$) is well-defined in a half-plane $\Im\xi\, \omega_-$ (resp., $\Im\xi < \omega_+$). We set $a_\pm(\xi) = \Lambda_\pm(\xi)^{\kappa_\pm} \exp b_\pm(\xi)$.

THEOREM 3.4. *a) a_+ (resp., a_-) is holomorphic in a half-plane $\Im\xi > \omega_-$ (resp., $\Im\xi < \omega_+$). It admits the analytic continuation into a wider half-plane $\Im\xi > \lambda_-$ (resp., $\Im\xi < \lambda_+$), and the continuous extension up to the boundary, by*

$$a_+(\xi) = a(\xi)/a_-(\xi), \quad \Im\xi \in [\lambda_-, \omega_-],$$

$$a_-(\xi) = a(\xi)/a_+(\xi), \quad \Im\xi \in [\omega_+, \lambda_+],$$

where $a(\xi) = d^{-1}(q + \psi(\xi))$;

b) on a strip $\Im\xi \in [\lambda_-, \lambda_+]$,

$$q + \psi(\xi) = da_+(\xi)a_-(\xi); \tag{3.21}$$

c) there exist $C, c > 0$ *and* $\rho_1 > 0$ *such that in a half-plane* $\Im\xi \ge \omega_-$,

$$c(1 + |\xi|)^{\kappa_+} \le |a_+(\xi)| \le C(1 + |\xi|)^{\kappa_+}; \tag{3.22}$$

$$|a_+(\xi)^{\pm 1} - \Lambda_+(\xi)^{\pm\kappa_+}| \le C(1 + |\xi|)^{\pm\kappa_+ - \rho_1}; \tag{3.23}$$

and in a half-plane $\Im\xi \le \omega_+$,

$$c(1 + |\xi|)^{\kappa_-} \le |a_-(\xi)| \le C(1 + |\xi|)^{\kappa_-}, \tag{3.24}$$

$$|a_-(\xi)^{\pm 1} - \Lambda_-(\xi)^{\pm\kappa_-}| \le C(1 + |\xi|)^{\pm\kappa_- - \rho_1}; \tag{3.25}$$

d) factors in (3.1) and (3.21) are related by

$$\phi_q^{\pm}(\xi)^{-1} = a_\pm(\xi)/a_\pm(0). \tag{3.26}$$

Proof. a) The first statement is straightforward from (3.20), and once c) is proven, the second one follows since $a(\xi)$ is holomorphic on a strip $\Im\xi \in (\lambda_-, \lambda_+)$, and admits the continuous extension up to the boundary of the strip.

b) By the residue theorem, we have for $\tau_1 \in (\omega_-, \Im\xi)$ and $\tau_2 \in (\Im\xi, \omega_+)$

$$b_+(\xi) = \frac{i}{2\pi}\left(\int_{-\infty+i\tau_1}^{+\infty+i\tau_1} - \int_{-\infty+i\tau_2}^{+\infty+i\tau_2}\right)\frac{b(\eta)}{\xi - \eta}d\eta + \frac{i}{2\pi}\int_{-\infty+i\tau_2}^{+\infty+i\tau_2}\frac{b(\eta)}{\xi - \eta}d\eta = b(\xi) - b_-(\xi).$$

Hence, $\exp b_+(\xi)\exp b_-(\xi) = B(\xi)$, and (3.21) is immediate on a narrow strip $\omega_- < \Im\xi < \omega_+$; on a wider strip $\Im\xi \in [\lambda_-, \lambda_+]$, it holds by construction.

c) By using (3.20), we obtain

$$|(\xi + i\tau - \eta)^{-1}b(\eta)| \le C(1 + |\xi - \eta|)^{-1}(1 + |\eta|)^{-\rho}.$$

By considering separately a region, where $|\xi - \eta| \ge |\xi|/2$, and its complement, it is easy to show that the RHS admits an upper bound via

$$C_1(1 + |\xi|)^{-\rho_1}(1 + |\eta|)^{-1-\rho_1} + C_1(1 + |\xi|)^{-\rho_1}(1 + |\xi - \eta|)^{-1-\rho_1},$$

where $\rho_1 = \min\{1, \rho\}/2 > 0$. By integrating, we obtain for ξ in a half-plane $\Im\xi \ge \omega_-$ (see (3.18)–(3.19))

$$|b_\pm(\xi)| \le C_3(1 + |\xi|)^{-\rho_1}, \tag{3.27}$$

and (3.22)–(3.25) follow from (3.27) and the definition of $a_\pm$.

d) Notice that $a_\pm$, $1/a_\pm$, $\phi_q^\pm$ and $1/\phi_q^\pm$ are bounded by a polynomial in the half-plane $\pm\Im\xi \ge \pm\omega_\mp$, therefore, by comparing (3.1) and (3.21), we conclude that $a_\pm\phi_q^\pm$ is holomorphic, polynomially bounded and non-vanishing on the complex plane. By the Liouville theorem, this is constant, and taking into account that $\phi_q^\pm(0) = 1$, we obtain (3.26). □

3.4. Approximate formulas for the factors in the case of NIG, HP and KoBoL. We can write these formulas down for the both representation (in (3.1) and (3.21)). In the case of the former, the argument and formulas are shorter. We use (3.9) and Lemma 2.7 to transform the line of integration into the contour along the banks of the cut $(-i\infty, i\lambda_-]$. In empirical studies (see e.g. [3] and [27]), the λ_+ and $-\lambda_-$ are usually large, of order 40-50, and then for typical values of other parameters, both roots $-i\beta_\pm$ in Lemma 2.7 exists. Therefore, in the process of transformation, the contour crosses the simple pole at $\eta = -i\beta_+$. By the residue theorem, we obtain, for ξ in the upper half-plane,

$$\phi_q^+(\xi) = \exp\left[\ln\frac{-i\beta_+}{-i\beta_+ - \xi} + \Phi_q^+(\xi)\right],$$

where

$$\Phi_q^+(\xi) = (2\pi)^{-1}\int_{-\infty}^{\lambda_-}\left[\frac{\psi'(iz-0)}{q+\psi(iz-0)} - \frac{\psi'(iz+0)}{q+\psi(iz+0)}\right]\ln\frac{-z-i\xi}{-z}dz. \tag{3.28}$$

Thus,

$$\phi_q^+(\xi) = \frac{\beta_+}{\beta_+ - i\xi}\exp\Phi_q^+(\xi). \tag{3.29}$$

Similarly, from (3.11), we deduce, for ξ in the lower half-plane,

$$\phi_q^-(\xi) = \frac{-\beta_-}{-\beta_- + i\xi}\exp\Phi_q^-(\xi), \tag{3.30}$$

where

$$\Phi_q^-(\xi) = (2\pi)^{-1}\int_{\lambda_+}^{+\infty}\left[\frac{\psi'(iz-0)}{q+\psi(iz-0)} - \frac{\psi'(iz+0)}{q+\psi(iz+0)}\right]\ln\frac{z+i\xi}{z}dz. \tag{3.31}$$

If $-\lambda_-$ (resp., λ_+) is large, $|\Phi_q^+(\xi)|$ (resp., $|\Phi_q^+(\xi)|$) is small uniformly in ξ in the upper (resp., lower) half-plane, which can be easily seen from the explicit formulas for the characteristic exponents and (3.28) (resp., (3.31)). Hence, we may calculate the integrals in (3.28) and (3.31) with large relative error and still obtain $\phi_q^+(\xi)$ from (3.29) and $\phi_q^-(\xi)$ from (3.30) with good accuracy. This observation can be used to developed effective numerical procedures. In fact, even the simple approximations

$$\phi_q^+(\xi) \sim \frac{\beta_+}{\beta_+ - i\xi}, \quad \phi_q^-(\xi) \sim \frac{-\beta_-}{-\beta_- + i\xi} \tag{3.32}$$

produce errors of several percent only, for many typical parameters values.

The comparison of (3.6) and (3.32) provides an analytical explanation why a simple adjustment of parameters of the gaussian model can give fairly good fit even in a very non-gaussian situation.

4. Pricing of the perpetual American put and similar perpetual options.

4.1. Sufficient conditions for the solution for the perpetual put-like options, in the class $\mathcal{M}_0$ of hitting times $\tau(a)$ of segments $(-\infty, a]$. Let $\mathbf{Q}$ be an EMM chosen by the market, and assume that X under $\mathbf{Q}$ is an RLPE with the characteristic exponent ψ and the infinitesimal generator L. For $g(X_t)$ the payoff, set

$$V(h,x) := E^x[e^{-q\tau(h)}g(X_{\tau(h)})], \tag{4.1}$$

where E^x is the expectation operator of the process X started at x, under $\mathbf{Q}$.

LEMMA 4.1. *Let there exist h_* with the following properties:*
a) if $h < h_*$, then there exists x such that

$$V(h,x) < g(x); \tag{4.2}$$

b) for any $x \ge h_*$,

$$V(h^*,x) \ge g(x); \tag{4.3}$$

c) if $h > h^*$, then for any $x \ge h$,

$$V(h^*,x) \ge V(h,x). \tag{4.4}$$

Then $\tau(h_*)$ is an optimal stopping time of the class $\mathcal{M}_0$.

Proof. Clearly, the rational price of the option must satisfy (4.3), hence (4.2) excludes $h < h_*$. Due to (4.3), h_* is an admissible choice, and (4.3)–(4.4) ensure that a choice $h > h_*$ is no better than h_*. □

To apply Lemma 4.1, we need an explicit formula for $V(h,x)$. We derive it by using the Dynkin's formula and the solution to the Wiener-Hopf equation. Let $U^q = U^q_X$ be the potential operator (the resolvent) of the process X:

$$U^q W(x) = E^x\left[\int_0^{+\infty} e^{-qt}W(X_t)dt\right].$$

If $V \in C_0$ is sufficiently regular, for instance, $(q-L)V \in C_0$, then

$$U^q(q-L)V = V. \tag{4.5}$$

(see e.g. [34], V.31). We will need (4.5) for not so regular V.

LEMMA 4.2. *Let $W := (q-L)V := (q+\psi(D))V$ belong to L_1 and*

$$(q+\Re\psi)^{-1}\hat{W} \in L_1. \tag{4.6}$$

Then (4.5) holds.

Proof. Since $W \in L_1$, we have

$$\begin{aligned}(U^q W)(x) &= \int_0^{+\infty} e^{-qt}(P_t W)(x)dt \\ &= \int_0^{+\infty} e^{-qt}(2\pi)^{-n}\int_{\mathbf{R}^n} e^{-i\langle x,\xi\rangle - t\psi(\xi)}\hat{f}(\xi)d\xi dt.\end{aligned}$$

Due to (4.6), the last integral computed in the reverse order $dtd\xi$ converges absolutely, and hence we can apply the Fubini theorem and obtain $U^q W = (q+\psi(D))^{-1}W$; (4.5) follows. □

If W is universally measurable, then for any stopping time τ, the Dynkin's formula is valid (see e.g. [34], (41.3)):

$$U^q W(x) = E^x\left[\int_0^{\tau} e^{-qt}W(X_t)dt\right] + E^x\left[e^{-q\tau}U^q W(X_\tau)\right]. \tag{4.7}$$

It follows that (4.7) holds for $g \in L_1 := L_1(\mathbf{R}^n)$, which admits a representation $g = g_1 + g_2$, where $g_1 \in C_0$ and g_2 is a non-negative (or non-positive) function of the class L_1. Denote the class of such sums by $UL := UL(\mathbf{R}^n)$. This class is sufficiently wide for all the applications, which we will need in the paper.

LEMMA 4.3. *Let $W := (q-L)V := (q+\psi(D))V \in UL$ satisfy (4.6). Then*

$$W(x) = E^x\left[\int_0^\tau e^{-qt}(q-L)W(X_t)dt\right] + E^x\left[e^{-q\tau}W(X_\tau)\right]. \tag{4.8}$$

Proof. Apply (4.7) to W. Due to (4.6), (4.5) holds, and hence, (4.7) becomes (4.8). □

Let $\sigma_+ > 0$ be from Lemma 2.4. Let $g^{(s)} = D^s g$, $s = 0, \ldots, m$, be measurable, and let

$$\sum_{0\le s\le m} |g^{(s)}(x)| \le Ce^{-\omega'_+ x}, \quad x \le 0, \tag{4.9}$$

$$\sum_{0\le s\le m} |g^{(s)}(x)| \le Ce^{-\omega'_- x}, \quad x \ge 0, \tag{4.10}$$

In Subsection 4.3, we will prove the following theorem.

THEOREM 4.4. *Let g satisfy (4.9)–(4.10) with $\omega'_- < \omega'_+ < \sigma_+$ and $m = 2$. Then*

I. for any $h \in \mathbf{R}$, a solution of the problem

$$(q-L)V(x) = 0, \quad x > h, \tag{4.11}$$

$$V(x) = g(x), \quad x \le h, \tag{4.12}$$

in the class of measurable functions, bounded on $[h, +\infty)$, exists.

II. (i) if $\kappa_- = 1$, then a continuous bounded solution is unique. It is given by

$$V = \phi_q^-(D)\mathbf{1}_{(-\infty,h)}\phi_q^-(D)^{-1}g; \tag{4.13}$$

(ii) if $\kappa_- \in (0,1)$, then a bounded solution is unique. It is given by (4.13), and it is continuous;

(iii) if $\kappa_- = 0$, then a bounded solution is unique. It is given by (4.13), and it is continuous if and only if $(\phi_q^-(D)g)(h) = 0$;

(iv) if $\kappa_- \in (0,1]$, then $V'(h-0) = V'(h+0)$ if and only if $(\phi_q^-(D)g)(h) = 0$.

Remark 4.1. a) The condition $\kappa_- = 0$ is equivalent to $\nu \in (0,1)$ and $\mu > 0$. This is the case of the process of bounded variation, with the positive drift.

b) The regularity condition on g can be relaxed: for some $s > \kappa_- + 1/2$, and $\omega'_- < \omega'_+ < \sigma_+$,

$$(e^{-\omega'_- x} + e^{-\omega'_+ x})^{-1} g \in H^s(\mathbf{R}).$$

c) (4.13) can be written as

$$V(h,\cdot) := V(\cdot) = qU_N^q \mathbf{1}_{(-\infty,h)} w(\cdot), \tag{4.14}$$

where

$$w := \phi_q^-(D)^{-1}g = U_M^q(q-L)g, \tag{4.15}$$

and U_M^q and U_N^q are the resolvents of the supremum and infimum processes, respectively.

We continue to study the optimal stopping problem. If $\kappa_- > 0$ or $\kappa_- = 0$ and $w(h) = 0$, then the next lemma provides the representation of $W := (q-L)V$, which implies that $W \in UL$ and (4.6) holds. The lemma is formulated and proven under simplifying assumptions, which hold for models classes. We also require more regularity of g.

Thus, in these cases, we may use (4.8) due to Lemma 4.2. If $\kappa_- = 0$, the condition $w(h) = 0$ can be used formally to find the optimal exercise boundary as the only boundary for which the solution is continuous, and in Section 6, we will show that this is really the optimal boundary (for model classes, at least). Otherwise, we cannot justify the usage of the Dynkin's formula for discontinuous V. This is the reason why we exclude the case $\nu \in (0,1)$ and $\mu > 0$ below.

LEMMA 4.5. *Assume that $\nu_1 = \nu - 1$ in (2.15), and (4.9)–(4.10) hold with $m = 3$. Then*

a) if $\kappa_+ < 1$, then W is continuous on $(h, +\infty)$, exponentially decays as $x \to +\infty$, and admits the following representation in the right neighbourhood of h:

$$W(x) = a_+(0)^{-1}w(h)\Gamma(1-\kappa_+)^{-1}(x-h)^{-\kappa_+}(1+O((x-h)^{\gamma_1}) + O((x-h)^{\gamma_2}), \tag{4.16}$$

for some $\gamma_1, \gamma_2 > 0$.

b) if $\kappa_+ = 1$ and $w(h) = 0$, then W is continuous on $(h, +\infty)$ and exponentially decays as $x \to +\infty$; in addition, $W(h+0)$ exists.

c) if $\kappa_+ = 1$, and (2.14) holds with $\nu = 2$ and $\nu_1 < 1$ (that is, the process X is a mixture of a Brownian motion with independent RLPE of order less than 1), then the statement in b) holds.

d) In all cases, W satisfies (4.6).

Proof will be given in Subsection 4.4.

Thus, our further considerations in this Section do not apply in the case of the mixture of a Brownian motion with RLPE of order ≥ 1. Notice that we will not use the additional conditions when we verify the sufficient optimality conditions in Section 6, for all mixtures of processes from model classes and put options.

Under conditions of Lemma 4.5, we can use (4.8); from (4.11)–(4.12), we conclude that V is nothing else but $V(h, x)$ given by (4.1). Hence, (4.13) is the formula for $V(h, x)$ we need, and we can formulate a simple sufficient optimality condition in the class $\mathcal{M}_0$.

THEOREM 4.6. *Let p_q^-, the (generalized) density of the distribution μ_q^- in Theorem 3.1, be continuous, let (4.9)–(4.10) hold with $\omega'_- < \omega'_+ < \sigma_+$ and $m = 3$, and let there exist $\tilde{h}_1 \leq \tilde{h}_2$ such that the following conditions are satisfied:*

$$w(x) > 0, \quad \forall\, x < \tilde{h}_1; \tag{4.17}$$

$$w(x) = 0, \quad \forall\, \tilde{h}_1 \leq x \leq \tilde{h}_2; \tag{4.18}$$

$$w(x) < 0, \quad \forall\, x > \tilde{h}_2; \tag{4.19}$$

Then for any $\tilde{h} \in [\tilde{h}_1, \tilde{h}_2]$, $\tau(\tilde{h})$ is an optimal stopping time in the class $\mathcal{M}_0$.

Proof. Write (4.13), for $x > h$, as

$$V(h, x) = \int_{-\infty}^{h} p_q^-(x-y)w(y)dy, \tag{4.20}$$

and as

$$V(h, x) = g(x) - \int_{h}^{+\infty} p_q^-(x-y)w(y)dy. \tag{4.21}$$

If $h < \tilde{h}_1$, then we notice that supp$p_q^- \subset [0, +\infty)$, and therefore from (4.17) and (4.21), we conclude that there exist x such that $V(h, x) < g(x)$, which violates the necessary optimality conditions. Now consider h on the half-axis $(\tilde{h}_2, +\infty)$, and $x > h$. By differentiating (4.20) w.r.t. h and using (4.19), we find

$$V'_h(h, x) = p_q^-(x-h)w(h) < 0,$$

hence for these h, x, $V(h, x) < V(\tilde{h}_2, x)$. Finally, for $\tilde{h}_1 \leq h \leq \tilde{h}_2$ and $x > h$, we have from (4.18)

$$V'_h(h, x) = p_q^-(x-h)w(h) = 0,$$

hence

$$V(h,x) = V(\tilde{h}_1, x), \quad \forall\ h \in [\tilde{h}_1, \tilde{h}_2] \text{ and } x > h.$$

We conclude that any $h \in [\tilde{h}_1, \tilde{h}_2]$ is an optimal exercise boundary. □

Let us show that if (2.15) holds with any $\nu_1 \in (\nu - 1, \nu)$ (this condition is satisfied for model processes), then p_q^- is continuous on $(0, +\infty)$. For $x > 0$,

$$p_q^-(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} \phi_q^-(\xi) d\xi. \tag{4.22}$$

By choosing ν_1 sufficiently closely to $\nu - 1$, it is possible to refine the proof of (3.25) and obtain (3.25) with any $\rho_1 \in (0,1)$ (and C depending on ρ_1). Then from (3.25) and (3.26), we deduce

$$\phi_q^-(\xi) = a_-(0)(\lambda + i\xi)^{-\kappa_-} + \hat{f}(\xi), \tag{4.23}$$

where $\hat{f}(\xi) = O((1+|\xi|)^{-s})$, as $\xi \to \infty$, with some $s > 1$. Hence, f, the inverse Fourier transform of $\hat{f}$, is a continuous function, and since for $\nu < 1$

$$\int_0^{+\infty} e^{-ix\xi} x^{-\nu-1} e^{-\lambda x} dx = \Gamma(-\nu)(\lambda + i\xi)^{\nu}, \tag{4.24}$$

we deduce from (4.22)–(4.23) that

$$p_q^-(x) = a_-(0)\Gamma(\kappa_-)^{-1} \mathbf{1}_{(0,+\infty)}(x) x^{\kappa_- - 1} e^{-\lambda x} + f(x) \tag{4.25}$$

is continuous on $(0, +\infty)$.

Example 4.1. Let g be given by (1.3). Then

$$w(x) = \sum_{j=1}^{l} c_j \phi_q^-(-i\gamma_j)^{-1} e^{\gamma_j x}, \tag{4.26}$$

and it is easy to verify the sufficient conditions of Theorem 4.6 in concrete cases. In particular, if they are satisfied then $\tilde{h}_1 = \tilde{h}_2$; call it $\tilde{h}$.

For instance, if the option owner has the right to sell a share of the stock for $K + a\sqrt{S}$, where S is the spot price, then $g(x) = K + ae^{x/2} - e^x$, $w(x) = K + a\phi_q^-(-i/2)^{-1}e^{-x/2} - \phi_q^-(-i)^{-1}e^x$, and the optimal exercise price is $\sqrt{Y}$, where Y is the only positive root of

$$K + a\phi_q^-(-i/2)^{-1} Y - \phi_q^-(-i)^{-1} Y^2 = 0.$$

When an optimal $\tilde{h}$ is found, we can calculate the rational price by using the explicit formulas for ϕ_q^-:

$$V(\tilde{h}, x) = (2\pi)^{-1} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \phi_q^-(\xi) \hat{u}(\tilde{h}, \xi) d\xi, \tag{4.27}$$

where $\sigma \in (\omega'_+, \lambda_+)$ is arbitrary, and $\hat{u}$ is the Fourier transform of

$$u(\tilde{h}, x) := \mathbf{1}_{(-\infty, \tilde{h})}(x) w(x)$$

w.r.t. x. If $\hat{u}(\xi)$ and ψ are holomorphic in the upper half-plane with pole(s) and/or cut(s), then we can reduce the calculation of the integral in (4.27) to the sum of terms corresponding to poles, and integrals over these cuts. This procedure allows one to derive more effective formulas. We illustrate this procedure for puts.

4.2. Perpetual American put. For puts, $g(x) = K - e^x$, (4.9) and (4.10) hold with $\omega'_+ = 0$ and $\omega'_- = -1$, respectively, and any m, and $\tilde{h}$ is defined from

$$K - \phi_q^-(-i)^{-1}e^x = 0,$$

that is,

$$e^{\tilde{h}} = K\phi_q^-(-i) = KqE\left[\int_0^\infty e^{-qt+N_t}dt \mid N_0 = 0\right]. \tag{4.28}$$

Take $\sigma \in (0, \lambda_+)$, and calculate for $\Im\xi = \sigma$:

$$\begin{aligned}\hat{u}(\tilde{h},\xi) &= \int_{-\infty}^{\tilde{h}} e^{-ix\xi}(K - \phi_q^-(-i)^{-1}e^x)dx \\ &= \frac{Ke^{-i\tilde{h}\xi}}{(-i\xi)(1-i\xi)} = \frac{-Ke^{-i\tilde{h}\xi}}{\xi(\xi+i)}.\end{aligned}$$

By substituting into (4.27), we obtain the formula for the rational perpetual put price

$$V(\tilde{h},x) = -\frac{K}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma}\frac{\exp[i(x-\tilde{h})\xi]\phi_q^-(\xi)}{\xi(\xi+i)}d\xi, \tag{4.29}$$

where $\sigma \in (0, \lambda_+)$ is arbitrary.

Assume that ϕ in (2.13) admits the analytic continuation into the upper half-plane $\Im\xi > 0$ with the cut $[i\lambda_+, +i\infty)$ and satisfies (2.20) there (if $\phi(\xi) = a\xi^2 + \phi_1(\xi)$, we assume that ϕ_1 satisfies (2.20) with $\nu_1 < 2$, in the upper half-plane with the cut). Assume also that $q + \psi$ has the only zero $-i\beta_-$ in the upper half-plane, $0 < -\beta_- < \lambda_+$; by Lemma 2.7, these conditions are satisfied for model processes. Then ϕ_q^- admits the analytic continuation into the upper half-plane with one simple pole at $-i\beta_-$, and the cut $[i\lambda_+, +i\infty)$, by

$$\phi_q^-(\xi) = q(q+\psi(\xi))^{-1}\phi_q^+(\xi)^{-1}. \tag{4.30}$$

For $z \in (\lambda_+, +\infty)$, set

$$\Phi_q^-(z) = iq[(q+\psi(iz+0))^{-1} - (q+\psi(iz-0))^{-1}]\phi_q^+(iz)^{-1}. \tag{4.31}$$

By transforming the contour in (4.29) into the integral over the banks of the cut $[i\lambda_+, +i\infty)$, we meet the simple pole, which gives the first term in (4.32) below; in the integral over the banks of the contour, we make the change of variables $\xi = iz$, and, finally, obtain for $x > \tilde{h}$

$$V(\tilde{h},x) = \frac{iqK\exp[\beta_-(x-\tilde{h})]}{\psi'(-i\beta_-)\phi_q^+(-i\beta_-)(-\beta_-)(1-\beta_-)} + (2\pi)^{-1}\int_{\lambda_+}^{+\infty}\frac{K\Phi_q^-(z)\exp[-(x-\tilde{h})z]}{z(1+z)}dz. \tag{4.32}$$

As the empirical studies of financial markets reveal, usually λ_+ is large, hence, the second term in (4.32) is small. Therefore, one may calculate it with a large relative error. This observation facilitates the numerical implementation of (4.32). The leading term is a decaying exponential function, as in the Gaussian case, when there is no cut at all, and the second term in (4.32) is zero.

In particular, in the Gaussian case,

$$(q+\psi(\xi))/q = ((-\beta_- + i\xi)/(-\beta_-))\,((\beta_+ - i\xi)/\beta_+) = \phi_q^-(\xi)^{-1}\phi_q^+(\xi)^{-1}, \tag{4.33}$$

and hence

$$q^{-1}\psi'(-i\beta_-) = i(-\beta_-)^{-1}\phi_q^+(-i\beta_-)^{-1}.$$

By substituting into (4.28) and (4.32), we obtain the optimal exercise price

$$e^{\tilde{h}} = \frac{K\beta_-}{\beta_- - 1} \tag{4.34}$$

and the rational put price, for $x > \tilde{h}$:

$$V(\tilde{h}, x) = \frac{K\exp[\beta_-(x-\tilde{h})]}{1-\beta_-} = \left(\frac{K}{1-\beta_-}\right)^{1-\beta_-} (-\beta_-)^{-\beta_-} e^{\beta_- x}. \tag{4.35}$$

This is Merton's result.

One can easily calculate $\hat{u}$ for payoffs of the form (1.3), and obtain the analogues of (4.29) and (4.32), and in the gaussian case, of (4.34) and (4.35) as well.

4.3. Proof of Theorem 4.4. We need several basic definitions and facts of the theory of PDO. For the sake of completeness, and in order to demonstrate the role of the conditions, which we impose, we give the proof of two crucial facts (for more details, see [20]).

$H^s(\mathbf{R}^n)$ is the space of generalized functions on $\mathbf{R}^n$ with the finite norm

$$||u||_s = \left(\int_{\mathbf{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\right)^{1/2}. \tag{4.36}$$

Denote by $\overset{o}{H}{}^s(\mathbf{R}_+)$ (resp., by $\overset{o}{H}{}^s(\mathbf{R}_-)$) the subspace of $H^s(\mathbf{R})$ consisting of generalized functions supported on $[0, +\infty)$ (resp., on $(-\infty, 0]$).

THEOREM 4.7. *Let $s, m \in \mathbf{R}$, and let ϕ be a measurable function, which admits the following estimate, for $\xi \in \mathbf{R}$:*

$$|\phi(\xi)| \le C(1+|\xi|)^m. \tag{4.37}$$

Then

$$\phi(D) : H^s(\mathbf{R}) \to H^{s-m}(\mathbf{R}) \quad \text{is bounded.} \tag{4.38}$$

Proof. Apply the Fourier transform and the definition of the norm (4.36). □

We call ϕ a symbol of order m, and $\phi(D)$ is called a PDO of order m.

THEOREM 4.8. *Let $s, m \in \mathbf{R}$. Let $\phi_\pm$ be holomorphic in the half-plane $\pm\Im\xi > 0$, continuous up to the boundary and admit the estimate (4.37) in the closed half-plane.*

Then

a) for any $v \in C_0^\infty((-\infty, 0))$ (resp., $v \in C_0^\infty((0, +\infty))$, the function $\phi_+(D)v$ (resp., $\phi_-(D)v$) is supported on $(-\infty, 0)$ (resp., on $(0, +\infty)$;

b) for any $s \in \mathbf{R}$, $\phi_\mp(D) : \overset{o}{H}{}^s(\mathbf{R}_\pm) \to \overset{o}{H}{}^{s-m}(\mathbf{R}_\pm)$ is bounded.

We call ϕ_+ (resp., ϕ_-) a positive (resp., negative) symbol of order m.

Proof. Consider $\phi^-(D)$ and $v \in C_0^\infty((0, +\infty))$. a) Let $x_0 := \inf \operatorname{supp} v (> 0)$. We will prove that $\phi_-(D)v(x) = 0$ for all $x \le x_0$. By changing the variable, we may assume $x_0 = 0$. Take $x \le 0$ and calculate

$$\phi_-(D)v(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi}\phi_-(\xi)\hat{u}(\xi)d\xi. \tag{4.39}$$

Change the line of integration in (4.39):

$$\phi_-(D)v(x) = (2\pi)^{-1} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{ix\xi}\phi_-(\xi)\hat{u}(\xi)d\xi, \tag{4.40}$$

where $\sigma < 0$. Since $u \in C_0^\infty((0,+\infty))$, its Fourier transform admits the following estimate in the half-plane $\Im\xi \le 0$:

$$|\hat{u}(\xi)| \le C_N(1+|\xi|)^{-N}, \tag{4.41}$$

for any N. From (4.37) and (4.41), we conclude that the integrand admits the bound via

$$C_N e^{-\sigma x}(1+|\xi|)^{m-N},$$

for any N. By choosing $N > m+1$ and passing to the limit $\sigma \to -\infty$ in (4.40), we obtain 0.

b) Since $C_0^\infty((0,+\infty))$ is dense in $\overset{o}{H}{}^s(\mathbf{R}_+)$, we deduce b) from Theorem 4.7 and a). □

By the change of the variable $x \mapsto h + x$, we reduce the proof of Theorem 4.4 to the case $h = 0$. Next, for $\omega'_+ < \sigma_+$ in (4.9), take any $\gamma \in (\omega'_+, \sigma_+)$, and set $V_\gamma(x) = e^{\gamma x}V(x)$. Denote

$$a(D) := q + \psi(D) = q - L,$$

insert $V(x) = e^{-\gamma x}V_\gamma(x)$ into (4.11), after that multiply (4.11) and (4.12) by $e^{\gamma x}$, and use the equality

$$e^{\gamma x}a(D)e^{-\gamma x} = a(D+i\gamma). \tag{4.42}$$

We obtain

$$a(D+i\gamma)V_\gamma(x) = 0, \quad x > 0; \tag{4.43}$$

$$V_\gamma(x) = g_\gamma(x), \quad x \le 0. \tag{4.44}$$

Notice that g_γ decays exponentially as $x \to -\infty$: from (4.9), on $(-\infty, 0]$,

$$\sum_{0\le s\le 2} |g_\gamma^{(s)}(x)| \le Ce^{-\epsilon|x|}, \tag{4.45}$$

where $\epsilon = \gamma - \omega'_+ > 0$. Construct G_γ, which coincides with g_γ on $\mathbf{R}_-$ and admits a bound (4.45) on $\mathbf{R}$, and set $u_\gamma = V_\gamma - G_\gamma$, $F_\gamma = -a(D+i\gamma)G_\gamma$. Then u_γ solves the problem

$$a(D+i\gamma)u_\gamma(x) = F_\gamma(x), \quad x > 0; \tag{4.46}$$

$$u_\gamma(x) = 0, \quad x \le 0. \tag{4.47}$$

(4.45) implies that $G_\gamma \in H^2(\mathbf{R})$, and from (2.13)–(2.14) we conclude that $F_\gamma \in H^{2-\bar\nu}(\mathbf{R})$, where $\bar\nu = \nu$, if $\nu \ge 1$ or $\mu = 0$, and $\bar\nu = \max\{\nu, 1\}$ otherwise. Recall that we are looking for V, which is measurable and bounded on $(0,+\infty)$. Hence, u_γ is measurable and admits a bound via $Ce^{\gamma x}$. We want to reduce to the case of an unknown function of the class $L_2(\mathbf{R}_+)$. Since $\sigma_- < 0$, we can choose $\gamma' \in (\sigma_- - \gamma, -\gamma)$. Set $u_{\gamma,\gamma'}(x) = e^{\gamma' x}u_\gamma(x)$, insert $u_\gamma(x) = e^{-\gamma' x}u_{\gamma,\gamma'}(x)$ into (4.46) and (4.47), and after that multiply (4.46) by $e^{\gamma' x}$. By using (4.42), we obtain

$$a(D+i(\gamma+\gamma'))u_{\gamma,\gamma'}(x) = F_{\gamma,\gamma'}(x), \quad x > 0; \tag{4.48}$$

$$u_{\gamma,\gamma'}(x) = 0, \quad x \le h. \tag{4.49}$$

Now $u_{\gamma,\gamma'} \in L_2(\mathbf{R}_+)$, and on the strength of (2.13)–(2.14), Theorem 4.7 gives $a(D+i(\gamma+\gamma'))u_{\gamma,\gamma'} \in H^{-\bar\nu}(\mathbf{R})$. Hence, we can write the Wiener-Hopf equation (4.48) in the form

$$a(D+i(\gamma+\gamma'))u_{\gamma,\gamma'} = F_{\gamma,\gamma'} + F_-, \tag{4.50}$$

where $F_- \in \overset{o}{H}{}^{-\bar\nu}(\mathbf{R}_-)$. Multiply by q^{-1}, and then apply $\phi_q^+(D+i(\gamma+\gamma'))$. Since in the strip $\Im\xi \in (\lambda_-, \lambda_+)(\supset (\sigma_-, \sigma_+) \supset (\sigma_-, 0))$

$$q^{-1}a(\xi) = \phi_q^+(\xi)^{-1}\phi_q^-(\xi)^{-1}$$

and by our choice, $\gamma+\gamma'\in(\sigma_-,0)$, we obtain

$$\phi_q^-(D+i(\gamma+\gamma'))^{-1}u_{\gamma,\gamma'}=K+K_-, \tag{4.51}$$

where

$$\begin{aligned}K:&=q^{-1}\phi_q^+(D+i(\gamma+\gamma'))F_{\gamma,\gamma'}\\&=q^{-1}\phi_q^+(D+i(\gamma+\gamma'))e^{\gamma' x}(-a(D+i\gamma))G_\gamma\\&=-\phi_q^-(D+i(\gamma+\gamma'))^{-1}e^{\gamma' x}G_\gamma,\end{aligned}$$

and

$$K_-:=q^{-1}\phi_q^+(D+i(\gamma+\gamma'))F_-.$$

By construction, $G_\gamma\in H^2(\mathbf{R})$, and $u_{\gamma,\gamma'}\in L_2(\mathbf{R}_+)=\overset{o}{H}{}^0(\mathbf{R}_+)$, $F_-\in\overset{o}{H}{}^{-\bar\nu}(\mathbf{R}_-)$. From Theorem 3.4, we know that for any $\sigma\in(\sigma_-,\sigma_+)$,

$$c(1+|\xi|)^{\kappa_\pm}\le|\phi_q^\pm(\xi)|\le C(1+|\xi|)^{\kappa_\pm},\quad \pm\Im\xi\ge\sigma,$$

therefore, by applying Theorem 4.7 and Theorem 4.8, we conclude that

$$\phi_q^-(D+i(\gamma+\gamma'))^{-1}u_{\gamma,\gamma'}\in\overset{o}{H}{}^{-\kappa_-}(\mathbf{R}_+),\quad K_-\in\overset{o}{H}{}^{-\kappa_-}(\mathbf{R}_-),\quad K\in H^{-\kappa_-}(\mathbf{R}_+).$$

Notice that $\kappa_-\in[0,1]$, and consider two cases: a) $\kappa_-\in[0,0.5)$; b) $\kappa_-\in[0.5,1]$.

In the case a), $H^{-\kappa_-}(\mathbf{R})$ is the direct sum of the subspaces $\overset{o}{H}{}^{-\kappa_-}(\mathbf{R}_\pm)$, the projections being $\theta_\pm$, the closures of the-multiplication-by-$\mathbf{1}_{\mathbf{R}_\pm}$-operators defined on a dense subset $L_2(\mathbf{R})\subset H^{-\kappa_-}(\mathbf{R})$ (see [20], Theorem 5.1 and Lemma 5.4). Hence, from (4.51), we deduce

$$\phi_q^-(D+i(\gamma+\gamma'))^{-1}u_{\gamma,\gamma'}=-\theta_+\phi_q^-(D+i(\gamma+\gamma'))^{-1}e^{\gamma' x}G_\gamma. \tag{4.52}$$

Next, we multiply (4.52) by $\phi_q^-(D+i(\gamma+\gamma'))$, which establishes an isomorphism between $\overset{o}{H}{}^{-\kappa_-}(\mathbf{R}_+)$ and $L_2(\mathbf{R}_+)$:

$$u_{\gamma,\gamma'}=-\phi_q^-(D+i(\gamma+\gamma'))\theta_+\phi_q^-(D+i(\gamma+\gamma'))^{-1}e^{\gamma' x}G_\gamma. \tag{4.53}$$

Then we multiply (4.53) by $e^{-\gamma' x}$ and use (4.42):

$$u_\gamma=-\phi_q^-(D+i\gamma)\theta_+\phi_q^-(D+i\gamma)^{-1}G_\gamma.$$

After that, we return to

$$\begin{aligned}V_\gamma&=G_\gamma+u_\gamma\\&=G_\gamma-\phi_q^-(D+i\gamma)\theta_+\phi_q^-(D+i\gamma)^{-1}G_\gamma\\&=\phi_q^-(D+i\gamma)\theta_-\phi_q^-(D+i\gamma)^{-1}G_\gamma,\end{aligned}$$

and notice that since G_γ coincides with g_γ on $\mathbf{R}_-$, Theorem 4.8 ensures that $\mathrm{supp}\phi_q^-(D+i\gamma)^{-1}(G_\gamma-g_\gamma)\subset[0,+\infty)$. Thus,

$$\theta_-\phi_q^-(D+i\gamma)^{-1}G_\gamma=\theta_-\phi_q^-(D+i\gamma)^{-1}g_\gamma,$$

and in the formula for V_γ, we may replace G_γ with g_γ. By using (4.42), we finally arrive at

$$V = \phi_q^-(D)\theta_-\phi_q^-(D)^{-1}g. \tag{4.54}$$

Due to (4.9)–(4.10) and Theorem 4.7, $w := \phi_q^-(D)^{-1}g \in H^{2-\kappa_-}(\mathbf{R})$. Since $2-\kappa_- > 1/2$, we can apply Lemma 5.5 in [20], and obtain

$$\theta_-\phi_q^-(D)^{-1}g = w(0)(1-iD)^{-1}\delta + (1-iD)^{-1}\theta_-(1-iD)\phi_q^-(D)^{-1}g, \tag{4.55}$$

where δ is the Dirac delta-function. Notice that for any $\epsilon > 0$, $\delta \in H^{-1/2-\epsilon}(\mathbf{R})$, and $\theta_-(1-iD)\phi_q^-(D)^{-1}g \in H^0(\mathbf{R})$. Hence, if $\kappa_- > 0$, we obtain $V \in H^{1/2+\rho}(\mathbf{R})$, for any $\rho \in (0,\kappa_-)$. But for $s > 1/2$, $H^s(\mathbf{R}) \subset C(\mathbf{R})$, and therefore, V is continuous. By using (4.42) and (4.9), it is easy to show that the RHS in (4.54) decays exponentially.

If $\kappa_- = 0$, we have from (3.25) and (3.26)

$$\phi_q^-(D) = a_-(0) + T(D), \tag{4.56}$$

where $T(\xi)$ admits an estimate (4.37) with $m < 0$, therefore from (4.54) and (4.55), we conclude that

$$V = a_-(0)w(0)(1-iD)^{-1}\delta + V_1, \tag{4.57}$$

where $V_1 \in C(\mathbf{R})$. It is straightforward to check that the Fourier transform of $\mathbf{1}_{(-\infty,0]}e^x$ is $(1-i\xi)^{-1}$, therefore $(1-iD)^{-1}\delta = \mathbf{1}_{(-\infty,0]}e^x$, and we conclude from (4.57), that V is continuous if and only if $w(0) = 0$.

In the case b), we notice that for $s \in (-3/2,-1/2)$, the decomposition of $H^s(\mathbf{R})$ into the sum of the subspaces $\overset{o}{H}{}^s(\mathbf{R}_\pm)$ is not direct, the intersection of the latter couple being $\mathbf{C}\cdot\delta$, where δ is the Dirac delta-function. It follows that in (4.52), an additional term $C\delta$ may appear, and in (4.54), the term $C\phi_q^-(D)\delta$, where C is a constant.

If $\kappa_- = 0$, we use (3.25) and (3.26), and conclude from (4.55), that $V = C\delta + V_1$, where $V_1 \in H^s(\mathbf{R})$, for some $s > -1/2$. Since $\delta \notin H^s(\mathbf{R})$, for $s > -1/2$, and V is bounded, we conclude that C must be 0.

If $\kappa_- \in (0,1)$, we can show with the help of (3.25) and (3.26), that

$$\phi_q^-(D)\delta(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}\phi_q^-(\xi)d\xi$$

is unbounded as $x \to +0$. Further, for $\kappa_- > 0$, V in (4.54) belongs to $H^s(\mathbf{R})$ for some $s > 1/2$, and hence, is continuous, we conclude that $C\phi_q^-(D)\delta + V$ is bounded only in the case $C = 0$, and so we are left with the same (4.54). Finally, if $\kappa_- = 1$, then the argument in part a) shows that V in (4.54) is continuous, and the same argument shows that $\phi_q^-(D)\delta(x)$ is discontinuous at 0. Hence, in order to get a continuous solution, we need $C = 0$. This finishes the proof of part I and part II, (i)-(iii). The last part II, (iv) can be proven by the same argument, after differentiating in (4.54).

Theorem 4.4 has been proven.

4.4. Proof of Lemma 4.5. As in the proof of Theorem 4.4, we change the variable so that $h = 0$, and the usage of (4.42) establishes the exponential decay at the infinity. The main difficulty is the regularity at 0. We have

$$(q-L)V = a(D)V = \phi^+(D)^{-1}\theta_-\phi_q^-(D)^{-1}g.$$

Under our regularity assumption on g, we can use Lemma 5.5 in [20] with one more term than in (4.55), and obtain

$$(q-L)V = W_1 + W_2 + W_3, \tag{4.58}$$

where

$$W_1 = \phi^+(D)^{-1}(1-iD)^{-1}(\phi_q^-(D)^{-1}g)(0)\delta,$$
$$W_2 = \phi^+(D)^{-1}(1-iD)^{-2}(\phi_q^-(D)^{-1}(1-iD)g)(0)\delta,$$
$$W_3 = \phi^+(D)^{-1}(1-iD)^{-2}\theta_+(1-iD)^2\phi_q^-(D)^{-1}g.$$

The equality (4.58) holds provided for some $s > 5/2$, $\phi_q^-(D)^{-1}g \in H^s(\mathbf{R})$ (locally). By using (4.9)-(4.10), we conclude that, locally, $g \in H^3(\mathbf{R})$, and by Theorem 4.7, $\phi_q^-(D)^{-1}g \in H^{3-\kappa_-}(\mathbf{R})$. Since $\kappa_- \in [0,1]$, (4.58) holds.

For $|s| < 1/2$, $\theta_- : H^s(\mathbf{R}) \to \overset{o}{H}{}^s(\mathbf{R}_-)$ is bounded, and since $\kappa_+ \le 1$, we conclude that $W_3 \in \overset{o}{H}{}^s(\mathbf{R})$ for some $s > 1/2$. For such s, $H^s(\mathbf{R}) \subset C^0(\mathbf{R})$, hence W_3 is continuous and vanishes on $[0,+\infty)$. Since $\delta \in \overset{o}{H}{}^{-s}(\mathbf{R})$ for any $s > 1/2$, and $\kappa_+ \le 1$, we obtain $W_2 \in \overset{o}{H}{}^{2-\kappa_+-s}(\mathbf{R}_-)$. If $\kappa_+ < 1$, we conclude that $W_2 \in \overset{o}{H}{}^s(\mathbf{R}_-)$. If $\kappa_+ = 1$, we use (3.23) and represent $\phi^+(D)^{-1}(1-iD)^{-2}$ in the form

$$\phi^+(D)^{-1}(1-iD)^{-2} = a_+(0)^{-1}(1-iD)^{-1} + T(D), \tag{4.59}$$

where $T(\xi)$ is a positive symbol of order less than one. Hence, $f := T(D)\delta$ is continuous and vanishes on $[0,+\infty)$, and

$$(\phi^+(D)^{-1}(1-iD)^{-2}\delta)(x) = a_+(0)^{-1}\mathbf{1}_{(-\infty,0]}(x)e^x + f(x). \tag{4.60}$$

It remains to consider W_1. Since in part b), $w(h) = 0$, and hence, $W_1 = 0$, the application of (4.60) finishes the proof of part a). Let $\kappa_+ < 1$. By using (2.14) with $\nu_1 = \nu - 1$ and the same sort of argument as in the proof of (4.25), we obtain

$$\phi^+(D)^{-1}(1-iD)^{-1}\delta = \Gamma(\kappa_+ - 1)^{-1}a_+(0)^{-1}\mathbf{1}_{(-\infty,0]}(x)(-x)^{-\kappa_+}e^x + f_1(x), \tag{4.61}$$

where f_1 is continuous and vanishes on $[0,+\infty)$, and (4.16) follows from (4.61).

The proof of the last part c) differs from the one above. We represent $q - L$ in the form $q - L = a_2(D) + a_{\nu'}(D)$, where $a_2(D)$ is a differential operator of order 2, and $a_{\nu'}(D)$ is a PDO of order $\nu' < 1$, and consider separately $a_2(D)V$ and $a_{\nu'}(D)V$ on $(-\infty,0)$. Since $a_2(D)$ is local, we obtain, for $x < 0$:

$$a_2(D)V(x) = \phi_q^-(D)\theta_+\phi^-(D)^{-1}a_2(D)g(x).$$

By Theorem 4.7, $\phi^-(D)^{-1}a_2(D)g \in H^{3-\kappa_--2}(\mathbf{R}) = L_2(\mathbf{R})$, since $\kappa_- = 2 - \kappa_+ = 1$, and hence, by the same theorem, $\phi_q^-(D)\theta_+\phi^-(D)^{-1}a_2(D)g \in H^1(\mathbf{R})$. Hence, this is a continuous function. Since $\nu' < 1$, and the order of $\phi_q^-(D)$ is $-\kappa_- = -1$, the continuity of $a_{\nu'}(D)V = a_{\nu'}(D)\phi_q^-(D)\theta_-\phi_q^-(D)^{-1}g$ is established as in the proof of part a) above.

It remains to verify d). In the course of the proof of a)-c), we have shown that

$$|\hat{V}(\xi)| \le C(1+|\xi|)^{-1-\kappa_-},$$

therefore from (2.13)–(2.14),

$$(q+\Re\psi(\xi))^{-1}|\hat{W}(\xi)| = (q+\Re\psi(\xi))^{-1}|(q+\psi(\xi))\hat{V}(\xi)| \le C(1+|\xi|)^{-1+\kappa_+-\nu}.$$

If $\kappa_- > 0$, then $\nu - \kappa_+ > 0$, and (4.6) holds.

Lemma 4.5 has been proven.

5. Pricing of the perpetual American call and similar perpetual options.

5.1. Sufficient conditions for the solution for the perpetual call-like options, in the class $\mathcal{M}_0$ of hitting times $\tau(a)$ of segments $[a, +\infty)$. The substitutions $-x$ for x, and the dual process $\tilde{X}$ for X transform a problem for an RLPE X on $(h, +\infty)$ into a problem for an RLPE $\tilde{X}$ on $(-\infty, -h)$. Therefore, all the statements and proofs for call-like options are obtained by changing the direction on the real axis and the reflection of the complex plane w.r.t. the real axis, from the corresponding statements for put-like options. The boundedness conditions (4.9) and (4.10) allow for the growth of the payoff in the direction to $-\infty$, so the growth of the payoff for the call option is not a problem as far as the main considerations in the proof of the analog of Theorem 4.4 can be restricted to a strip $\Im\xi \in (\sigma_-, \sigma_+)$, where the real part of $a(\xi) = q + \psi(\xi)$ is positive. In the case of calls, the payoff $g(x) = e^x - K$, hence we need $a(-i) = q + \psi(-i)$ to be positive. If there is no dividend, $q = r$, and the EMM-condition for the measure means that $q + \psi(-i) = 0$. This provides the formal explanation why we need the condition $q > r$ in the case of calls; and standard considerations can be used to show that in the no-dividend case, it is non-optimal to exercise the call option ever.

The analog of Lemma 4.1 is obtained by inverting signs in all the inequalities there, and we will not state it explicitly. The reformulation of Theorem 4.4 is also straightforward; in addition to changing signs of inequalities and reflections, the condition $\omega'_- < \omega'_+ < \sigma_+$ must be replaced with $\sigma_- < \omega'_- < \omega'_+$.

Notice only that we consider a problem

$$(q - L)V(x) = 0, \quad x < h, \tag{5.1}$$

$$V(x) = g(x), \quad x \geq h, \tag{5.2}$$

in the class of measurable functions, bounded on $(-\infty, h]$, and its solution is

$$V = \phi_q^+(D)\mathbf{1}_{(h,+\infty)}\phi_q^+(D)^{-1}g. \tag{5.3}$$

The solution can be written as

$$V(h, \cdot) := V(\cdot) = qU_M^q\mathbf{1}_{(h,+\infty)}w(\cdot), \tag{5.4}$$

where

$$w = U_N^q(q - L)g, \tag{5.5}$$

and U_M^q and U_N^q are the resolvents of the supremum and infimum processes, respectively.

THEOREM 5.1. *Assume that $\nu_1 = \nu - 1$ in (2.15), and (4.9)–(4.10) hold with $\sigma_- < \omega'_- < \omega'_+$ and $m = 2$, and let there exist $\tilde{h}_1 \leq \tilde{h}_2$ such that the following conditions are satisfied:*

$$w(x) < 0, \quad \forall\ x < \tilde{h}_1; \tag{5.6}$$

$$w(x) = 0, \quad \forall\ \tilde{h}_1 \leq x \leq \tilde{h}_2; \tag{5.7}$$

$$w(x) > 0, \quad \forall\ x > \tilde{h}_2. \tag{5.8}$$

Then for any $\tilde{h} \in [\tilde{h}_1, \tilde{h}_2]$, $\tau(\tilde{h})$ is an optimal stopping time in the class $\mathcal{M}_0$.

5.2. Perpetual American call. For calls, $g(x) = e^x - K$, (4.9) and (4.10) hold with $\omega'_+ = 0$ and $\omega'_- = -1$, respectively, and any m. So, we have to assume that $\sigma_- < -1$, which is equivalent to $q > r$.

$\tilde{h}$ is defined from $\phi_q^+(-i)^{-1}e^x - K = 0$, that is,

$$e^{\tilde{h}} = K\phi_q^+(-i) = KqE\left[\int_0^\infty e^{-qt+M_t}dt \mid M_0 = 0\right]. \tag{5.9}$$

When an optimal $\tilde{h}$ is found, we can calculate the rational price by using the explicit formulas for ϕ_q^+:

$$V(\tilde{h}, x) = (2\pi)^{-1}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \phi_q^+(\xi)\hat{u}(\tilde{h}, \xi)d\xi, \tag{5.10}$$

where $\sigma \in (\lambda_-, -1)$ is arbitrary, and $\hat{u}(\xi)$ is the Fourier transform of $u(\tilde{h}, x) := \mathbf{1}_{(\tilde{h},+\infty)}(x)w(x)$:

$$\hat{u}(\tilde{h}, \xi) = \int_{\tilde{h}}^{+\infty} e^{-ix\xi}(\phi_q^+(-i)^{-1}e^x - K)dx = \frac{Ke^{-i\tilde{h}\xi}}{(-i\xi)(1-i\xi)} = \frac{-Ke^{-i\tilde{h}\xi}}{\xi(\xi+i)}.$$

By substituting into (5.10), we obtain the formula for the rational perpetual call price

$$V(\tilde{h}, x) = -\frac{K}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[i(x-\tilde{h})\xi]\phi_q^-(\xi)}{\xi(\xi+i)}d\xi, \tag{5.11}$$

where $\sigma \in (\lambda_-, -1)$ is arbitrary. One can easily calculate $\hat{u}$ for payoffs of the form (1.3), and obtain the analogue of (5.11).

Assume that ϕ in (2.13) admits the analytic continuation into the lower half-plane $\Im\xi < 0$ with the cut $(-i\infty, i\lambda_-]$ and satisfies (2.20) there (if $\phi(\xi) = a\xi^2 + \phi_1(\xi)$, then we require that ϕ_1 satisfies (2.20) with $\nu_1 < 2$, in the lower half-plane with the cut). Assume also that $q + \psi$ has the only zero $-i\beta_+$ in the lower half-plane, $\lambda_- < -\beta_+ < 0$; by Lemma 2.7, these conditions are satisfied for model processes. Then ϕ_q^+ admits the analytic continuation into the the lower half-plane with one simple pole at $-i\beta_+$, and the cut $(-i\infty, i\lambda_-]$, by

$$\phi_q^+(\xi) = q(q+\psi(\xi))^{-1}\phi_q^-(\xi)^{-1}. \tag{5.12}$$

For $z \in (\lambda_+, +\infty)$, set

$$\Phi_q^+(z) = iq[(q+\psi(iz-0))^{-1} - (q+\psi(iz+0))^{-1}]\phi_q^-(iz)^{-1}. \tag{5.13}$$

By transforming the contour in (5.11) into the integral over the banks of the cut $(-i\infty, i\lambda_-]$, we meet the simple pole, which gives the first term in (5.14) below; in the integral over the banks of the contour, we make the change of variables $\xi = iz$, and, finally, obtain for $x < \tilde{h}$

$$V(\tilde{h}, x) = \frac{iqK\exp[\beta_+(x-\tilde{h})]}{\psi'(-i\beta_+)\phi_q^-(-i\beta_+)\beta_+(\beta_+-1)} + (2\pi)^{-1}\int_{-\infty}^{\lambda_-} \frac{K\Phi_q^+(z)\exp[-(x-\tilde{h})z]}{z(1+z)}dz. \tag{5.14}$$

As the empirical studies of financial markets reveal, usually $-\lambda_-$ is large, hence, the second term in (5.14) is small. Therefore, one may calculate it with a large relative error. This observation facilitates the numerical implementation of (5.14). The leading term is a decaying exponential function, as in the Gaussian case, when there is no cut at all, and the second term in (5.14) is zero.

6. Reduction to the free boundary value problem and verification of optimality in the class $\mathcal{M}$.

6.1. Main Lemma. Consider the following free boundary value problem:
Given a non-negative continuous function g, find an open set $\mathcal{C}$ and a function V such that

$$(q-L)V(x) = 0, \quad x \in \mathcal{C}, \tag{6.1}$$

$$V(x) = g(x), \quad x \notin \mathcal{C}; \tag{6.2}$$

$$V(x) \geq g(x), \quad x \in \mathcal{C}; \tag{6.3}$$

$$(q-L)V(x) \geq 0, \quad x \notin \overline{\mathcal{C}}. \tag{6.4}$$

Lemma 6.1. *Let $(\tilde{\mathcal{C}}, \tilde{V})$ be a solution of (6.1)–(6.4), let τ_* be the hitting time of $\mathcal{C}$, and let*

$$\tilde{W} := (q-L)\tilde{V} \quad \textit{be universally measurable}; \tag{6.5}$$

$$U^q\tilde{W} = \tilde{V}. \tag{6.6}$$

Then τ_* and $V_* = \tilde{V}$ solve the optimization problem (1.1).

Proof. Due to (6.1) and (6.4), $\tilde{W}$ is non-negative, and by (6.5), it is universally measurable, therefore for any stopping time τ, (4.7) holds, and by substituting from (6.6), we obtain (4.8). From (4.8) and (6.4) and (6.1), we conclude that for any stopping time τ,

$$\tilde{V}(x) \geq E^x\left[e^{-q\tau}\tilde{V}(X_\tau)\right], \tag{6.7}$$

and from (4.8) and (6.1), for a chosen stopping time τ_*,

$$\tilde{V}(x) = E^x\left[e^{-q\tau_*}\tilde{V}(X_{\tau_*})\right]. \tag{6.8}$$

By using (6.3) and (6.2), we deduce from (6.7) and (6.8)

$$\tilde{V}(x) \geq E^x[e^{-q\tau}g(X_\tau)],$$

$$\tilde{V}(x) = E^x[e^{-q\tau_*}g(X_{\tau_*})].$$

But this means that a pair (τ_*, V_*), where $V_* = \tilde{V}$, is the optimal stopping time and the rational price. □

6.2. Verification of conditions of Lemma 6.1 for puts and options with payoffs (1.3). Assume that conditions of Theorem 4.4 hold. Let $\tilde{h}$ be defined by conditions (4.17)–(4.19), and set $\mathcal{C} = (\tilde{h}, +\infty)$. Define $\tilde{V}(x) = V(h, x)$ by (4.11). Then (6.1)–(6.2) hold by Theorem 4.4, and by repeating a part of the proof of Theorem 4.6, we see that (6.3) hold. It remains to verify (6.4)–(6.6). We formulate sufficient conditions, which hold for puts and many other payoffs of the form (1.3).

Of the process, we require

(i) the function ϕ in (2.14) admits the analytic continuation into the lower half-plane with the cut $(-i\infty, i\lambda_-]$, and admits the bound (2.20) in this half-plane, outside a vicinity of $i\lambda_-$; and if $\nu = 2$, there exist c and $\nu_1 < 2$ such that $\phi(\xi) - c\xi^2$ satisfy (2.20) with ν_1 instead of ν;

(ii) in a neighborhood of $i\lambda_-$, ϕ may have a weak singularity:

$$|\phi(\xi)| \leq C|\xi - i\lambda_-|^{-\alpha}, \tag{6.9}$$

for some $\alpha < 1$;

(iii) for any $z \in (-\infty, \lambda_-)$, the limit

$$\Psi_-(z) = i[\psi(iz - 0) - \psi(iz + 0)] \quad \text{exists and is non-positive.} \tag{6.10}$$

LEMMA 6.1. *Let X be a mixture of independent BM, HP, NIG and KoBoL. Then (i)-(iii) hold.*

Proof. See Appendix. □

THEOREM 6.2. *Let X be an RLPE of exponential type $[\lambda_-, \lambda_+]$, let (i)-(iii) hold, and let*

$$g(x) = K - \sum_{j=1}^{l} c_j e^{\gamma_j x}, \tag{6.11}$$

where $K > 0, c_j > 0$, and $-i\gamma_j \in [i\sigma_-, 0)$, $j = 1, \dots, l$.

Then a) the solution of the optimal stopping problem (1.1) in the class $\mathcal{M}$ exists and belongs to $\mathcal{M}_0$;

b) the optimal exercise price, $\tilde{h}$, is the solution to the equation

$$K = \sum_{j=1}^{l} c_j \phi_q^-(-i\gamma_j)^{-1} e^{\gamma_j h}; \tag{6.12}$$

c) the price of the option can be calculated from (4.27) with any $\sigma \in (0, \sigma_+)$, and

$$\hat{u}(\xi) = K(-i\xi)^{-1} - \sum_{j=1}^{l} c_j \phi_q^-(-i\gamma_j)^{-1}(\gamma_j - i\xi)^{-1}. \tag{6.13}$$

Proof. Notice that $\hat{u}$ admits the meromorphic continuation into the complex plane with the finite number of simple poles at points $\{0, -i\gamma_1, \dots, -i\gamma_l\} \subset (i\lambda_-, i\sigma)$, and it has the following asymptotics, as $\xi \to \infty$:

$$\hat{u}(\xi) = c_1\xi^{-1} + c_2\xi^{-2} + O(|\xi|^{-3}). \tag{6.14}$$

Under condition (6.12), $c_1 = 0$, which means that the candidate for the optimal solution is more smooth, than a solution for a generic h. Hence, much simpler considerations than in the proof of Lemma 4.5 for a generic h, and without additional conditions, show that $(q - L)V \in UL$ (in fact, it turns out to be even bounded), and therefore, (6.5) and (6.6) hold. It remains to verify (6.4). The conditions (6.4) and (6.12) are evidently "additive" w.r.t. g in the sense that if g_1 and g_2 satisfy (6.4) (resp., (6.12)), then $g_1 + g_2$ satifies satisfy (6.4) (resp., (6.12)) as well. Hence, it suffices to prove that if g is a payoff of the form

$$g(x) = A - Be^{\gamma x}, \tag{6.15}$$

where $A, B > 0$, $\sigma_- \le \gamma < 0$, and

$$A - B\phi_q^-(-i\gamma)^{-1}e^{\gamma h} = 0, \tag{6.16}$$

then $V = \phi_q^-(D)\mathbf{1}_{(-\infty,h)}\phi_q^-(D)^{-1}g$ satisfies $(q - L)V \ge 0$. The payoff (6.15) being essentially the same as the one for puts, the calculations leading to (4.29) give

$$V(x) = \frac{A}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[i(x-h)\xi]\phi_q^-(\xi)}{(-i\xi)(\gamma - i\xi)}d\xi, \tag{6.17}$$

where $\sigma \in (0, \lambda_+)$ is arbitrary. By applying $(q - L) = q + \psi(D)$ to (6.17), we see that it suffices to prove that the following function is non-negative on $(0, +\infty)$:

$$W(x) = (2\pi)^{-1}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[ix\xi](q + \psi(\xi))\phi_q^-(\xi)}{(-i\xi)(\gamma - i\xi)}d\xi.$$

By using (i)-(iii), we can transform the contour of integration, and reduce to the integral over the banks of the cut $(-i\infty, i\lambda_-]$. In the process of the transformation, the contour crosses two simple poles at $\xi = 0$ and $\xi = -i\gamma$ (if $-i\gamma$ is a root of $q + \psi(\xi)$, as it the case of puts on a non-dividend paying stock, there is no second pole, but there is no need to consider this case separately: the corresponding term below will be automatically 0), which gives the first two terms; in the integral over the banks of the cut, we make the change of the variable $\xi = iz$. The result is

$$W(x) = q/\gamma - (1/\gamma)(q + \psi(-i\gamma))\phi^-(-i\gamma)e^{\gamma x} + (2\pi)^{-1}\int_{-\infty}^{\lambda_-} \frac{\Psi_-(z)\phi_q^-(iz)\exp[-zx]}{z(z+\gamma)}dz. \tag{6.18}$$

For $z \le 0$, $\phi_q^-(iz) > 0$, and since $\lambda_- < -\gamma$, the denominator of the integrand is positive. From (6.10), the integrand is negative, hence it is a decreasing function of x on $(-\infty, 0)$. Since $0 < \gamma \le -\sigma_-$, $(1/\gamma)(q + \psi(-i\gamma))\phi^-(-i\gamma) \ge 0$. It follows that W is decreasing on $(-\infty, 0)$, and hence it suffices to show that $W(+0) \ge 0$.

If $\kappa_+ < 1$, the integrand is absolutely integrable uniformly in $x \in (-\infty, 0]$, and therefore

$$W(+0) = W(0) = (q/\gamma) - (1/\gamma)(q + \psi(-i\gamma))\phi^-(-i\gamma) + (2\pi)^{-1} \int_{-\infty}^{\lambda_-} \frac{\Psi_-(z)\phi_q^-(iz)}{z(z+\gamma)} dz.$$

By transforming the contour of integration back, and taking into account that $(q + \psi(\xi))\phi_q^-(\xi) = q\phi_q^+(\xi)^{-1}$, we arrive at

$$W(+0) = W(0) = q(2\pi)^{-1} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{d\xi}{\phi_q^+(\xi)(-i\xi)(\gamma - i\xi)}.$$

The integrand is holomorphic in the upper half-plane $\Im\xi > 0$ and admits an estimate via $C(1+|\xi|)^{-2+\kappa_+}$, for $\Im\xi \ge \sigma > 0$. Hence, we can push the line of integration up, and in the limit $\sigma \to +\infty$ obtain zero. This finishes the proof in the case $\kappa_+ < 1$.

If $\kappa_+ = 1$, we can represent $(q + \psi(\xi))\phi_q^-(\xi) = q\phi_q^+(\xi)^{-1}$ in the form

$$(q + \psi(\xi))\phi_q^-(\xi) = qa_+(0)^{-1}(-i\xi) + \chi(\xi),$$

where $a_+(0) > 0$, and χ enjoys all the properties of $(q + \psi(\xi))\phi_q^-(\xi)$ in the case $\kappa_+ < 1$, which have been used above. Hence, if we use χ instead of $(q + \psi(\xi))\phi_q^-(\xi)$ in the constructions above, we obtain a non-negative function. To finish the proof, it remains to notice that

$$\begin{aligned} W_1(x) &:= (2\pi)^{-1} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[ix\xi]qa_+(0)^{-1}(-i\xi)}{(-i\xi)(\gamma - i\xi)} d\xi \\ &= qa_+(0)^{-1}\mathbf{1}_{(-\infty,0]}(x)e^{\gamma x} \ge 0. \end{aligned}$$

□

Now we consider the general case of payoffs of the form (1.3).

THEOREM 6.3. *Let the following conditions be satisfied*

1) the equation

(6.19) $$\sum_{j=1}^{m} c_j \phi_q^-(-i\gamma_j)^{-1} e^{\gamma_j h} = 0$$

has the unique solution, call it $\tilde{h}$;

2) g *can be represented in the form*

(6.20) $$g(x) = \sum_{k=1}^{l} c_k^+ \exp[\gamma_k^+ x] - \sum_{k=1}^{l} c_k^- \exp[\gamma_k^- x],$$

where $c_k^\pm$ *are positive,* $\gamma_k^\pm \in (-\sigma_+, -\sigma_-]$ *are not necessarily different, and satisfy*

(6.21) $$\gamma_k^+ \le \gamma_k^-, \quad \forall\, k,$$

(6.22) $$c_k^+ \phi_q^-(-i\gamma_k^+)^{-1} = c_k^- \phi_q^-(-i\gamma_k^-)^{-1}, \quad \forall\, k.$$

Then a) the solution of the optimal stopping problem (1.1) in the class $\mathcal{M}$ *exists and belongs to* $\mathcal{M}_0$*, the* $\tilde{h}$ *being the optimal exercise price;*

b) the price of the option can be calculated from (4.27) with any $\sigma \in (\max_j \gamma_j, \sigma_+)$, and

$$\hat{u}(\xi) = \sum_{j=1}^{l} c_j \phi_q^-(-i\gamma_j)^{-1}(\gamma_j - i\xi)^{-1}. \tag{6.23}$$

Proof. First, notice that for payoffs (6.11), 2) follows from 1).

Second, the conditions 1) and (6.4) are evidently "additive" w.r.t. g in the sense that if g_1 and g_2 satisfy 1) (resp., (6.4)), then $g_1 + g_2$ also satisfies 1) (resp., (6.4)). The condition 2) allows one to reduce to the case of the payoff of the form

$$g(x) = Ae^{\gamma^+ x} - Be^{\gamma^- x},$$

where $A, B > 0$, and $\sigma_- \le -\gamma^+ < -\gamma^- < \sigma_+$. Further, by using (4.42), we can reduce the verification to the case of the payoff (6.15), where $\gamma = \gamma^- - \gamma^+ > 0$, and $\psi(\cdot - i\gamma^+)$ instead of ψ. Since the conditions (i)-(iii) are invariant under such a shift in the argument of the characteristic exponent, we can repeat the end of the proof of Theorem 6.2. □

6.3. Verification of conditions of Lemma 6.1 for calls and options with payoffs (1.3). In all formulations and proofs above, change the signs and make the reflection w.r.t. the origin.

7. The smooth fit principle. Consider the case of put options.

THEOREM 7.1. *Let ϕ_q^- satisfy*

$$\int_{-\infty+i\sigma}^{+\infty+i\sigma} |\phi_q^-(\xi)(1 - i\xi)^{-1}| d\xi < +\infty, \tag{7.1}$$

for some $\sigma \in (0, \sigma_+)$.

Then the price of the perpetual American put satisfies the smooth fit principle.

Proof. By differentiating under the integral sign in (4.27), we obtain $V' = v$, where

$$v(x) := -\frac{K}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{i(x-\tilde{h})\xi} \phi_q^-(\xi)(1 - i\xi)^{-1} d\xi < +\infty.$$

Hence, V is smooth if and if only v is continuous. Under condition (7.1), the Fourier transform of v is of the class $L_1(\mathbf{R})$, hence v is continuous, and the smooth fit principle holds. □

THEOREM 7.2. *Let ϕ_q^- admit a representation $\phi_q^-(\xi) = c + \chi(\xi)$, where $c \neq 0$, and χ satisfies (7.1). Then the principle of the smooth fit fails.*

Proof. This time we obtain that $v = v_1 + v_2$, where v_1 is continuous, and

$$v_2(x) = -\frac{cK}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[i(x - \tilde{h})\xi] d\xi}{1 - i\xi} = -cK \mathbf{1}_{(-\infty, \tilde{h})}(x) e^{x - \tilde{h}},$$

which is discontinuous. □

Notice that for RLPE, (7.1) fails if and only if $\mu > 0$ and $\nu \in (0, 1)$.

As our results in Sections 4 and 6 show, the natural candidate is determined from the equation $w(x) := \phi_q^-(D)^{-1} g(x) = 0$, and it can be singled out formally in one of the following forms:

I. If there is a unique h such that $V(h; \cdot)$ is continuous, this h is the candidate; if $V(h; \cdot)$ is continuous for all h, the candidate is chosen by the standard smooth fit principle.

(This observation was used in [31] for a jump process with the drift).

II. If there is h such that $V'_x(h; H \pm 0)$ are finite, then h is the optimal boundary.

The second principle works for purely non-Gaussian RLPE, i.e. for RLPE of order $\nu < 2$.

In all cases, one may say that the optimal choice of h makes $V(h, \cdot)$ "more smooth" at h than generically.

8. Appendix.

8.1. Proof of Lemma 6.1. The verification of (i) for NIG (and more generally, Normal Tempered Stable Lévy processes (NTSLP)) and KoBoL is trivial due to the simplicity of the analytic expressions (2.18) and (2.17). In both cases, the characteristic exponents are continuous at the ends of the cuts, and there is no singularity mentioned in (ii).

Verification of (iii) for NTSLP: here $\lambda_- = -\alpha+\beta$, and for $z < -\alpha+\beta$,

$$\begin{aligned}\Psi_-(z) &= i\delta[(\alpha^2-(\beta+i(iz-0)^2)^{\nu/2}-(\alpha^2-(\beta+i(iz+0)^2)^{\nu/2}]\\ &= i\delta[((\alpha+\beta-z-i0)(\alpha-\beta+z+i0))^{\nu/2}-((\alpha+\beta-z+i0)(\alpha-\beta+z-i0))^{\nu/2}]\\ &= i\delta(\alpha+\beta-z)^{\nu/2}[(\alpha-\beta+z+i0)^{\nu/2}-(\alpha-\beta+z+i0)^{\nu/2}]\\ &= i\delta(\alpha+\beta-z)^{\nu/2}(-\alpha+\beta-z)^{\nu/2}[e^{i\pi\nu/2}-e^{-i\pi\nu/2}]\\ &= -\delta(\alpha+\beta-z)^{\nu/2}(-\alpha+\beta-z)^{\nu/2}2\sin[\pi\nu/2]<0.\end{aligned}$$

Verification for KoBoL: for $z < \lambda_-$,

$$\begin{aligned}\Psi_-(z) &= ic\Gamma(-\nu)[-(-\lambda_- - i(iz-0))^\nu + (-\lambda_- - i(iz+0))^\nu]\\ &= -ic\Gamma(-\nu)[(-\lambda_- + z + i0)^\nu - (-\lambda_- + z - i0)^\nu]\\ &= -ic\Gamma(-\nu)(-z+\lambda_-)^\nu[e^{i\pi\nu}-e^{-i\pi\nu}]\\ &= c\Gamma(-\nu)(-z+\lambda_-)^\nu 2\sin(\pi\nu)<0,\end{aligned}$$

since $\Gamma(-\nu)\sin(\pi\nu)<0$.

The equation (2.19) being more involved, the verification of (i)-(iii) for Hyperbolic Processes is rather long, and we omit it here to save space.

8.2. Proof of Lemma 2.7. Part (i) for the lower half-plane is a part of the statement (i) of Lemma 6.1 proven above, so it remains to prove that there is no roots of $q+\psi(\xi)$ outside the imaginary axis. Take a large R and small $\epsilon>0$ so that all roots of $q+\psi(\xi)$ on $(i\lambda_-, i\lambda_+)$ lie on $(i(\lambda_-+2\epsilon), i(\lambda_+-2\epsilon))$, and construct a contour

$$\mathcal{L}_{\epsilon,R} = \mathcal{L}^+_{\epsilon,R}\cup\mathcal{L}^l_{\epsilon,R}\cup\mathcal{L}^-_{\epsilon,R}\cup\mathcal{L}^r_{\epsilon,R},$$

where

$$\begin{aligned}\mathcal{L}^+_{\epsilon,R} &= \{\xi \mid |\xi|\le R,\ \Im\xi\ge\lambda_+-\epsilon,\ \mathrm{dist}(\xi,[i\lambda_+,+i\infty))=\epsilon\},\\ \mathcal{L}^-_{\epsilon,R} &= \{\xi \mid |\xi|\le R,\ \Im\xi\le\lambda_-+\epsilon,\ \mathrm{dist}(\xi,(-i\infty,i\lambda_-])=\epsilon\},\\ \mathcal{L}^l_{\epsilon,R} &= \{\xi \mid |\xi|= R,\ \Re\xi<0,\ \mathrm{dist}(\xi,[i\lambda_+,+i\infty)\cup(-i\infty,i\lambda_-])\ge\epsilon\},\\ \mathcal{L}^r_{\epsilon,R} &= \{\xi \mid |\xi|= R,\ \Re\xi>0,\ \mathrm{dist}(\xi,(-i\infty,i\lambda_-]\cup[i\lambda_+,+i\infty))\ge\epsilon\}.\end{aligned}$$

Let $U_{\epsilon,R}$ be a part of the complex plane, bounded by $\mathcal{L}_{\epsilon,R}$, and $N\in\{0,1,2\}$ (resp., $N_{\epsilon,R}$) the number of roots of $q+\psi(\xi)$ on $(i(\lambda_-+2\epsilon), i(\lambda_+-2\epsilon))$ (resp., on $U_{\epsilon,R}$). Since the complex plane with cuts $(-i\infty,i\lambda_-]$ and $[i\lambda_+,+i\infty)$ is a union of all $U_{\epsilon,R}$, and $U_{\epsilon,R}\subset U_{\epsilon',R'}$ whenever $\epsilon\ge\epsilon'$ and $R\le R'$, it suffices to show that for sufficiently large R and small $\epsilon>0$, $N=N_{\epsilon,R}$.

We will do this for KoBoL; the other cases can be considered similarly. First we check that

$$q+\psi(\xi)\neq 0 \tag{8.1}$$

for any $\xi\in\mathcal{L}_{\epsilon,R}$ provided $\epsilon>0$ is sufficiently small and R is sufficiently large:

1) for $\xi\in\mathcal{L}_{\epsilon,R}$ such that $\Im\xi\in[\lambda_-,\lambda_+]$, (8.1) holds by continuity of ψ, for all $\epsilon\in(0,\epsilon_0)$ and $R\ge R_0$, provided $\epsilon_0>0$ is sufficiently small and R_0 large;

2) as $R\to+\infty$, and $|\xi|=R$

$$q+\psi(\xi)\sim -i\mu\xi+o(|\xi|),\quad \nu\in(0,1), \tag{8.2}$$

and if $\nu\in(1,2)$ or $\mu=0$ and $\nu\in(0,1)$,

$$q+\psi(\xi)\sim c\Gamma(-\nu)[(-i\xi)^\nu+(i\xi)^\nu]+o(|\xi|^\nu), \tag{8.3}$$

hence (8.1) holds for these R and ξ;

3) fix such R; then on parts of $\mathcal{L}_{\epsilon,R}$ near the cuts (8.1) holds since the limits of the imaginary part of $q+\psi(\xi)$ are non-zero, namely

for $z>\lambda_+$,

$$\Im(q+\psi(iz\mp 0))=-c\Gamma(-\nu)(z-\lambda_+)^\nu\sin(\mp\pi\nu), \tag{8.4}$$

and for $z<\lambda_-$,

$$\Im(q+\psi(iz\mp 0))=-c\Gamma(-\nu)(-z+\lambda_-)^\nu\sin(\pm\pi\nu). \tag{8.5}$$

Thus, (8.1) has been proven, and now, to show that $N=N_{\epsilon,R}$, it suffices to verify an equality

$$\frac{1}{2\pi}\int_{\partial U_{\epsilon,R}} d\arg(q+\psi(\xi))=N. \tag{8.6}$$

One can check (8.6) by considering various N and $\nu\in(0,1)$, $\nu\in(1,2)$; if $N=1$, one has to distinguish cases $q+\psi(i\lambda_-)>0$, $q+\psi(i\lambda_-)<0$, and if $\nu\in(0,1)$, cases $\mu=0,\mu>0$ and $\mu<0$.

We write down the argument for two cases; others can be considered similarly.

1. If $\nu\in(1,2)$ or $\nu\in(0,1)$ and $\mu=0$, and $N=2$, then at $\xi=i(\lambda_-+\epsilon)$ and $\xi=i(\lambda_+-\epsilon)$, $q+\psi(\xi)$ is negative, and (8.3) holds. When ξ moves from $i(\lambda_+-\epsilon)$ along $\mathcal{L}_{\epsilon,R}$ counterclockwise till an intersection point with a circle $|\xi|=R$, and $\epsilon>0$ is small enough, $q+\psi(\xi)$ moves to the right-half plane due to (8.3), passing below the origin in the complex plane due to (8.4) and an inequality $-\Gamma(-\nu)\sin(-\pi\nu)<0$. At the intersection point, it is (approximately) equal to $2c\Gamma(-\nu)\cos(\pi\nu)R^\nu$, due to (8.3). When ξ moves along $\mathcal{L}^l_{\epsilon,R}$ till the intersection with a line $\Re\xi=-\epsilon$, $q+\psi(\xi)$ remains in an angle of less than 2π and arrives at approximately the starting point $2c\Gamma(-\nu)\cos(\pi\nu)R^\nu$, due to (8.3). After that ξ moves to $i(\lambda_-+\epsilon)$; due to (8.5), $q+\psi(\xi)$ passes above the origin till a point on the negative real axis. In the result, we obtain

$$\frac{1}{2\pi}\int_{\xi\in\mathcal{L}_{\epsilon,R},\Re\xi\le 0} d\arg(q+\psi(\xi))=1. \tag{8.7}$$

Similarly, we obtain (8.7) with $\Re\xi\ge 0$, and by adding the two integrals, we finish the proof of (8.6).

2. Let $N=1$, $q+\psi(i\lambda_-)<0$ and $\nu\in(0,1)$, $\mu>0$. Then $q+\psi(i\lambda_+)>0$, and therefore the first part of the journey described above is in the right half-plane $\Re\xi>0$, due to (8.2) and an assumption $\mu>0$. During the second part of the journey, $q+\psi(\xi)$ moves above the origin and arrives at (approximately) $-\mu R$, and after that moves to $i(i\lambda_-+\epsilon)$ remaining in the left half-plane. Thus, this time we obtain (8.7) with 1/2 in the RHS, and after completing the full circle, we obtain (8.6) with $N=1$.

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REFERENCES

[1] O. E. Barndorff-Nielsen, *Exponentially decreasing distributions for the logarithm of particle size*, Proc. Roy. Soc. London. Ser. A, 353 (1977), pp. 401–419.

[2] ———, *Processes of Normal Inverse Gaussian Type*, Finance and Stochastics, 2 (1998), pp. 41–68.

[3] O. E. Barndorff-Nielsen and W. Jiang, *An initial analysis of some German stock price series*, Working Paper Series, 15, Aarhus: CAF Univ. of Aarhus/Aarhus School of Business (1998).

[4] O. E. Barndorff-Nielsen and S. Levendorskiĭ, *Feller Processes of Normal Inverse Gaussian type*, Quantitative Finance 1 (2001), pp. 1–14.

[5] J. Bertoin, *Lévy processes (Cambridge tracts in mathematics, vol. 121)*, Cambridge University Press, Cambridge, 1996.

[6] J-P. Bouchaud and M. Potters, *Theory of Financial Risk. Paris*, Aléa-Saclay, Eurolles, 1997.

[7] S. I. Boyarchenko and S. Z. Levendorskiĭ, *Models of investment under uncertainty when shocks are non-gaussian*, Working Paper Series EERC, 98/02, Moscow: EERC/Eurasia Foundation (1998).

[8] ———, *On rational pricing of derivative securities for a family of non-Gaussian processes*, Preprint 98/7, Potsdam: Universität Potsdam, Insititut für Mathematik.

[9] ———, *Generalizations of the Black-Scholes equation for truncated Lévy processes*, Working Paper (1999).

[10] ———, *Pricing of a perpetual American put for truncated Lévy processes*, Working paper (1999).

[11] ———, *Option pricing for truncated Lévy processes*, In Applications of Physics in Financial Analysis (Europhysics conference abstracts, Doublin, July 15-17 (1999)), P. Alstrom, K.B. Lauritsen, eds, European Physical Society, 1999, S5.

[12] ———, *Option pricing for truncated Lévy processes*, Intern. Journ. Theor. and Appl. Finance, 3:3 (2000), pp. 549–552.

[13] A. A. Borovkov, *Stochastic processes in Queuing Theory*, Nauka, Moscow, 1972 (Transl. Springer, New York, 1976).

[14] R. Cont, M. Potters and J.-P. Bouchaud, *Scaling in stock market data: stable laws and beyond*, In Scale Invariance and beyond (Proceedings of the CNRS Workshop on Scale Invariance, Les Houches, March 1997), B. Dubrulle, F. Graner, and D. Sornette, eds., Springer, Berlin, pp. 75–85, 1997.

[15] D. A. Darling, T. Liggett and H. M. Taylor, *Optimal stopping for partial sums*, Annals of Mathem. Statistics, 43 (1972), pp. 1363-1368.

[16] F. Delbaen and W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, Math. Annalen, 300 (1994), pp. 463–520.

[17] E. Eberlein and U. Keller, *Hyperbolic distributions in finance*, Bernoulli, 1 (1995), pp. 281–299.

[18] E. Eberlein, U. Keller and K. Prause, *New insights into smile, mispricing and value at risk: The hyperbolic model*, Journ. of Business, 71 (1998), pp. 371–406.

[19] E. Eberlein and S. Raible, *Term structure models driven by general Lévy processes*, Math. Finance, 9 (1999), pp. 31–53.

[20] G. I. Eskin, *Boundary problems for elliptic pseudo-differential equations*, Nauka, Moscow 1973 (Transl. of Mathematical Monographs, 52, Providence, Rhode Island: Amer. Math. Soc., 1980).

[21] H. P. Mc Kean, Jr., *Appendix: A free boundary problem for the heat equation arising from a problem in Mathematical Economics*, Indust. Mang. Rev., 6 (1965), pp. 32–39.

[22] I. Koponen, *Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process*, Phys. Rev. E, 52 (1995), pp. 1197–1199.

[23] D. B. Madan, *Purely discontinuous asset price processes*, Working paper.

[24] D. B. Madan, P. Carr and E. C. Chang, *The variance Gamma process and option pricing*, European Finance Review, 2 (1998), pp. 79–105.

[25] R. N. Mantegna and H.E. Stanley, *Stochastic process with ultraslow convergence to a gaussian: the truncated Lévy flight*, Phys.Rev.Lett., 73 (1994), pp. 2946–2949.

[26] ———, *Physics investigation of financial markets*, Proceedings of the International School of Physics "Enrico Fermi", Course CXXXIV, F.Mallamace and H.E.Stanley, eds., IOS Press, Amsterdam, 1997, pp. 473–489.

[27] A. Matacz, *Financial modeling and option theory with the Truncated Levy Process*, University of Sydney Report 97-28, October 1997.

[28] R. C. Merton, *The theory of rational option pricing*, Bell Journ. of Economics, 4 (1973), pp. 141–183.

[29] E. Mordecki, *Optimal stopping for a diffusion with jumps*, Finance Stochast., 3 (1999), pp. 227-236.

[30] ———, *Optimal Stopping and perpetual options for Lévy processes*, Talk presented at the 1 World Congress of the Bachelier Finance Society, June 2000.

[31] G. Peskir and A.N. Shiryaev, *Sequential testing problems for Poisson processes*, Ann. Statist., 1998.

[32] N. U. Prabhu, *Stochastic storage processes. Queues, insurance risks and dams*, Springer, New York, 1980.

[33] P. Protter, *Stochastic integration and differential equations. A new approach*, Springer-Verlag, Berlin, 1990.

[34] K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.

[35] A. N. Shiryaev, *Essentials of Stochastic Finance. Facts, Models, Theory*, World Scientific, Singapoure, New Jersey, London, Hong Kong, 1999.