

A Note on the Multiplicity of Monetary Equilibria: Green-Zhou Meets Lagos-Wright

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1 Introduction

Green and Zhou (1998) make a contribution to monetary economics by showing how to relax the assumption of indivisible money in search-based models along the lines of Kiyotaki and Wright (1993), Shi (1995), and Trejos and Wright (1995).¹ In their paper and its various extensions, including Zhou (1999, 2003), Green and Zhou (2002), Kamiya and Sato (2004), Kamiya, Morishita and Shimizu (2005), and Kamiya and Shimizu (2006, 2007*a*, 2007*b*), significant advances are made along two fronts. First, these papers make progress on deriving analytic results in the Green-Zhou framework, which is a technical contribution because search-based models with divisible money are not especially tractable, mainly because one has to keep track of a nondegenerate distribution of money holdings. Second, they discuss some interesting substantive economic results; in particular they show that there is a real indeterminacy of steady-state, single-price, monetary equilibria.

In this note, we reconsider the substantive economic results in Green and Zhou in the framework of Lagos and Wright (2005), which is much more tractable than earlier search-based monetary models, because at least in the baseline version of the model, in equilibrium the distribution of money across agents of a given type is degenerate (at least, given quasi-linear utility). Here we use price posting rather than bargaining, and assume certain goods are indivisible, consistent with at least some versions of the Green-Zhou model, but otherwise stay close to the Lagos-Wright framework. Since it is easy to analyze, one does not need all the machinery developed for the Green-Zhou model to derive very

¹It is obviously desirable to allow divisible money because, among other reasons, it is otherwise hard to do much serious quantitative or policy analysis, but it is difficult because one has to track the distribution of money balances across agents as an endogenous state variable. Other approaches, different from Green and Zhou, to allowing divisible money in these types of models include Shi (1997) or Lagos and Wright (2005), who make special assumptions on the environment to render the distribution of money degenerate, and Molico (2006) or Dressler (2008), who attack the problem with a nondegenerate distribution numerically.

similar substantive results. In particular, the same multiplicity of steady-state, single-price, monetary equilibria arises, and the results are simple to derive and understand.

To preview the intuition, note that if sellers think all buyers will carry exactly m dollars, over some range of parameters, their best response is to charge $p = m$; and if buyers think all sellers will charge exactly p , over some range of parameters, their best response is to carry $m = p$. Hence, many values for $p = m$ are potentially consistent with equilibrium. This argument is simplistic, however, for two reasons. First, it might make it appear the indeterminacy is nominal when in fact it is real: given the nominal money supply, there is in fact an interval of different price levels and hence different real balances consistent with equilibrium, and these equilibria have different welfare implications. Second, one has to work out the exact bounds on the interval, since when m is too low or p too high, sellers or buyers will not participate in the market. Still, the above intuition is basically valid. Although it may seem obvious ex post, we think our results are useful because they illustrate concisely the economics behind the Green-Zhou multiplicity, and they also provide insights into a class of nonmonetary price-posting models emanating from the work of Diamond (1970), as discussed below.

2 The Model

There is a $[0, 1]$ continuum of agents who live forever in discrete time with discount factor $\beta \in (0, 1)$. Each period is divided into two subperiods. In the second subperiod all agents consume a general good X and supply labor H in a centralized market, CM, with competitive (Walrasian) pricing. In the first subperiod they trade specialized goods x and labor h in a decentralized market, DM, with random bilateral matching. In a random match between agents i and j , the probability i wants to consume a specialized good j can produce but not vice-versa, a so-called single-coincidence meeting, is $\sigma \in (0, \frac{1}{2}]$, and the probability i wants to consume a good i can produce and vice-versa, a so-called double-coincidence meeting, is $\delta = 0$ (it is a simple generalization to allow $\delta > 0$ and the main results would not change at all). When i wants to consume what j produces, we call the former a buyer and the latter a seller. We assume with no loss in generality H produces X and h produces x one-for-one (it is a simple generalization to allow general concave technologies).

Trade in the DM requires money m because of now well-known reasons in monetary theory: we assume consumption goods are nonstorable, which rules out barter; and we assume agents are anonymous, which rules out credit. See Kocherlakota (1997), Wallace (2001), Corbae et al. (2003), Araujo (2004), and Aliprantis et al. (2006) for formal discussions of these ideas. This specification of the DM

corresponds to what one sees in almost all recent work on the foundations of monetary theory, including the papers mentioned in the introduction, although the earlier models restricted individual money holdings to $m \in \{0, 1\}$ while here we allow, as in most modern models, $m \in [0, \infty)$.

We make two additional assumptions about the DM, consistent with at least some versions of the Green-Zhou framework, but different from other models such as Lagos-Wright: we assume the decentralized market good x is indivisible; and we assume the terms of trade are determined by price posting, as opposed to bargaining. Thus, each agent sets a p such that in any meeting where a buyer likes his specialized good, he commits to producing it as long as the buyer hands over p dollars. One interpretation, discussed e.g. in Curtis and Wright (2004) in a different but related model, is that each agent stocks a vending machine with a unit of x and programs it to accept p dollars, then agents sample each other's machines (this story may or may not be helpful in explaining how sellers can commit to p and not succumb to bargaining). Given these assumptions, if we shut down the CM our model collapses to basically Green-Zhou. But as in Lagos-Wright, the CM can simplify the analysis dramatically, mainly because it allows agents to rebalance their money holdings after each round of DM trade.

Let u and c be the utility of consuming and producing an indivisible specialized good x , conditional on it being one the buyer consumes and the seller produces, in the DM. Let $U(X) - H$ be the utility of consuming X and working H in the CM, which are perfectly divisible, where $U(X)$ satisfies standard assumptions. In general, an equilibrium is described by, among other things, a distribution of prices $F(p)$ and a distribution of money holdings $G(m)$ across agents in the DM.² Let $V(m, p)$ be the expected payoff of an agent entering the DM with m dollars and a posted price of p , and $W(m)$ the expected payoff of an agent entering the CM with m . Let ϕ be the CM price of m in terms of X . Then

$$\begin{aligned} W(m) &= \max_{X, m_+, p_+} \{U(X) - H + \beta V_+(m_+, p_+)\} \\ \text{st } X &= H + \phi(m + T - m_+), \end{aligned}$$

where in general a subscript $+$ indicates next period, so e.g. m_+ is the money with which the agent exits the CM and enters the next DM. Also, as is standard, T is a lumpsum monetary transfer (or tax if negative) consistent with increases (or decreases) in the aggregate money supply described by $M_+ = (1 + \pi)M$, with $\pi > \beta - 1$, so that in steady state $\phi/\phi_+ = 1 + \pi$ gives the inflation rate.³

Substituting H from the budget constraint and taking the FOC for m_+ , we get $\phi = \beta \partial V_+ / \partial m_+$ (see

²More generally, one should specify the joint distribution of m and p , although it is natural to think they would be independent. We ignore this because we are interested here mainly in equilibria with degenerate distributions.

³There is no equilibrium with $\pi < \beta - 1$. We can consider the limit as $\pi \rightarrow \beta - 1$, which is the Friedman rule. Defining the nominal interest rate i by the Fisher equation $1 + i = (1 + \pi)/\beta$, the Friedman rule means $i = 0$.

Lagos-Wright for a discussion of conditions to guarantee an interior solution and the differentiability and concavity of V with respect to m_+). Thus the solution to the problem of choosing m_+ is independent of the m one brought to the CM, because V_+ is. This does not of course mean that all agents choose the same m_+ since there could be multiple solutions to this problem. Indeed, in a related but different model – different because, first, they have multilateral meetings, and second, they use auctions instead of price posting – Galenianos and Kircher (2007) and Dutu, Julien and King (2007) show there is a continuum of solutions to their analogous problem, and equilibrium implies a nondegenerate distribution $G(m)$, but we look for equilibria here with degenerate distributions. Also, notice that W is linear with $\partial W/\partial m = \phi$.

In the DM, a buyer generally meet a seller with a random price p and has to decide whether to buy or not. Let $R(m)$ be his reservation price, above which he will not buy, given his money balances m . It is obvious and simple to show $m = R$: why take more money than you will ever spend, given it is costly when $\pi > \beta - 1$? Then

$$\begin{aligned} V(m, p) &= (1 - 2\sigma)W(m) + \sigma \left\{ \int_0^m [u + W(m - \tilde{p})] dF(\tilde{p}) + [1 - F(m)] W(m) \right\} \\ &\quad + \sigma \{ [1 - G(p)] [-c + W(m + p)] + [1 - \sigma + \sigma G(p)] W_s(m) \} \\ &= \sigma \int_0^m (u - \phi \tilde{p}) dF(\tilde{p}) + \sigma [1 - G(p)] (\phi p - c) + W(m), \end{aligned}$$

using the linearity of W .⁴ Assuming, for the moment, for the sake of illustration, that F is differentiable, notice $\partial V/\partial m = \sigma(u - \phi m) F'(m) + \phi$. Updating this one period and inserting into the FOC $\phi = \beta \partial V_+/\partial m_+$, we have

$$\phi = \beta \phi_+ + \beta \sigma (u - \phi_+ m) F'(m).$$

The second term on the right is a liquidity premium on cash, which means agents may carry $m > 0$ despite the fact that it has bad rate of return given $\pi > \beta = 1$.

Here, as in most of the literature on the Green-Zhou model, we focus on single-price equilibria, where F is degenerate. We consider only equilibria where G is also degenerate, which makes sense in the Lagos-Wright framework. Now it should be obvious that if F were degenerate at p^* , all agents should bring either $m = 0$ or $m = p^*$ to the DM: given it is costly to carry money, when $\pi > \beta - 1$, there is no incentive to bring $m > p^*$ or $m \in (0, p^*)$. It should also be obvious that if everyone brings m^* , posting $p < m^*$ would not be a best response, and posting $p > m^*$ results in no trade. Therefore, we look for equilibria with $p^* = m^*$. What needs to be established is: when do agents prefer to post

⁴In the first equation, the first term is the payoff to no meeting, the second the payoff in a meeting when the agent wants to buy, and the third the payoff when the agent wants to sell.

$p = m^*$, as opposed to posting some high p and not selling at all; and when do they prefer bringing $m = p^*$, as opposed to bringing 0 and not buying at all.⁵

At this point we invoke market clearing and set $m = M$ at every date. Given a path for M induced by an exogenous value for π , a steady-state monetary equilibrium is a path for ϕ such that real balances ϕM are constant. A sale at $p_+ = M_+$ in the next DM covers the cost of production iff $c \leq W_+(M_+) - W_+(0) = M_+\phi_+$. Also, conditional on having $m_+ = M_+$, in the DM, agents prefer buying at $p_+ = M_+$ to not trading iff $u \geq W_+(M_+) - W_+(0) = \phi_+M_+$. Hence, we need $\phi_+M_+ \in [c, u]$ for trade to be profitable for sellers and buyers ex post. We actually need something more for buyers: we need to check that $m_+ = M_+ > 0$ makes them better off than $m_+ = 0$ ex ante when they choose m_+ in the CM. So we must check $\beta V_+(M_+, p_+) - \phi M_+ \geq \beta V_+(0, p_+)$, which reduces to $\sigma u \geq M_+ [\phi/\beta - (1 - \sigma)\phi_+]$. Letting the real interest rate r be given by $1 + r = 1/\beta$. and the nominal rate i by $i = (1 + \pi)(1 + r) - 1$, this reduces to $M_+\phi_+ \leq \sigma u/(i + \sigma)$, which provides an upper bound on ϕ_+M_+ less than the one given above for ex post trade as long as $i > 0$.

What we have shown is that any value of real balances in the interval $\left[c, \frac{\sigma}{i+\sigma}u \right]$ is consistent with all the equilibrium conditions. As a special case, with $\pi = 0$ and M constant, the interval in question becomes $\left[c, \frac{\sigma}{r+\sigma}u \right]$; as another, at the Friedman rule $\pi = \beta - 1$ it becomes $[c, u]$. In general, notice that higher π or i reduces the size of the interval, and for big enough i the equilibrium collapses. It is also important to emphasize that this is a real indeterminacy: although in this simple context every DM meeting results in a trade and the good is indivisible, the ex post split of the gains from trade between buyers and sellers depends on the equilibrium we choose. We also point out that this indeterminacy is essentially the same as that discussed in the Green-Zhou framework, even if our model is much easier, because they have to worry about the endogenous distribution $G(m)$, even when $F(p)$ is degenerate, while we do not.

Proposition 1 *For $i < \sigma(u - c)/c$ there exists a continuum of steady-state, single-price monetary equilibria where real balances ϕM can be anything in the interval $\left[c, \frac{\sigma}{i+\sigma}u \right]$; for $i > \sigma(u - c)/c$ there is no such equilibrium.*

Although this completes the main point we wanted to make, to develop some further intuition and connect to an additional literature it is useful to present a slightly different version of the above model along the lines of Rocheteau and Wright (2005). To this end, suppose agents come in two *permanently* different types called buyers and sellers, distinguished as follows: in the DM buyers can consume good

⁵There is of course always a nonmonetary equilibrium where m is not valued.

x but cannot supply h , while sellers can supply h but do not consume x . We can normalize the measure of buyers to 1 and let the measure of sellers be n . In all other respects the model is the same. One cannot do this in the basic Green-Zhou framework because sellers would have no incentive to produce for m if they never get to be buyers in a future DM, but here we can since sellers can always spend their money in the CM. It should also be clear that sellers in this version of the model never carry m into the DM, but simply set p , while buyers carry m and have no p to set.

Let $V^b(m)$ and $V^s(p)$ be the type-dependent DM payoffs, and similarly for $W^b(m)$ and $W^s(m)$. Then the CM problems look like the previous model, except buyers do not set p_+ and sellers choose $m_+^s = 0$, while in the DM

$$\begin{aligned} V^b(m) &= \sigma_b \int_0^m (u - \phi\tilde{p})dF(\tilde{p}) + W^b(m) \\ V^s(p) &= \sigma_s [1 - G(p)] (\phi p - c) + W^s(0), \end{aligned}$$

where σ_j is the probability of a single-coincidence DM meeting for j , satisfying the identity $\sigma_b = n\sigma_s$. As in the previous model, while there could potentially be other equilibria, consider single-price, steady-state monetary equilibria. Again, if F is degenerate at p^* all buyers should bring $m_+ = 0$ or $m_+ = p^*$ to the DM, and if G is degenerate at m^* all sellers will post $p = m^*$ if they want to trade at all. In fact Proposition 1 goes through as stated except σ_b replaces σ .⁶

The reason this alternative model is interesting is that it looks like a monetary version of the large literature on price posting in nonmonetary economies emanating from Diamond (1970). In Diamond, sellers post prices and buyers search, as in our model, but he gets a unique equilibrium, in this equilibrium all sellers post the same p^* , and this p^* extracts all the surplus from buyers. The analog here is $\phi M = u$. Several things are different in our monetary economy, however. First, $\phi M = u$ is *not* an equilibrium here, since buyers have to make a costly ex ante choice to bring money to the market, except at the Friedman rule $i = 0$. Second, there are lots of other single-price equilibria. The reason is that the usual Diamond-style logic does not apply: in his model, if all sellers charge the same price p , buyers would in principle be willing in a given meeting to pay $p + \varepsilon$ for some $\varepsilon > 0$ rather than continue searching. Hence, sellers will want to shade up their price until in real terms it equals u . Here buyers are also *willing* in a given meeting to pay $p + \varepsilon$ but are *not able* since they only have $m = p$.

The liquidity constraints implied by a monetary model can therefore have a dramatic effect on the set of equilibria. Moreover, it is well known how to get around Diamond's result that there must be

⁶It appears sellers have only a static trade off, since they produce only after meeting a buyer; it is however easy to alternatively assume they produce in the CM, in which case we need $\beta\sigma_s\phi M \geq c$ instead of $\phi M \geq c$, which is basically a renormalization. The difference between sellers and buyers is the former do not make an ex ante investment in cash.

a single price using a variety of technical devices, including multilateral matching or heterogeneous preferences (see e.g. Burdett and Judd 1983 or Albrecht and Axell 1984). Already related monetary economies have been shown to generate interesting endogenous distributions of money and prices with similar devices (see e.g. Head and Kumar 2005, Dutu, Julien and King 2007, Galenianos and Kircher 2007 or Curtis and Wright 2004). But it is interesting to conjecture that there might exist equilibria with dispersion in economies like the one considered here *without* devices like heterogeneous preferences or multilateral matching. This is a subject of ongoing research. The main point of this note was to show some well-known and interesting results in the Green-Zhou model can be derived with much less effort once one integrates it with Lagos-Wright.

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