

# The Venture Capital Cycle\*

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## Abstract

We propose a model of venture capital markets, characterized by the following cycle: (i) capitalists raise funds; (ii) capitalists and entrepreneurs match; (iii) after matching both parties take active roles in implementation; (iv) when ventures mature capitalists leave to start a new cycle. We analyze entry by capitalists, the flow of funds into the market, the decisions to start a venture, the duration of each phase in the cycle, and the returns to the parties.

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# 1 Introduction

The venture capital (private equity) market works like this: Some entrepreneurs with viable ideas for projects, or ventures, have trouble getting financing from banks or other conventional sources, typically because their ideas are technically difficult for non-experts to evaluate and/or because they lack sufficient collateral. Venture capitalists, or VC's, are agents with access to funds that specialize in locating, evaluating, and selecting high-risk but potentially high-return projects. They also have expertise in implementation that is a key input into the project; more than just lending money, VC's enter into partnerships with entrepreneurs, taking an active role in development, management, monitoring, etc. As projects mature, especially once they can be more easily evaluated and operated by others, VC's exit to raise funds for new projects. This is called *the venture capital cycle*.<sup>1</sup>

The economics of this market seems worth investigation. There have been rapid increases in the size of the venture capital industry over the past two decades: private equity increased from under \$5 billion in 1980 to over \$300 billion in 2004, and in the past 25 years more than \$1 trillion has passed through these funds (Lerner 2003; Lerner et al. 2004). Despite the 2001-2003 downturn, commitments to US venture capital funds increased from \$8.7 billion in 2003 to \$16-17 billion in 2004. Although it still is only a fraction of the total financial sector, this market is considered an engine for economic growth. For example, a significant fraction of blue-chip firms, including Apple, Compaq, FedEx, Intel, Microsoft and Cisco Systems,

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<sup>1</sup>We're not making this up. In the words of the experts, "To understand the venture capital industry, one must understand the whole 'venture cycle.' The venture capital cycle starts with raising a venture fund; proceeds through the investment in, monitoring of, and adding value to firms; continues as the venture capital firm exits successful deals and returns capital to its investors; and renews itself with the venture capitalist raising additional funds." (Gompers and Lerner 2001).

received venture capital in their early stages, and generally venture capital-backed firms perform quite well, by a variety of criteria.<sup>2</sup>

There is much research on particular aspects of this market, surveyed e.g. in Gompers and Lerner (1999, 2001) and Kaplan and Stromberg (2001). But we think there is much more to be done. Two questions that we think especially interesting are, what determines the duration of each phase of venture cycle, and what determines the size of the fund? The former (timing) is a major concern; as Cochrane (2004) puts it, e.g., “The risk facing a venture capital investor is as much when his or her return will occur as it is how much that return will be.” The latter (fund size) is a decision affected by various factors discussed in the empirical literature; but scholars still seem uncertain about the determinants of the optimal level of investment in the industry, or, at the individual level, optimal fund size. To address these issues, we develop an equilibrium model that makes explicit various frictions and imperfections in the market, using some tools from search theory.<sup>3</sup>

The idea that frictions may be relevant here is not new. According to Gompers and Lerner (2000), e.g., there is a positive relation between the valuation of venture capital investment and liquidity inflows in the market not driven by improvements in potential prospects, as would be predicted

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<sup>2</sup>A study by NVCAA (2002) found that during 1970-2000 “venture capital-backed companies had approximately twice the sales, paid almost three times the federal taxes, generated almost twice the exports, and invested almost three times as much in R&D as the average non-venture capital-backed public company, per each \$1,000 of assets.” Kortum and Lerner (2000) estimated that although such firms account for only about 3 percent of all R&D spending, they generate about 14 percent of the innovation.

<sup>3</sup>Search has proved a useful way to study many markets with frictions; see Rogerson et al. (2005) for a recent survey of applications to labor markets, and the references therein for applications to the theory of marriage, monetary economics, and industrial organization. Recent applications in finance include Duffie, Garleanu and Pedersen (2002), Weill (2004), and Lagos (2006). Formally, although different on several dimensions, our framework shares some features with the macro-labor model in Pissarides (1990) or Mortensen and Pissarides (1994) and the monetary model in Lagos and Wright (2005).

by classic asset pricing theory, and interpret this as evidence against the assumption of a perfect and frictionless market. Gompers and Lerner (1999) and Lerner (2002) also describe recurrent imbalances between supply and demand for funds. Our model helps clarify these observations by explaining capital flows into this market, the decision to enter (or not) into a venture, and how we can simultaneously have entrepreneurs in search of funding and capitalists with idle funds. The model also determines the lengths of the different phases of the venture cycle, and how these depend on various factors.

There is a considerable literature in financial economics that explores how VC's screen, select, finance, monitor and advise their companies.<sup>4</sup> At the risk of generalizing, it is mainly partial equilibrium, and focuses on optimal contracts between VC's and entrepreneurs in the presence of adverse selection and moral hazard; there is little consideration of other market frictions. Inderst and Muller (2004), Michelacci and Suarez (2004), and Keuschnigg (2003) do consider search-based models, but our approach and focus are very different. For instance, while they assume that every match between an entrepreneur and capitalist is the same, we take seriously the notion of heterogeneity, which seems critical for analyzing project selection. We also model explicitly liquidity in a way that is realistic and has not been done before, and is critical for analyzing the duration of the different phases of the cycle and optimal fund size.<sup>5</sup>

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<sup>4</sup>In addition to research summarized in the surveys cited above, we mention Aghion and Tirole (1994), Bascha and Walz (2001), Bergemann and Hege (1998), Casamatta (2002), Cumming and MacIntosh (2002), Hellmann (1998), Hellmann and Puri (2000,2002), Kaplan and Strömberg (2000,2001), Schmidt (2002), and Babarino and Jovanovic (2006).

<sup>5</sup>In Silveira and Wright (2006) we also endogenized liquidity in the market for "ideas" in a way that is similar to the approach here, but that paper was not about the venture capital market and had nothing to say about the venture cycle.

## 2 Basic Assumptions

There are two types of risk neutral agents: entrepreneurs labeled  $e$ , and capitalists labeled  $k$ . They need each other to start a venture. The notion is that  $e$  has an idea and some technical ability, but lacks the funds and/or the expertise to implement it, while  $k$  is a specialist in finding and evaluating potentially profitable ideas that can provide funds, expertise and other inputs.<sup>6</sup> For simplicity, assume that  $k$  can enter into one and only one project at a time, and symmetrically  $e$  can deal with only one VC. Let  $\ell_e$  and  $\ell_k$  denote inputs, or investments, by the two parties during the implementation stage, which are taken as fixed for now, but endogenized below; the marginal cost of this investments is normalized to 1 for each agent without loss of generality. Agents meet bilaterally in continuous time according to a standard Poisson matching process, with arrival rates  $\alpha_e$  and  $\alpha_k$  that they take as given, but are endogenized below.

In general, when  $e$  and  $k$  meet, they draw  $(R, C)$  from a distribution with CDF  $F(R, C) = pr(\tilde{R} \leq R, \tilde{C} \leq C)$ , where  $C$  is the start-up cost and  $R$  the return on the project when it comes to fruition in the future. Although one could proceed differently, we assume that agents and projects are ex ante homogenous, but  $(R, C)$  differs ex post across matches, say because  $k$  may have more or less expertise in the area corresponding to the idea of a particular  $e$ . We abstract from private information, not because we think it is uninteresting, but because we want to focus on different issues; thus, when they meet,  $e$  and  $k$  both know the realization  $(R, C)$ . In this section

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<sup>6</sup>As we said in the Introduction,  $e$  cannot easily access banks or other conventional sources of funds because he lacks significant tangible assets and because his idea may be hard for non-experts to evaluate. We abstract from the question of whether  $e$  should, instead of entering the venture capital market, try to save up and finance the project himself (Babarino and Jovanovic 2006); even if  $e$  has access to his own funds, however, he may still value the expertise of  $k$  in terms of implementation.

we begin with the simplest case where  $C = 0$  and  $R > 0$  are constant across matches. With  $C = 0$ , funding is not an issue, but  $k$  still brings expertise to the table. In any event, we will soon allow random  $C$  and  $R$ .

If they agree to start a venture, a payment  $p$  from  $e$  to  $k$  is negotiated, due when the project comes to fruition. We can trivially reinterpret  $k$  as receiving an equity share  $s$ , as is typically the case in reality, by setting  $s = p/R$ ; even if returns are random, it does not matter if they bargain over a fixed payment  $p$  or the equity share  $s$ , since agents are risk neutral. We assume that the agreement is binding – e.g. an enforceable contract, not subject to opportunistic renegotiation. The venture comes to fruition at some random date after implementation begins. Let  $\sigma$  be the Poisson rate at which this happens as a function of the investments made during implementation,  $\sigma = \sigma(\ell_e, \ell_k)$ . When the venture comes to fruition,  $k$  receives  $p$  and returns to the market, while  $e$  keeps the residual  $R - p$ , exits, and is replaced by a clone of himself in order to maintain steady state.<sup>7</sup>

In our framework there are four recurring stages in the life of a VC, corresponding to the cycle described in the Introduction: (i) fund raising; (ii) search for a partner/project; (iii) implementation; and (iv) exit upon maturation. For now we ignore the first stage, since  $C = 0$ , but fund raising will be a crucial part of the general model discussed below. The second stage is random in duration because search takes time, and depends on market tightness, we discuss below. The third stage is also random because the rate at which projects come to fruition depends on investment inputs, as we also

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<sup>7</sup>In reality the parties may actually sell the venture – i.e. *go public* – but again this does not matter because of risk neutrality, so we assume  $e$  simply pays off  $k$ . Also, although it is realistic to have  $k$  start the cycle anew, but  $e$  exit the market after a project matures, the model also works if we allow  $e$  to go back with a new idea. Having new  $e$  agents enter only when old ones leave, as opposed to specifying an exogenous inflow, say, simplifies the analysis, but is not essential, as in many search models (see e.g. Burdett and Coles 1997).

discuss below. The exit stage can also be random, but since this is the one phase that has been analyzed in a related model, by Michelacci and Suarez (2004), we assume for now that  $k$  can exit and reap his return costlessly once the project matures.

### 3 Search and Bargaining

Let the value function for type  $i$  in state  $j$  be  $V_i^j$ , where  $j = 0$  means that  $i$  is in the partner search stage and  $j = 1$  means that  $i$  is in the implementation stage. Let  $\bar{p}$  be the value of  $p$  prevailing in the market. Given the discount rate  $r$ , the value functions for  $e$  satisfy<sup>8</sup>

$$rV_e^0 = \alpha_e(V_e^1 - V_e^0) \quad (1)$$

$$rV_e^1 = \sigma(R - \bar{p} - V_e^1) - \ell_e, \quad (2)$$

and those for  $k$  satisfy

$$rV_k^0 = \alpha_k(V_k^1 - V_k^0) \quad (3)$$

$$rV_k^1 = \sigma(\bar{p} + V_k^0 - V_k^1) - \ell_k. \quad (4)$$

It is simple to solve these (linear) equations for

$$V_e^0 = \frac{\alpha_e[\sigma(R - \bar{p}) - \ell_e]}{(r + \sigma)(r + \alpha_e)} \text{ and } V_e^1 = \frac{\sigma(R - \bar{p}) - \ell_e}{r + \sigma} \quad (5)$$

$$V_k^0 = \frac{\alpha_k(\sigma\bar{p} - \ell_k)}{r(r + \sigma + \alpha_k)} \text{ and } V_k^1 = \frac{(r + \alpha_k)(\sigma\bar{p} - \ell_k)}{r(r + \sigma + \alpha_k)}. \quad (6)$$

When  $e$  and  $k$  meet, they negotiate over  $p$  for their particular project, taking as given the market  $\bar{p}$ . We adopt the generalized Nash bargaining solution

$$\max_p S_k^\theta S_e^{1-\theta} \quad (7)$$

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<sup>8</sup>These standard programming equations have simple interpretations: e.g. the per period value of the venture to  $e$  is  $rV_e^1$ , which equals the rate at which it comes to fruition  $\sigma$ , times the capital gain  $R - \bar{p} - V_e^1$ , minus the flow cost  $\ell_e$ .

where  $\theta$  is the bargaining power of  $k$ , while  $S_k = V_k^1 - V_k^0$  and  $S_e = V_e^1 - V_e^0$  are the surpluses. It is convenient rearrange (4) as  $V_k^1 = \frac{\sigma(p+V_k^0)-\ell_k}{r+\sigma}$  and (2) as  $V_e^1 = \frac{\sigma(R-p)-\ell_e}{r+\sigma}$ , to express

$$S_k = \frac{\sigma p - rV_k^0 - \ell_k}{r + \sigma} \quad (8)$$

$$S_e = \frac{\sigma(R-p) - (r+\sigma)V_e^0 - \ell_e}{r + \sigma} \quad (9)$$

as functions of the  $p$  in the current venture, as well as the outside options  $V_k^0$  and  $V_e^0$ , which are themselves functions of  $\bar{p}$ . For future reference, write the total surplus  $S = S_k + S_e$  as

$$S = \frac{\sigma R - (r + \sigma)V_e^0 - rV_k^0 - \ell_e - \ell_k}{r + \sigma}. \quad (10)$$

Inserting (8) and (9) into (7), taking the first order condition, and simplifying, we find the  $p$  for a particular venture in terms of its expected return and costs, plus the outside options:

$$\sigma p = \theta \sigma R - \theta \ell_e + (1 - \theta)\ell_k - \theta(r + \sigma)V_e^0 + (1 - \theta)rV_k^0 \quad (11)$$

Inserting (5)-(6), setting  $\bar{p} = p = p^*$ , and simplifying, we get

$$\sigma p^* = \frac{\theta r (r + \sigma + \alpha_k) (\sigma R - \ell_e) + (1 - \theta)(r + \sigma)(r + \alpha_e)\ell_k}{\theta r (r + \sigma + \alpha_k) + (1 - \theta)(r + \sigma)(r + \alpha_e)}. \quad (12)$$

This yields  $p^*$  as a weighted average of  $\sigma R - \ell_e$  and  $\ell_k$ . There are, however, two participation constraints to check:  $S_k \geq 0$  and  $S_e \geq 0$ . It turns out that these both hold iff  $S \geq 0$ , which simplifies to  $\sigma R - \ell_k - \ell_e \geq 0$ . Hence, as long as we satisfy this obviously necessary condition for a venture to be worthwhile, the constraints are not binding.<sup>9</sup>

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<sup>9</sup>This is standard in search-and-bargaining models with transferable utility, and implies that one side wants to form a relationship iff the other side wants to form a relationship (see e.g. Burdett and Wright 1998). Of course, for now all relationships are the same, but the same result holds with  $(R, C)$  random.

Taking the parameters on the RHS of (12) as exogenous, for now, the theory makes some interesting predictions. For instance,  $\partial p^*/\partial R > 0$ , so more lucrative projects yield higher payments to the VC. However, his equity share  $s^* = p^*/R$  could go up or down, since  $\partial s^*/\partial R > 0 \iff \theta r(r + \sigma + \alpha_k)\ell_e > (1 - \theta)(r + \sigma)(r + \alpha_e)\ell_k$ ; in particular,  $\ell_e = \ell_k = 0 \implies \partial s^*/\partial R = 0$ . Also,  $\partial \sigma p^*/\partial \sigma > 0$ , so projects that take less time yield the VC a higher payoff per period of implementation  $\sigma p^*$ ; but  $\partial p^*/\partial \sigma < 0$ , at least if  $\ell_e$  and  $\ell_k$  are not too big, so the final realized payment  $p^*$  is actually lower. Also,  $\partial p^*/\partial \theta > 0$ ,  $\theta = 1 \implies p^* = R - \ell_e/\sigma$ , and  $\theta = 0 \implies p^* = \ell_k/\sigma$ . The effect of  $r$  is ambiguous (e.g.  $\alpha_e \approx 0 \implies \partial p^*/\partial r < 0$  and  $\alpha_k \approx 0 \implies \partial p^*/\partial r > 0$ ). Also,  $\partial p^*/\partial \ell_k > 0$  and  $\partial p^*/\partial \ell_e < 0$ , so that when the project requires greater investment by the capitalist (entrepreneur)  $p^*$  goes up (down). And  $\partial p^*/\partial \alpha_k > 0$  and  $\partial p^*/\partial \alpha_e < 0$ , so higher arrival rates and hence better outside options for the capitalist (entrepreneur) increase (decrease)  $p^*$ . These predictions seem reasonable, relevant, and potentially testable.<sup>10</sup>

## 4 Arrival Rates

The fact that these predictions take the parameters in (12) as exogenous may be appropriate for some purposes – e.g., for some empirical work, one can perhaps measure some of them directly – but our goal is to determine more things endogenously. The first things to discuss are the arrival rates  $\alpha_e$  and  $\alpha_k$ . Let the measure of type  $e$  be normalized to 1, and let the measure of type  $k$  be  $n$ , which is fixed for now but endogenized below. Let the

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<sup>10</sup>We also tried other bargaining solutions, including the case where the threat point is 0 rather than  $V_j^0$  (which is equally valid in the sense that either case can be derived from an underlying strategic bargaining model; see e.g. Osborne and Rubinstein 1990). Details are available upon request, but the predictions were similar.

number of ventures in the implementation stage be  $m$ . Since each venture takes one  $e$  and one  $k$  off the market, there are  $1 - m$  and  $n - m$  of each type in the partner-search phase at any point in time. See Figure 1. We have the identity  $\alpha_k(n - m) = \alpha_e(1 - m)$ , since both sides give the flow of new ventures, which in steady state equals  $\sigma m$ . Therefore  $m = \frac{n\alpha_k}{\alpha_k + \sigma}$ , and the measures of  $e$  and  $k$  in the search process are  $1 - m = \frac{(1-n)\alpha_k + \sigma}{\alpha_k + \sigma}$  and  $n - m = \frac{n\sigma}{\alpha_k + \sigma}$ .

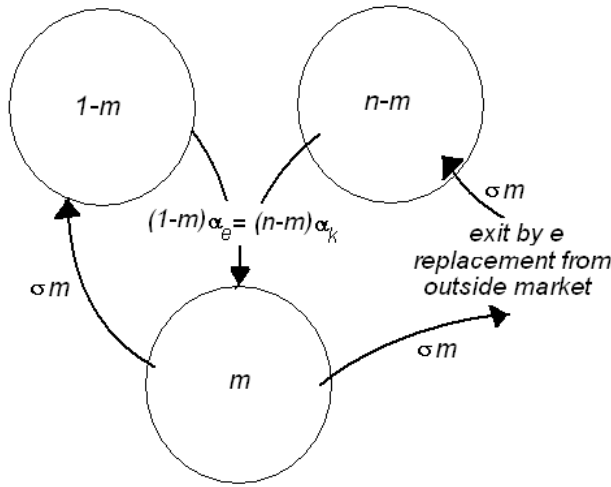


Figure 1: Flows across states

Define market tightness as the ratio of  $k$  to  $e$  in the search process,  $\tau = \frac{n-m}{1-m} = \frac{n\sigma}{(1-n)\alpha_k + \sigma}$ . Assume that the total number of meetings between unmatched  $e$  and  $k$  agents is given by a constant returns, technology  $\mu = \mu(1 - m, n - m)$ , satisfying the usual assumptions: i.e.,  $\mu$  is strictly increasing and concave, with  $\mu(0, 0) = 0$ ,  $\mu_1(0, \cdot) = \infty$ ,  $\mu_1(\infty, \cdot) = 0$ ,  $\mu_2(\cdot, 0) = \infty$ , and  $\mu_2(\cdot, \infty) = 0$ . The arrival rate for  $k$  satisfies

$$\alpha_k = \frac{\mu(1 - m, n - m)}{n - m} = \mu\left(\frac{1}{\tau}, 1\right) = \mu\left[\frac{(1 - n)\alpha_k + \sigma}{n\sigma}, 1\right]. \quad (13)$$

Given  $n$  and  $\sigma$ , there is a unique solution  $\alpha_k$  to (13).<sup>11</sup> Given this,

$$\alpha_e = \alpha_k \frac{n - m}{1 - m} = \mu(1, \tau) = \mu \left[ 1, \frac{n\sigma}{(1 - n)\alpha_k + \sigma} \right]. \quad (14)$$

We often write  $\alpha_k = \alpha_k(\tau)$  and  $\alpha_e = \alpha_e(\tau)$ , where  $\alpha_k(\tau) \equiv \mu(1/\tau, 1)$  and  $\alpha_e(\tau) \equiv \mu(1, \tau) = \tau\alpha_k(\tau)$ ; clearly,  $\alpha'_k(\tau) < 0$  and  $\alpha'_e(\tau) > 0$ .

This gives us an internally consistent way to determine market tightness and the arrival rates as a function of  $n$  and  $\sigma$ , from which we then determine  $p$  as before. Increasing  $n$  obviously increases  $\tau$ . Increasing  $\sigma$  increases (decreases)  $\tau$  when  $n < 1$  ( $n > 1$ ), and when  $n = 1$ ,  $\alpha_k = \alpha_e = \mu(1, 1)$  is independent of  $\sigma$ . One can redo the comparative statics discussed above, except instead of changing  $\alpha_e$  and  $\alpha_k$  directly, changing  $n$  and letting  $\alpha_e$  and  $\alpha_k$  respond. One can also allow  $\alpha_e$  and  $\alpha_k$  to change when  $\sigma$  changes, although it is still legitimate to change  $\sigma$  and keep  $\alpha_e$  and  $\alpha_k$  constant by assuming  $n = 1$ . One can also determine how the parameters affect the measures of agents in the implementation and search phases, as well as the hazard rate  $\alpha_j$  (the instantaneous probability of exiting a given state) and expected duration  $1/\alpha_j$  for agents in partner search.

The next step is to endogenize  $n$ , following Pissarides' (1990) approach to macro-labor theory, by allowing entry. The idea is that, over the relevant range, there is a perfectly elastic supply of potential VC's who can enter the market as long as they pay some fixed cost  $\kappa$  – say, a cost to acquiring expertise (later we introduce explicitly the cost of raising funds). Then

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<sup>11</sup>The RHS of (13) is a function of  $\alpha_k$  with intercept  $\mu[1/n, 1] > 0$ . It is increasing if  $n > 1$  and decreasing if  $n < 1$ . In the latter case there clearly exists a unique  $\alpha_k$  solving (13), and in the former case the same is true because  $\mu_1(\infty, 1) = 0$ . Another way to proceed is to rewrite (13) as  $\mu(1, \tau) = \sigma(n - \tau)/(\tau(1 - n))$ . It is then clear that  $\partial\tau/\partial n > 0$  and  $\partial\tau/\partial\sigma \simeq 1 - n$ . One can also write  $m = (n - \tau)/(1 - \tau)$  and from this show, e.g.,  $\partial m/\partial\sigma < 0$ .

equilibrium requires  $\kappa = V_k^0$ , which by (6) is equivalent to

$$\kappa = \frac{\alpha_k(\sigma p - \ell_k)}{r(r + \sigma + \alpha_k)}. \quad (15)$$

Using (12) to eliminate  $p$ , then inserting  $\alpha_j = \alpha_j(\tau)$ , we arrive at

$$\kappa = \frac{\alpha_k(\tau)\theta(\sigma R - \ell_k - \ell_e)}{\theta r[r + \sigma + \alpha_k(\tau)] + (1 - \theta)(r + \sigma)[r + \alpha_e(\tau)]}. \quad (16)$$

Equilibrium with entry is a pair  $(\tau^*, p^*)$  solving (16) and (12).

Conveniently, equilibrium is recursive: we first look for a  $\tau^*$  solving (16); given  $\tau^*$  we know  $\alpha_k^* = \alpha_k(\tau^*) = \mu(1/\tau^*, 1)$  and  $\alpha_e^* = \alpha_e(\tau^*) = \mu(1, \tau^*)$ ; we also know  $m^*$ ; the actual measure of VC's  $n^*$  is then pinned down from  $\alpha_k^*$  by (13); and, finally, we get  $p^*$  from (12), as in the model with no entry. The usual assumptions on the matching technology  $\mu(\cdot)$  imply that  $\alpha_k(0) = \infty$ ,  $\alpha_k(\infty) = 0$ ,  $\alpha_e(0) = 0$ , and  $\alpha_e(\infty) = \infty$ . Given this, it is easy to check that as long as  $\theta > 0$ , the RHS of (16) is decreasing in  $\tau$ , approaches 0 as  $\tau \rightarrow \infty$ , and approaches  $\bar{\kappa} = (\sigma R - \ell_k - \ell_e)/r$  as  $\tau \rightarrow 0$ . Hence, given  $\sigma R - \ell_k - \ell_e \geq 0$ , which we obviously need for ventures to be profitable in the first place, there exists a unique equilibrium, with  $\tau^* > 0$  iff  $\kappa < \bar{\kappa}$ . Given  $\kappa < \bar{\kappa}$ , it is easy to verify  $\partial\tau^*/\partial\kappa < 0$ ,  $\partial\tau^*/\partial R > 0$ ,  $\partial\tau^*/\partial\sigma > 0$ ,  $\partial\tau^*/\partial\ell_k < 0$ ,  $\partial\tau^*/\partial\ell_e < 0$ ,  $\partial\tau^*/\partial\theta > 0$  and  $\partial\tau^*/\partial r < 0$ .

The framework is actually somewhat more tractable once we add entry (as in the Pissarides labor model). It implies that market tightness will increase when the cost of entry falls, when ventures are more lucrative, when ventures are less costly in terms of either time or other investments, when VC's have more bargaining power, and when they are more patient. Anything that increases  $\tau$  increases entry, since  $n$  is increasing in  $\tau$ .<sup>12</sup> Anything

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<sup>12</sup>To verify this, use  $\tau = n\sigma/[(1-n)\alpha_k + \sigma]$  to rewrite  $\alpha_k = \mu(1/\tau, 1)$  as  $(n-\tau)\sigma = \mu(1, \tau)(1-n)$ . Then  $\partial n/\partial\tau \simeq \mu_2 + \sigma + \mu - \tau\mu_2 > 0$ , where  $a \simeq b$  means that  $a$  and  $b$  have the same sign, since  $\mu > \tau\mu_2$ , by our assumptions on the matching function.

that affects  $n$  and  $\tau$  affects the arrival rates  $\alpha_e$  and  $\alpha_k$ , and hence the duration of partner-search on both sides of the market, as well as the measures of ventures in the implementation and search stages. In equilibrium, these changes affect  $p$  in ways that can be easily computed. Again, even this simple model can be used to make predictions that seem reasonable, relevant, and testable empirically.

## 5 Investment in Implementation

Having determined the duration of search, we now endogenize the length of the implementation stage, by letting agents choose the investments  $(\ell_e, \ell_k)$ . We assume that these inputs affect not the return  $R$ , but the rate at which the venture comes to fruition  $\sigma = \sigma(\ell_e, \ell_k)$ . We assume  $\sigma(\cdot)$  is concave; satisfies  $\sigma(0, \ell_k) = 0$ ,  $\sigma_e(0, \ell_k) = \infty$  and  $\sigma_e(\infty, \ell_k) = 0$  for a given  $\ell_k$ ; and something similar for a given  $\ell_e$ . We restrict  $(\ell_e, \ell_k) \in \mathcal{L} = [0, \hat{\ell}]^2$  for some  $\hat{\ell} < \infty$ . When  $e$  and  $k$  meet, they bargain over  $(\ell_e, \ell_k)$  as well as  $p$ . We can find the outcome by inserting (8)-(9) into (7) and maximizing with respect to all three variables. However, an easier route to the same end is to first maximize the total surplus  $S$  (which does not depend on  $p$ ) with respect to  $(\ell_e, \ell_k)$ , and then split  $S$  by maximizing the Nash product with respect to  $p$ .

The problem of maximizing the surplus is:

$$\max_{(\ell_e, \ell_k) \in \mathcal{L}} S = \frac{\sigma(\ell_e, \ell_k)R - [r + \sigma(\ell_e, \ell_k)] V_e^0 - rV_k^0 - \ell_e - \ell_k}{r + \sigma(\ell_e, \ell_k)} \quad (17)$$

Because  $S$  is continuous and  $\mathcal{L} = [0, \hat{\ell}]^2$  a solution exists. Standard curvature conditions can be imposed to guarantee it is interior, and hence satisfies

the first-order conditions

$$\ell_e : [rR + rV_k^0 + \ell_e + \ell_k]\sigma_e - (r + \sigma) = 0 \quad (18)$$

$$\ell_k : [rR + rV_k^0 + \ell_e + \ell_k]\sigma_k - (r + \sigma) = 0. \quad (19)$$

It is easy to check that  $S$  is strictly concave at any point where the first-order conditions hold, and so there is a unique choice  $(\ell_e, \ell_k)$  that solves (17). From (18)-(19) we immediately see that  $\sigma_e = \sigma_k$ , which is natural, given that we normalized  $\ell_e$  and  $\ell_k$  to have the same marginal cost.

This choice of  $(\ell_e, \ell_k)$  takes all the market values  $V_i^j$  as given, although from (18)-(19) it actually depends only on  $V_k^0$ . In equilibrium,  $V_k^0$  is given by (6) as a function of the market  $\bar{p}$ , which is in turn given by (12). Inserting these results and simplifying, we get

$$\sigma_e = \sigma_k = \frac{\theta r(r + \sigma + \alpha_k) + (1 - \theta)(r + \sigma)(r + \alpha_e)}{[r + (1 - \theta)\alpha_e + \alpha_k]rR + [r + (1 - \theta)\alpha_e](\ell_e + \ell_k)}. \quad (20)$$

The two equalities in (20) determine  $(\ell_e^*, \ell_k^*)$ , and then we get  $p^*$  from (12) as in the model with  $\ell_j$  fixed. Notice that the solution  $(\ell_e, \ell_k)$  to bargaining problem (17) does not depend on  $\theta$ , since agents at this stage maximize the joint surplus; in equilibrium, however, (20) indicates that  $(\ell_e^*, \ell_k^*)$  does depend on  $\theta$ , since  $\theta$  influences the continuation value  $V_k^0$ .

Let us assume for the moment that  $\alpha_e$  and  $\alpha_k$  are fixed, independent of  $\sigma$ , as would be the case if  $n = 1$ , e.g., since as we saw in the last section  $n = 1 \implies \alpha_k = \alpha_e = \mu(1, 1)$ . This gives a parallel to the model in the previous section, where  $\sigma$  was fixed and the arrival rates were endogenous. Then an equilibrium is simply a solution  $(\ell_e^*, \ell_k^*)$  to (20), although this still entails solving a fixed-point problem. Thus, consider an arbitrary pair  $(\bar{\ell}_e, \bar{\ell}_k) \in \mathcal{L}$  prevailing in the market. Clearly the values of  $\sigma$  and  $V_j^i$  prevailing in the market are continuous in  $(\bar{\ell}_e, \bar{\ell}_k)$ . Given this, the unique solution  $(\ell_e, \ell_k) \in$

$\mathcal{L}$  to (17) discussed above is continuous in  $(\bar{\ell}_e, \bar{\ell}_k)$  by the Theorem of the Maximum. Since  $\mathcal{L} = [0, \hat{\ell}]^2$  there exists a fixed point  $(\ell_e^*, \ell_k^*)$ . Hence, an equilibrium always exists when the  $\alpha_e$  and  $\alpha_k$  are fixed.

Before we endogenize  $\alpha_e$  and  $\alpha_k$ , for the sake of illustration, consider the limiting case  $\theta \rightarrow 1$ . Then (20) reduces to

$$\sigma_e = \sigma_k = \frac{r + \sigma + \alpha_k}{(r + \alpha_k)R + \ell_e + \ell_k}. \quad (21)$$

It is easy to check  $\partial \ell_e / \partial R \simeq \sigma_{ek} - \sigma_{kk}$ , where  $a \simeq b$  means  $a$  and  $b$  have the same sign, which indicates that  $\partial \ell_e / \partial R > 0$  iff  $\ell_e$  is a normal input.<sup>13</sup> Similarly,  $\partial \ell_e / \partial r > 0$  and  $\partial \ell_e / \partial \alpha_k > 0$  under the same conditions. Symmetrically,  $\partial \ell_k / \partial R > 0$ ,  $\partial \ell_k / \partial r > 0$  and  $\partial \ell_k / \partial \alpha_k > 0$  iff  $\ell_k$  is a normal input. The intuition is clear: increasing  $R$  raises the value of the venture once it comes to fruition, while increasing  $r$  or  $\alpha_k$  makes  $k$  more impatient for this to happen, and in either case it is desirable to increase  $\sigma$ . This means either  $\ell_e$  or  $\ell_k$  must increase, and both increase if they are normal inputs.

The next step is to combine the models that endogenize the search and implementation phases. To ease the presentation, suppose that  $\ell_e$  is variable but  $\ell_k = \bar{\ell}_k$  is given – e.g. the VC contributes some fixed factor – so that we can define equilibrium in terms of  $(\ell_e, \tau)$  and represent the outcome graphically in  $\mathbb{R}^2$ . Also, we present the limiting case  $\theta \rightarrow 1$  here and relegate the general case to the Appendix. A model that combines  $\theta = 1$  and free entry has the following extreme feature:  $e$  gets no surplus from the match, since he is compensated only for the cost of  $\ell_e$ , and  $k$  gets no ex ante profit, although he does get profit ex post once  $k$  meets  $e$  match and implementation begins. If taken literally, this implies that all ex ante gains from trade go

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<sup>13</sup>In standard price theory, saying  $\ell_e$  is a normal input means that in the problem  $\min \{p_k \ell_k + p_e \ell_e\}$  s.t.  $\sigma(\ell_e, \ell_k) = \bar{\sigma}$ , the solution satisfies  $\partial \ell_e / \partial \bar{\sigma} \simeq \sigma_k \sigma_{ek} - \sigma_e \sigma_{kk} > 0$ . In our problem, since  $\sigma_k = \sigma_e$ , this reduces to  $\sigma_{ek} > \sigma_{kk}$ .

to whoever is selling expertise to  $e$  at price  $\kappa$  – say, economics professors – although we do not need to take it literally, since by continuity the substantive conclusions hold for  $\theta \approx 1$ . Anyway, the general case is presented in the Appendix.

Equilibrium is characterized by the first order condition (20) for  $\ell_e$ , and the free entry condition (16) for  $\tau$  – as well as the price, but we can first solve independently for  $\ell_e$  and  $\tau$  and then get  $p$  from (12), as in the model with  $\alpha_j$  and  $\sigma$  fixed. When  $\theta = 1$ , after algebraic manipulation to isolate  $\alpha_k(\tau)$  on the LHS, the two conditions can be written

$$\alpha_k(\tau) = \frac{\sigma_e(\ell_e)(rR + \ell_e + \bar{\ell}_k) - r - \sigma(\ell_e)}{1 - \sigma_e(\ell_e)rR} \quad (22)$$

$$\alpha_k(\tau) = \frac{\kappa r [r + \sigma(\ell_e)]}{\sigma(\ell_e)R - \ell_e - \bar{\ell}_k - \kappa r}. \quad (23)$$

We prove in the Appendix the following that there exists a unique equilibrium; if  $\kappa \geq \bar{\kappa}$ , where  $\bar{\kappa}$  is defined in terms of parameters, then the market shuts down; and if  $\kappa < \bar{\kappa}$  then  $\tau^* > 0$  and  $\ell_e = \ell_e^*$ , where interestingly enough,  $\ell_e^*$  is the value of  $\ell_e$  that maximizes  $\tau$  subject to free entry.

These results are illustrated in Figure 2 in  $(\ell_e, \alpha_k)$  space, since it is obviously equivalent to solve for either  $\tau$  or  $\alpha_k(\tau)$ . The FE curve is the locus of points satisfying the free entry condition (23) and the OI curve is the locus of points satisfying the optimal investment condition (22), drawn assuming  $\kappa < \bar{\kappa}$ . These curves intersect uniquely in the positive quadrant, at the minimum of the FE curve over the interval  $(\ell_1, \ell_2)$ , where  $\ell_1$  and  $\ell_2$  are the two solutions to  $\sigma(\ell_e)R - \ell_e - \bar{\ell}_k - r\kappa = 0$ .<sup>14</sup> These curves make it relatively easy to analyze the full equilibrium. For example, as  $\kappa$  increases, OI does not change while FE shifts up, increasing  $\ell_e^*$  and  $\alpha_k(\tau^*)$

<sup>14</sup>Thus,  $\ell_1$  and  $\ell_2$  imply 0 profit given the VC gets a project without having to search – which means it is the limit of the free entry condition when  $\alpha_k \rightarrow \infty$ . Note that the Figure shows what happens when  $\alpha_k < 0$ , although obviously this cannot be an equilibrium.

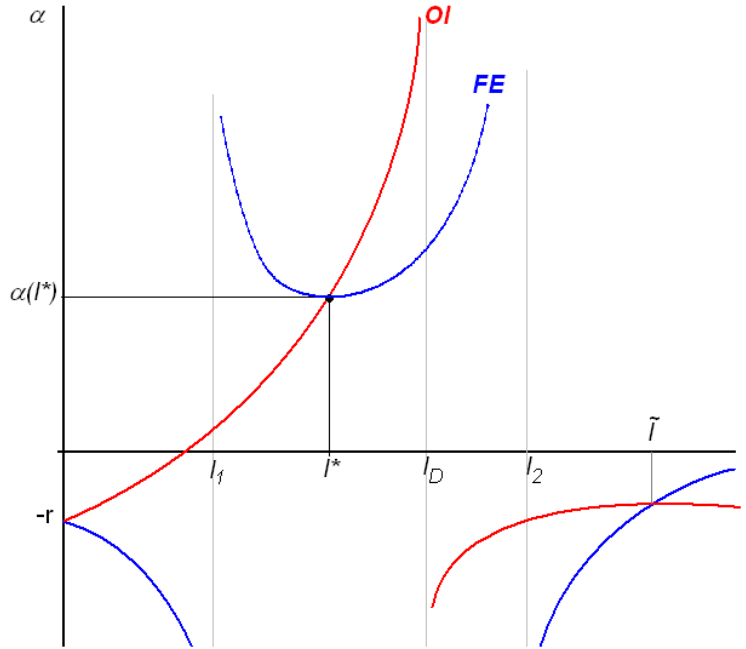


Figure 2: Conditions (22) and (23)

(i.e., decreasing entry). As  $\kappa$  increases, however, the interval  $(\ell_1, \ell_2)$  gets smaller, and eventually vanishes, so that we lose equilibrium.

Similarly, one can ask about a change in  $R$ . This shifts FE down as entry rises, reducing  $\alpha_k$  and  $\ell$  along a given OI curve; but the IO curve also shifts down, causing  $\ell$  to move back up and further reducing  $\alpha_k$  since we know the new equilibrium still lies at the minimum of FE. The net effect is a fall in  $\alpha_k$  due to increased entry, and apparently an ambiguous effect on  $\ell$ . But because we know that the intersection of FE and OI occurs at the minimum of the FE curve, we can conclude that in fact  $\ell^*$  must actually rise after an increase in  $R$ , because it is easy to see that the minimum point on FE shifts right. A similar argument can be used to establish that  $\ell^*$  and  $\alpha_k(\tau^*)$  must

rise after an increase in  $r$ . See Figure 2 for parametric examples.

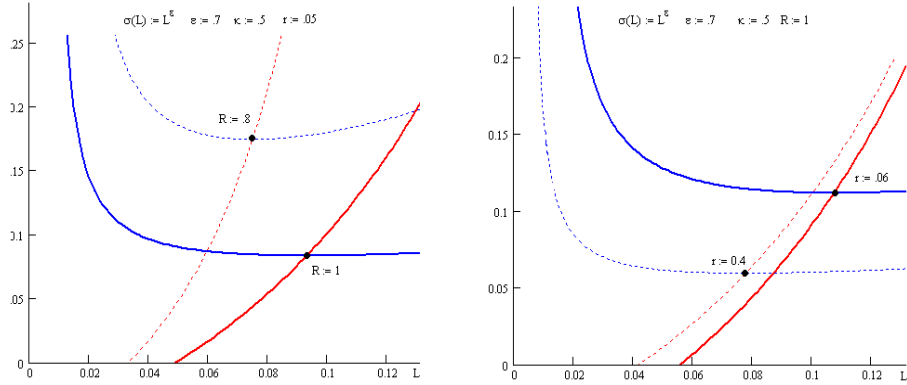


Figure 3: Changes in  $R$  and  $r$  in examples

## 6 Random Returns

The models in the previous section generate as an equilibrium outcome the return  $p$ , as well as the duration of the partner-search and implementation stages of the VC cycle. The duration of the partner-search phase is made endogenous through entry by VC's, which affects the arrival rates  $\alpha_e$  and  $\alpha_k$  through market tightness. Another way to make this stage endogenous is to let the quality of the match between  $e$  and  $k$  project be random. To be precise, a non-negative  $(R, C)$  is observed when agents meet, but it can differ across meetings, even though all agents are ex ante homogeneous. We can summarize a general joint distribution of  $(R, C)$  by the marginal  $F_C(C)$  and the conditional  $F_R(R|C)$ . However, we start with  $C = 0$  in all matches and  $F_R(R|C = 0)$  simple as  $F_R(R|0)$ . We take  $\ell_e$  and  $\ell_k$  as fixed for now.

The value functions for implementation satisfy the same conditions as above, (2) and (4), except  $V_e^1 = V_e^1(R)$  and  $V_k^1 = V_k^1(R)$  can now vary

across matches. The value functions for partner search are slightly more complicated. For an entrepreneur, e.g.,

$$rV_e^0 = \alpha_e \int_0^{\bar{R}} \max\{V_e^1(\tilde{R}) - V_e^0, 0\} dF_R(\tilde{R}) = \alpha_e \int_{R^*}^{\bar{R}} [V_e^1(\tilde{R}) - V_e^0] dF_R(\tilde{R}) \quad (24)$$

where, to clarify notation,  $\tilde{R}$  is the random variable (a dummy variable for integration),  $\bar{R}$  is the upper bound of its support, and  $R^*$  is the endogenous reservation return below which partnerships do not form. The reservation return obviously satisfies  $S(R^*) = S_e(R^*) = S_k(R^*) = 0$ ; in other words, it generates 0 surplus, which from (10) means

$$\sigma R^* = rV_k^0 + (r + \sigma)V_e^0 + \ell_e + \ell_k. \quad (25)$$

Inserting (9) into (24), we get

$$rV_e^0 = \frac{\alpha_e}{r + \sigma} \int_{R^*}^{\bar{R}} [\sigma(\tilde{R} - p) - (r + \sigma)V_e^0 - \ell_e] dF_R(\tilde{R}). \quad (26)$$

The bargaining problem is the same as in the simpler model, and yields the same solution (11), which we can insert into (26) to get

$$\begin{aligned} rV_e^0 &= \frac{\alpha_e(1 - \theta)}{r + \sigma} \int_{R^*}^{\bar{R}} [\sigma\tilde{R} - rV_k^0 - (r + \sigma)V_e^0 - \ell_e - \ell_k] dF_R(\tilde{R}) \\ &= \frac{\alpha_e(1 - \theta)\sigma}{r + \sigma} \int_{R^*}^{\bar{R}} (\tilde{R} - R^*) dF_R(\tilde{R}), \end{aligned}$$

after using (25). Integrating by parts yields

$$rV_e^0 = \frac{\alpha_e(1 - \theta)\sigma}{r + \sigma} \int_{R^*}^{\bar{R}} [1 - F_R(\tilde{R})] d\tilde{R}. \quad (27)$$

Similarly,

$$rV_k^0 = \frac{\alpha_k \theta \sigma}{r + \sigma} \int_{R^*}^{\bar{R}} [1 - F_R(\tilde{R})] d\tilde{R}. \quad (28)$$

Inserting (27)-(28) into (25), we arrive at

$$\sigma R^* = \Pi \int_{R^*}^{\bar{R}} [1 - F_R(\tilde{R})] d\tilde{R} + \ell_e + \ell_k, \quad (29)$$

where to save space we introduce

$$\Pi = \Pi(\sigma, r, \theta, \alpha_e \alpha_k) \equiv \frac{\sigma [\alpha_e (1 - \theta)(r + \sigma) + \alpha_k \theta r]}{r(r + \sigma)} \quad (30)$$

Given fixed  $n_e$  and  $n_k$ , and fixed  $\ell_e$  and  $\ell_k$ , equilibrium is characterized simply as a solution  $R^*$  to (29), since given this we can compute  $p^* = p^*(R)$  for any venture with  $R \geq R^*$  from (12) (notice that  $p^*$  is an increasing function). It is easy to verify that a unique equilibrium exists. The duration of partner search for  $k$  is endogenous, with hazard rate  $H_k \equiv \alpha_k [1 - F_R(R^*)]$ , because the parties agree to start a venture iff  $R \geq R^*$ . The expected duration of partner search for the VC is  $1/H_k$ . It is straightforward to show  $\partial R^*/\partial r < 0$  and  $\partial R^*/\partial \sigma < 0$ ; when impatience increases or projects becomes less time consuming, agents get less picky about  $R$ , and the expected duration of search falls. Also,  $\partial R^*/\partial(\ell_e + \ell_k) > 0$ , since agents get more picky when ventures are more costly. We cannot sign  $\partial R^*/\partial \theta$ , in general, as a change in  $\theta$  raises the outside option for one side and lowers it for the other.

One can show  $\partial R^*/\partial \alpha_j > 0$ ; more frequent meetings make agents more picky. However,  $\partial H_k/\partial \alpha_k$  is ambiguous, precisely because agents get more picky. One can show  $\partial H_k/\partial \alpha_k > 0$  if the density is log concave.<sup>15</sup> Using

<sup>15</sup>That is,  $\log F'(R)$  is concave in  $R$ , which holds for many but not all common distributions, and has been a common tool in search since Burdett (1981). See Sec. 2.5 of the notes at <http://www.ssc.upenn.edu/~rwright/courses/oss.pdf> for details concerning results discussed in this paragraph.

some other tricks from job search theory, one can study changes in the distribution. For instance, a translation of  $F_R$  (increase every  $R$  to  $R + \varepsilon$ ) raises  $R^*$ , but by less than the amount of the translation as long as  $\ell_e + \ell_k > 0$ , which means that  $H_k$  increases. Perhaps surprisingly, the expected return conditional on implementation  $E[R|R \geq R^*]$  can fall after a translation, but we can rule this out again by log concavity. Similar results apply to a scale transformation of  $F_R$  (increase every  $R$  to  $R + R\varepsilon$ ). One can also show that an increase in risk (a mean-preserving spread of  $F_R$ ) increases  $R^*$ ; intuitively, more risk makes search more attractive due to the option to pass on poor ventures.

As in the previous section, instead of assuming fixed  $n_e$  and  $n_k$  we can add free entry, which in this model reduces to

$$r\kappa = \frac{\alpha_k(\tau)\theta\sigma}{r + \sigma} \int_{R^*}^{\bar{R}} [1 - F_R(\tilde{R})] d\tilde{R}. \quad (31)$$

Given this, we can simplify (29) using  $\alpha_e = \alpha_k/\tau$  to

$$\theta\sigma R^* = [\theta r + (1 - \theta)(r + \theta)\tau] \kappa + \theta(\ell_e + \ell_k). \quad (32)$$

Equilibrium is now a pair  $(\tau, R^*)$  solving (31)-(32), two curves labeled FE and RR (for Free Entry and Reservation Return) in Figure 4. It is clear that FE is decreasing and approaches the axes under the usual assumptions on the  $m(\cdot)$ , and that RR gives  $R^*$  as a linearly increasing function of  $\tau$ .

Existence and uniqueness results are now routine. It is also easy to perform comparative static exercises. For example, if  $\kappa$  increases, both curves shift left, which reduces  $\tau$  by choking off entry, but has an ambiguous effect on  $R^*$ . The same thing happens if  $r$  increases; for a fixed  $\tau$  we earlier showed that  $R^*$  unambiguously decreases with  $r$ , but now the impact of changing entry on the arrival rates clouds the issue. The same thing happens

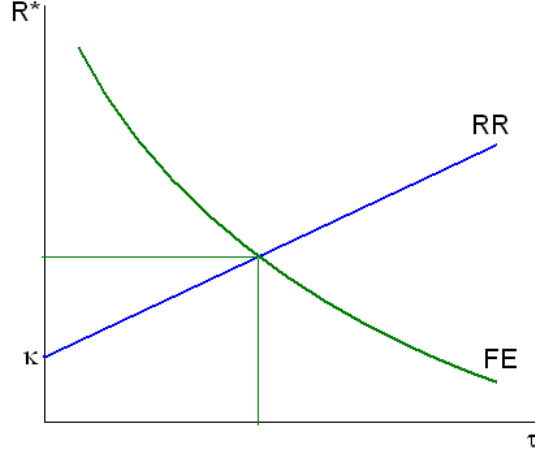


Figure 4: Equilibrium with  $R$  random and free entry

if  $\ell_e + \ell_k$  increases. If  $\theta$  or  $\sigma$  increases, both curves shift right, which increases entry and  $\tau$  and has an ambiguous effect on  $R^*$ . All of this says that there can be interesting general equilibrium effects on the duration of the VC cycle.

## 7 Random Returns and Costs

The case where  $R$  and  $C$  are both random is not much more difficult. In general,  $V_j^1 = V_j^1(R, C)$  now depends on the joint realization. Thus, e.g., we have

$$rV_k^0 = \alpha_k \int_0^{\bar{C}} \int_0^{\bar{R}} \max\{V_k^1(\tilde{R}, \tilde{C}) - \tilde{C} - V_k^0, 0\} dF_R(\tilde{R}|\tilde{C}) dF_C(\tilde{C}). \quad (33)$$

Notice  $k$  has to pay  $C$  up front. The reservation return  $R^*(C)$  now depends on the realized value of  $C$ , and satisfies a generalized version of (25)

$$\sigma R^*(C) = rV_k^0 + (r + \sigma)V_e^0 + (r + \sigma)C + \ell_e + \ell_k. \quad (34)$$

The bargaining problem yields a generalized version of (11),

$$\sigma p = \theta \sigma R - \theta \ell_e + (1 - \theta) \ell_k + (1 - \theta) (r + \sigma) C - \theta (r + \sigma) V_e^0 + (1 - \theta) r V_k^0 \quad (35)$$

where we note that  $p$  is increasing in the realization of  $C$  as long as  $\theta = 1$ .

This leads to a generalization of (28), and similarly, of (27), given by

$$r V_k^0 = \frac{\alpha_k \theta \sigma}{r + \sigma} \int_0^{\bar{C}} \int_{R^*(C)}^{\bar{R}} [1 - F_R(\tilde{R}|\tilde{C})] d\tilde{R} dF_C(\tilde{C}) \quad (36)$$

$$r V_e^0 = \frac{\alpha_e (1 - \theta) \sigma}{r + \sigma} \int_0^{\bar{C}} \int_{R^*(\tilde{C})}^{\bar{R}} [1 - F_R(\tilde{R}|\tilde{C})] d\tilde{R} dF_C(\tilde{C}). \quad (37)$$

Inserting (36)-(37) into (34), we get the analog of (29),

$$\sigma R^*(C) = (r + \sigma) C + \Pi \int_0^{\bar{C}} \int_{R^*(\tilde{C})}^{\bar{R}} [1 - F_R(\tilde{R}|\tilde{C})] d\tilde{R} dF_C(\tilde{C}) + \ell_e + \ell_k, \quad (38)$$

where  $\Pi = \Pi(\sigma, r, \theta, \alpha_e, \alpha_k)$  was define in (30). Thus  $R^*(C)$  is linear in  $C$ , with slope  $(r + \sigma)/\sigma$  and intercept  $R_0^*$ . This is technically useful because we can describe the outcome in terms of a number  $R_0^*$  instead of the function  $R^*(C)$ . With  $n_e$  and  $n_k$  fixed, then, we define equilibrium simply in terms of the number  $R_0^*$ . Or, with free entry, we get generalizations of (31)-(32),

$$r \kappa = \frac{\alpha_k(\tau) \theta \sigma}{r + \sigma} \int_0^{\bar{C}} \int_{R_0^* + C(r + \sigma)/\sigma}^{\bar{R}} [1 - F_R(R|C)] dR dF_C(C) \quad (39)$$

$$\theta \sigma R_0^* = [\theta r + (1 - \theta)(r + \sigma)\tau] \kappa + \theta (\ell_e + \ell_k), \quad (40)$$

which pin down  $(\tau, R_0^*)$ . Comparing these to (31)-(32), we see that making  $C$  random does not change the basic model much – but this will not be true in the next section when we introduce liquidity considerations.

## 8 Fund Raising: Liquidity

In this section we start to take seriously the fund raising stage of the VC cycle. The essential idea is not that fund raising takes time, but rather that it involves some cost, and therefore a VC will not enter the next stage with an arbitrarily large amount of money. Moreover, we will assume that once he enters the next stage he cannot easily go back and raise additional funds. To capture this succinctly, we take the position that he simply cannot spend more than the amount he brings to partner search. This assumption is quite realistic, based on what we read in the institutional literature, and not hard to model rigorously based on modern monetary theory. What is important is the idea that one may have to strike when the iron is hot: a VC who says, “that’s a good idea, let me see if I can raise some money” may lose the deal. This is what the notion of liquidity is all about.<sup>16</sup>

To make this interesting we need  $C$  is random – otherwise the VC would bring in exactly enough to cover the start-up cost. Given  $C$  is random, we can assume  $R$  is constant without much sacrifice. We also assume  $F_C(C)$  is continuously differentiable when we need it. We also set  $\ell_e = \ell_k = 0$  here to reduce notation. To reiterate, the key part of the specification is that  $k$  needs to choose a fund size  $m$  before the partner-search stage; any match with  $C > m$  is out of reach and cannot be implemented. While this is obviously extreme, it captures the idea that VC’s do commit to a fund size ex ante and typically have difficulty extending themselves beyond this commitment ex post. One can think of as also choosing a reservation cost  $C^*$ , but it adds nothing to the outcome, since  $m$  is the binding constraint:

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<sup>16</sup>In Silveira and Wright (2005) we introduced into a different but related model the idea that agents can *try* to raise additional funds after they meet someone, but with some probability  $\delta$  the deal falls through. To make the essential point here we assume  $\delta = 1$ , but the results hold more generally.

as long as raising funds is costly there is no sense choosing  $m > C^*$ . Hence we do not need to worry about  $C^*$ , and we can focus on the choice of  $m$ .

The value function for  $k$  at the fundraising stage is given by

$$W_k(m_{-1}) = \max_m \{m_{-1} - m + V_k^0(m)\}, \quad (41)$$

where  $m_{-1}$  denotes funds he starts with, typically given by his returns from exiting the previous cycle, and  $m$  denotes what he takes to the next cycle. If  $m_{-1} > m$  he consumes the difference; else he has to raise funds. We assume he can raise  $m$  instantly – although this would be easy to relax – at unit cost, and we restrict  $m \in [0, R]$ , obviously without loss in generality. One interpretation is that he can borrow at the same real interest rate that he uses to discount, which means that the cost of borrowing  $m$  between  $t$  and  $t'$  in present value terms is simply  $m$ .<sup>17</sup> The important point is that  $m$  is taken into the next stage as a state variable, which is why we write  $V_k^0(m)$ . The envelope condition is  $\partial W_k / \partial m_{-1} = 1$ . The first-order condition from the maximization problem is  $\partial V_k^0 / \partial m = 1$ , although as we show below, this condition may or may not characterize the solution.

Given the fund size  $m$ , the value to partner search is

$$rV_k^0(m) = \alpha_k \int_0^m \left[ V_k^1(\tilde{C}) + m - \tilde{C} - V_k^0(m) \right] dF_C(\tilde{C}). \quad (42)$$

Note that if the venture has startup cost  $C < m$  then  $k$  cashes in the difference immediately, and, again, if  $C > m$  he simply cannot afford it. Note also that the implementation value  $V_k^1(C)$  may depend on  $C$ , even though it

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<sup>17</sup>Because agents are risk neutral, there is a perfectly elastic supply of funds available at rate  $r$ . Alternatively, we can say that if  $m > m_{-1}$  then  $k$  takes the difference out of an endowment – it does not matter with risk neutrality. In any case, with  $R$  deterministic  $m > m_{-1}$  it does not arise in steady state – no one is going to take funds to the market in excess of his share of the value of the venture – although, of course, a new entrant VC has to either borrow or work to raise the initial fund.

is a sunk cost at the implementation stage, because the payment  $p = p(C, m)$  in general can depend on  $C$ . Emulating the analysis of bargaining in the previous sections, we get the generalization of (35):

$$\sigma p = \theta \sigma R - \theta(r + \sigma)V_e^0 + (1 - \theta)rV_k^0 + (1 - \theta)(r + \sigma)C - (1 - \theta)rm \quad (43)$$

As long as  $\theta < 1$ ,  $p$  is again increasing in  $C$ , and now it is also decreasing in  $m$  if we hold  $V_k^0$  constant at some given level; but if the first-order condition for  $m$  holds then  $\partial V_k^0 / \partial m = 1$ , and hence  $\partial p / \partial m = 0$ .<sup>18</sup>

Further emulating the analysis in the previous sections, we get the generalized versions of (27)-(28)

$$\begin{aligned} rV_e^0 &= \frac{\alpha_e(1 - \theta)}{r + \sigma} \int_0^M \left[ \sigma R + rM - (r + \sigma)\tilde{C} - rV_k^0 - (r + \sigma)V_e^0 \right] dF_C(\tilde{C}) \\ rV_k^0 &= \frac{\alpha_k\theta}{r + \sigma} \int_0^m \left[ \sigma R + rm - (r + \sigma)\tilde{C} - rV_k^0 - (r + \sigma)V_e^0 \right] dF_C(\tilde{C}). \end{aligned} \quad (44)$$

Note that we write  $V_e^0$  in terms of the value  $M$  that  $e$  expects the representative  $k$  to hold in equilibrium, while  $V_k^0$  depends directly on the  $m$  that  $k$  himself chooses. In equilibrium, of course, we can set  $m = M$ , and solve these (linear) equations for

$$V_e^0 = \frac{\alpha_e(1 - \theta)F(m) [\sigma R + rm - (r + \sigma)E(C|C \leq m)]}{r(r + \sigma) + \alpha_k\theta F(m)r + \alpha_e(1 - \theta)F(m)(r + \sigma)} \quad (45)$$

$$V_k^0 = \frac{\alpha_k\theta F(m) [\sigma R + rm - (r + \sigma)E(C|C \leq m)]}{r(r + \sigma) + \alpha_k\theta F(m)r + \alpha_e(1 - \theta)F(m)(r + \sigma)}, \quad (46)$$

where  $E(C|C \leq m) = \int_0^m \frac{CdF_C(C)}{F_C(m)}$ .

Now consider the fund-raising decision in (41). Differentiation of both sides of (44) (which gives  $V_k^0$  before we impose the equilibrium condition

<sup>18</sup>The key point is that the realization of  $C$  applies only to the current match, while the value of  $m$  is carried forward in the outside option  $V_k^0$ .

$m = M$ ) leads to

$$\begin{aligned} r \frac{\partial V_k^0}{\partial m} &= \frac{\alpha_k \theta}{r + \sigma} \int_0^m \left( r - r \frac{\partial V_k^0}{\partial m} \right) dF_C(\tilde{C}) \\ &\quad + \frac{\alpha_k \theta}{r + \sigma} \left[ \sigma (R - m) - r V_k^0(\hat{m}) - (r + \sigma) V_e^0 \right] F'_C(m), \end{aligned}$$

using Leibnez's rule. By virtue of the first-order condition  $\partial V_k^0 / \partial m = 1$ , this reduces to

$$r = \frac{\alpha_k F'_C(m) \theta \left[ \sigma (R - m) - r V_k^0 - (r + \sigma) V_e^0 \right]}{r + \sigma}. \quad (47)$$

This equates the marginal cost of funds,  $r$ , to the marginal benefit, which is given by the probability per unit time that an additional dollar just allows some venture to be funded,  $\alpha_k F'_C(m)$ , times the VC's share of the surplus, appropriately discounted.

This marginal calculation takes  $V_k^0$  and  $V_e^0$  as given. To describe equilibrium, we can insert (45)-(46) into (47) and reduce the system to

$$T(m) = T_1(m)T_2(m) - T_3(m)T_4(m) = 0, \quad (48)$$

where

$$\begin{aligned} T_1(m) &= \alpha_k \theta F_C(m) F'_C(m) [r \alpha_k \theta + (r + \sigma) \alpha_e (1 - \theta)] \\ T_2(m) &= [\sigma R + r m - (r + \sigma) E(C|C \leq m)] \\ T_3(m) &= [r(r + \sigma) + \alpha_k \theta F_C(m) r + \alpha_e (1 - \theta) F_C(m) (r + \sigma)] \\ T_4(m) &= [\alpha_k \theta \sigma (R - m) F'_C(m) - r(r + \sigma)]. \end{aligned}$$

This is *almost* our final answer. Since  $T(m)$  equals (in sign) marginal cost minus marginal benefit,  $T(m) = 0$  is the first-order condition for an interior solution for fund size in equilibrium; but we also must check the second-order condition and the participation condition  $V_k^0(m) \geq m$ . One can show

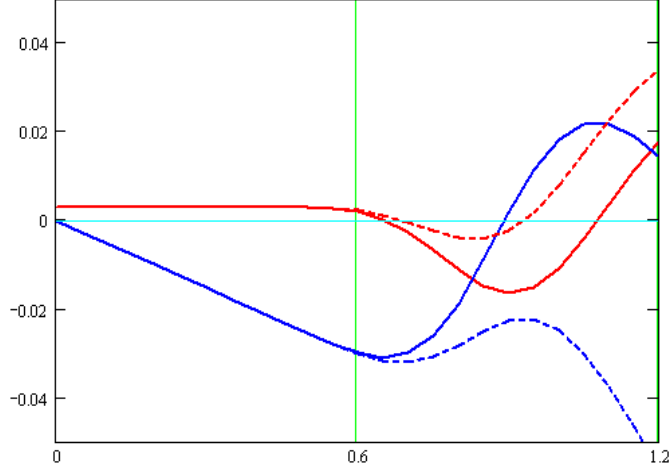
the second-order condition holds iff  $T'(m) > 0$  (which simply says marginal cost is increasing faster than marginal benefit). The participation condition turns out to hold iff  $R$  is above some threshold, as we now discuss.

The first thing to do is to graph  $T(m)$  (see Figure 8 below). The relevant range is  $m \in [0, R]$ , and notice that  $T(R) > 0$ . Suppose  $T(m) \geq 0 \forall m \in [0, R]$ , which must be true if  $R$  is small enough; then  $m = 0$  is an equilibrium, since the marginal cost exceeds the benefit  $\forall m \in [0, R]$ . So suppose  $T(m) < 0$  for some  $m \in [0, R]$ . Then two things are clear: there is at least one solution to  $T(m) = 0$  in  $(0, R)$  with  $T'(m) \geq 0$ , and hence there is a value of  $m$  that satisfies first- and second-order conditions; and if there are multiple solutions, one of them  $\bar{m}$  achieves the maximum over  $[0, R]$  of  $w(m) = V_k^0(m) - m$ , where  $V_k^0(m)$  is given by (??).<sup>19</sup> If  $w(\bar{m}) \geq 0$  then  $\bar{m}$  is an equilibrium, since it satisfies the first- and second-order conditions, plus participation. If  $w(\bar{m}) < 0$  then no solution to the first- and second-order conditions satisfies participation, and hence  $m = 0$  is the unique equilibrium. In either case we have existence of an equilibrium.

In Figure 8, an example with  $F_c$  log-normal shows  $T(m)$  in red, with the dashed curve corresponding to  $R = 1$  and the solid curve corresponding to  $R = 1.2$ . The dashed and solid blue curves show  $w(m)$  for the same two values of  $R$ . For each case, there are two solutions to  $T(m) = 0$  (two places where each red curve cuts the horizontal axis), and hence two values of  $m$  that satisfy the first-order condition; but only the higher value of  $m$  satisfies the second-order condition, which requires  $T'(m) > 0$ . With the lower  $R = 1$  shown as the dashed curve, the relevant solution violates participation since  $w(m) < 0$  and hence the only equilibrium is  $m = 0$ . With the higher  $R = 1.2$  shown as the solid curve, the relevant solution satisfies participation since

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<sup>19</sup>We assume that  $T(m)$  is continuous, and that there are at most a finite number of solutions to  $T(m) = 0$ , which can be guaranteed with primitive assumptions on  $F_C(\cdot)$ .



$w(m) > 0$  and hence it is an equilibrium.

The key point is that equilibrium always exists, but if  $R$  is sufficiently small then the only equilibrium is  $m = 0$  – i.e. the venture capital market shuts down – and if  $R$  is sufficiently small then equilibrium exists with  $m > 0$ . Since an increase in  $R$  shifts  $T(m)$  up, it is also clear that this will increase the equilibrium value of  $m$ , assuming equilibrium with  $m > 0$  exists, since  $T'(m) > 0$  in equilibrium by the second-order condition.

## 9 Appendix

Here we prove some claims made about the OI and FE curves defined by (22)-(23). First, we work with  $\alpha_k$ , instead of  $\tau$ , since  $\alpha_k = \alpha_k(\tau)$ . Second, given  $\bar{\ell}_k$  is fixed, we normalize it to 0. Then equilibrium is a pair  $(\ell_e, \alpha_k)$  satisfying the following versions of the investment and entry conditions:

$$\alpha_k = \alpha_k^I(\ell_e) \equiv \frac{\sigma_e(\ell_e)(rR + \ell_e) - [r + \sigma(\ell_e)]}{1 - \sigma_e(\ell_e)R}$$

$$\alpha_k = \alpha_k^F(\ell_e) \equiv \frac{r\kappa[r + \sigma(\ell_e)]}{\sigma(\ell_e)R - \ell_e - r\kappa}.$$

As  $\ell_e \rightarrow 0$ , by assumption,  $\sigma(\ell_e) \rightarrow 0$ ,  $\sigma_e(\ell_e) \rightarrow \infty$ , and l'Hopital's Rule implies  $\alpha_k^I(\ell_e) \rightarrow -r$ . Also, clearly,  $\alpha_k^F(0) = -r$ .

One can check that  $\alpha_k^F(\ell_e) > 0$  iff  $\ell_e \in (\ell_1, \ell_2)$ , where  $\ell_1$  and  $\ell_2$  are the two solutions to  $\sigma(\ell_e)R - \ell_e = r\kappa$ , which means  $(\ell_1, \ell_2)$  is nonempty iff  $\kappa < \bar{\kappa} = \max\{\sigma(\ell_e)R - \ell_e\}$ . It is also easy to verify that  $\alpha_k^F$  is strictly convex in  $(\ell_1, \ell_2)$ , and that  $\alpha_k^I$  is strictly increasing iff  $\ell_e \in (0, \tilde{\ell})$ , with  $\tilde{\ell} > \ell_2$ , and  $\alpha_k^I \rightarrow \pm\infty$  as  $\ell_e \rightarrow \ell_D$  from the left or right, where  $\ell_D$  is the zero of the denominator of  $\alpha_k^I$ . Notice that as long as  $\kappa < \bar{\kappa}$ , we have  $\ell_D \in (\ell_1, \ell_2)$ , because  $\ell_D = \arg \max\{\sigma(\ell_e)R - \ell_e - r\kappa\}$  while  $(\ell_1, \ell_2)$  are the zeros of  $\sigma(\ell_e)R - \ell_e - r\kappa$ . Hence the situation is as depicted in Figure 2 in the text.

Indeed we can prove the following:  $\alpha_k^I$  meets  $\alpha_k^F$  at  $\ell_e = \ell^*$  where  $\ell^*$  minimizes  $\alpha_k^F$  over  $(\ell_1, \ell_2)$ . To verify this, notice that since  $\alpha_k^F$  is strictly convex over this interval, a necessary and sufficient condition for the minimization is  $\partial\alpha_k^F/\partial\ell_e = 0$ , which can be rearranged as

$$\sigma_e(\ell^*) = \frac{r + \sigma(\ell^*)}{rR + \ell_e + r\kappa}.$$

Minor algebra reveals  $\alpha_k^I(\ell^*) = \alpha_k^F(\ell^*)$ . This establishes the desired result: when  $\theta = 1$ , equilibrium with  $\alpha_k > 0$  exists iff  $\kappa < \bar{\kappa}$ , since this is required for  $(\ell_1, \ell_2)$  to be nonempty, and when it exists it is uniquely determined by  $\ell_e = \ell^*$  and  $\alpha_k = \alpha_k^F(\ell^*)$ .

Next we consider the same model for a general value of  $\theta \in [0, 1]$ . The generalized OI and FE curves are given by

$$\begin{aligned} \alpha_k(\tau) &= \frac{[r + \sigma(\ell_e)] \{\theta r + (1 - \theta) [r + \alpha_e(\tau)]\}}{r [\sigma_e(\ell_e) R - \theta]} \\ &\quad - \frac{[r + (1 - \theta)\alpha_e(\tau)] (rR + \ell) \sigma_e(\ell_e)}{r [\sigma_e(\ell_e) R - \theta]} \\ \alpha_k(\tau) &= \frac{\kappa \{\theta r [r + \sigma(\ell_e)] + (1 - \theta) [r + \sigma(\ell_e)] [r + \alpha_e(\tau)]\}}{\theta [\sigma(\ell_e) R - \ell_e - r\kappa]}. \end{aligned}$$

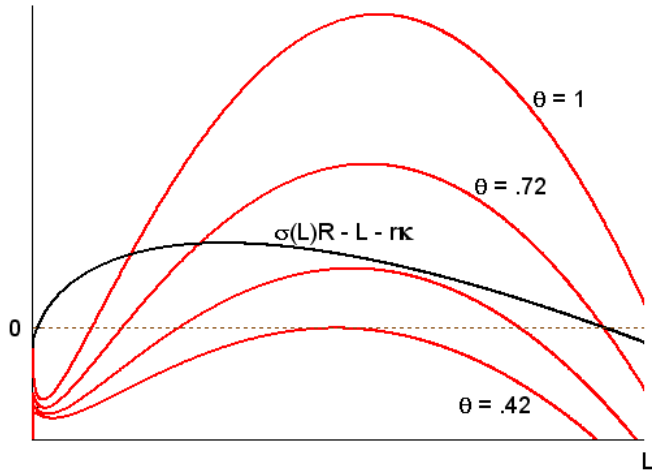
These are not especially useful, since  $\alpha_e(\tau)$  appears on the RHS; but if we equate the two expressions, after some algebra, a minor miracle occurs and  $\alpha_e(\tau)$  vanishes, leaving  $T(\ell_e) = 0$  where

$$\begin{aligned} T(\ell_e) &= \theta [r + \sigma(\ell_e) - (rR + \ell_e) \sigma_e(\ell_e)] [\sigma(\ell_e) R - \ell_e] \\ &\quad - r\kappa [(1 - \theta)rR + \sigma(\ell_e) R - \theta\ell_e] \sigma_e(\ell_e). \end{aligned}$$

An equilibrium with  $\ell_e > 0$  has to satisfy at least  $T(\ell_e) = 0$ , which is one equation in  $\ell_e$  independent of other endogenous variables. Now, if  $\theta = 0$  there are obviously no solutions to  $T(\ell_e) = 0$ , and if  $\theta = 1$  we already know there exists a solution iff  $\kappa$  is not too big. Hence, given  $\kappa$  not too big, there is a solution to  $T(\ell_e) = 0$  iff  $\theta$  is not too small. Given  $\ell^*$  such that  $T(\ell^*) = 0$ ,  $\ell^*$  is a *candidate* equilibrium, but we still have to check that no side conditions are violated. In particular, the expression for  $\alpha_k^E(\tau)$  derived above indicates that  $\alpha_k(\tau) \geq 0$  at the candidate solution iff  $\sigma(\ell^*) R - \ell^* \geq r\kappa$ , which is a participation constraint conditional on having found a partner; or a non-negative profit condition when  $\alpha_k(\tau) \rightarrow \infty$ . It is clear that  $\sigma(\ell_e) R - \ell_e \geq r\kappa$  iff  $\ell_e \in (\ell_1, \ell_2)$ , where  $\ell_1 > 0$ , and  $\ell_2 > \ell_1$  iff  $\kappa < \bar{\kappa} = \max_{\ell} \{\sigma(\ell) R - \ell\}$ , exactly as in the case  $\theta = 1$ .

Notice that at  $\ell_e = \ell_1$  or  $\ell_e = \ell_2$ , we have  $\sigma(\ell_e) R - \ell_e = r\kappa$ , and  $T(\ell_e) = r\kappa [r + \sigma(\ell_e)] [\theta - \sigma_e(\ell_e) R]$ . For small  $\theta$ ,  $T(\ell_e) < 0 \forall \ell_e \in (\ell_1, \ell_2)$ , so  $T(\ell_e) = 0$  has no solution in  $(\ell_1, \ell_2)$ ; for  $\theta = 1$  we already know  $T(\ell) = 0$  has a solution iff  $\kappa$  is not too big; therefore, in general, there is a solution iff  $\theta$  is not too small and  $\kappa$  not too big. As shown in the next Figure, given  $\kappa$  not too big there are two solutions to  $T(\ell_e) = 0$  when  $\theta < \hat{\theta}$  and no solutions when  $\theta > \hat{\theta}$ . We now check whether these solutions satisfy the participation constraint,  $\sigma(\ell_e) R - \ell_e - r\kappa \geq 0$ . From the case  $\theta = 1$  and continuity, for

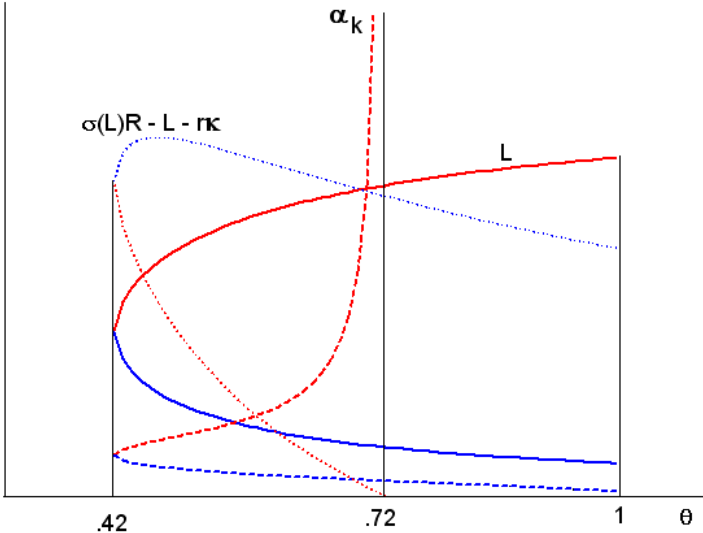
any  $\theta \approx 1$  we know the participation condition holds at the lower but not the upper solution – i.e., we know the lower solution is in  $(\ell_1, \ell_2)$  and the upper solution is not. As we lower  $\theta$ ,  $T(\ell_e)$  unambiguously shifts down, until we reach some  $\tilde{\theta}$  such that  $T(\ell_2) = 0$ . As we continue to lower  $\theta$  from  $\tilde{\theta}$  to  $\hat{\theta}$  both solutions are in  $(\ell_1, \ell_2)$  and hence satisfy the participation constraint. And, again, once we lower  $\theta$  past  $\hat{\theta}$  there are no solutions to  $T(\ell_e) = 0$ .



The above argument, based on the parametric example in the Figure, proceeds as if  $T$  were concave in the relevant interval  $(\ell_1, \ell_2)$ , which we cannot establish in general, since it depends on third derivatives. This does not affect the basic logic, however, except there could potentially be more than two solutions to  $T(\ell_e) = 0$  for some  $\theta$ ; the general point is that the lowest solution (rather than the lower solution) satisfies participation and the highest one (rather than the higher one) does not for big  $\theta$ , while all of them (rather than both of them) satisfy participation for intermediate  $\theta$ .

Going back to the parametric example, where there are exactly two solutions, the next figure shows the candidate equilibrium values of  $\ell_e$  – i.e.

the upper and lower solutions of  $T(\ell_e) = 0$  – when they exist as we vary  $\theta$ . Also shown are the values of  $\alpha_k$  and  $\sigma(\ell_e)R - \ell_e - r\kappa$  implied by the candidate values of  $\ell_e$ . When  $\theta > \tilde{\theta} \approx 0.72$ , the upper solution implies  $\sigma(\ell_e)R - \ell_e - r\kappa < 0$ , or equivalently, there is no  $\alpha_k$  consistent with free entry because  $\alpha_k \rightarrow \infty$  as  $\theta \rightarrow \tilde{\theta}$ . For  $\theta \in (\tilde{\theta}, \hat{\theta}) \approx (0.44, 0.72)$  both solutions satisfy participation; notice that the upper solution implies a much higher value for  $\alpha_k$  and lower value for  $\sigma(\ell_e)R - \ell_e - r\kappa$ . At  $\theta = \hat{\theta} \approx 0.44$  the upper and lower solutions coalesce; below  $\hat{\theta}$  there is no solution, and market shuts down.



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