Chapter 7

Search and Bargaining Models of Exchange

7.1 Introduction

In this chapter we consider models of the exchange process where, as in the previous chapter, once agents meet they exchange and part company (they do not form enduring relationships); but we now generalize things so that the rate at which they exchange is determined by bilateral bargaining. We take for granted that the reader understands the relation between strategic bargaining models and the Nash solution, as detailed in the Appendix, and already used in other chapters. We less interested here in this issue than in showing how the models can be used to address substantive questions.

In Section 2, we begin with a model due to Rubinstein and Wolinsky (1985), in which there are large numbers of homogeneous buyers and homogeneous sellers who meet at random and trade according to terms given by bargaining theory. They constructed this model in order to contrast the predictions of competitive (Walrasian) equilibrium theory with the predictions of a model with explicit frictions and price setting. They argue that standard competitive equilibrium theory is useful to the extent that it provides a good approximation to more "reasonable micromechanisms" describing the trading process. That is, if the real world is characterized by real frictions, then competitive theory is a useful description to the real world if it generates similar outcomes when the frictions are not too big.

As frictions vanish the unique equilibrium of the Rubinstein-Wolinsky model converges to something that does not generally coincide with a Walrasian equilibrium of a simple static model with demand and supply defined using the steady state numbers of buyers and sellers in the market. Hence, Rubinstein and Wolinsky conclude that models with small frictions are *not* Walrasian. However, this is not the last word on the matter. In Section 3, we consider a version of a model due to Gale (1987) that, in a sense, implies the opposite conclusion.

First, we generalize the Rubinstein-Wolinsky model to allow heterogeneity. Equilibrium in his model generally involves a price distribution, with, for example, high valuation buyers paying higher prices. As the frictions vanish, the price distribution collapses to a point, consistent with competitive theory.[1] This could not be observed in the Rubinstein-Wolinsky model, since with homogeneous buyers and sellers the equilibrium never yields a nondegenerate price distribution. However, the limiting price is again different from the prediction of a simple static Walrasian model where demand and supply are defined using set of agents in the market in steady state, just as Rubinstein and Wolinsky found in their special case.

What is one to make of this? Our view is that it is not very informative to compare the predictions of a static Walrasian model with those of a dynamic model, even as the frictions vanish in the latter. In particular, the set of buyers and sellers in the dynamic model is not the same as the set one observes in a steady state cross-section: the fact that new buyers and sellers always enter in the future is relevant to agents in the dynamic model but not the static model. Hence, following Gale, we endogenize the decision of

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[1] In Gale's (1987) original model, with heterogeneity on both sides of the market, one can only characterize the limit as frictions vanish, at which point the equilibrium price distribution is degenerate. We show here that, in the special case with heterogeneity on only one side of the market, one can explicitly solve for the equilibrium price distribution.

agents to participate in the market. As in Gale, we show there is a well-defined sense in which the limit of the equilibrium in the dynamic model as the frictions vanish does correspond to a notion of Walrasian equilibrium defined by equating the flows of new buyers and sellers into the market.

In Section 4, we consider a different model by Rubinstein and Wolinsky (1987), which is designed to ask if the frictions inherent in a search and bargaining model can generate a role for *middlemen* to help facilitate exchange. This is part of the general research program that asks what sorts of institutions can emerge endogenously in search models. Middlemen, in this model, are agents whose function it is to buy from sellers and sell to buyers, making profit as long as the spread between the prices at which they buy and sell is positive. We show that there is a role for middlemen (that is, there exists an equilibrium where middlemen are active) if and only if they can locate buyers more efficiently than sellers can directly locate buyers.

As we have seen earlier, a different institution that can emerge endogenously in search models is *money.* However, the monetary models presented to this point assume that if money has value at all it has some exogenously specified value (i.e., all monetary exchange involved a one-for-one swap of a unit of cash for an indivisible good). In Section 5, we consider the search model of fiat money presented by Shi (1995) and Trejos and Wright (1995), in which bargaining theory is used to determine the nominal price level. This not only addresses an important deficiency in those earlier models, it also leads, as we shall see, to new insights into monetary economics. In Section 6, we consider a special case of the model which is sufficiently simple that it allows us to derive several results with relative ease.

7.2 The Rubinstein-Wolinsky Model

Rubinstein and Wolinsky (1985) consider a model with a large number of homogeneous buyers, who each want to buy 1 unit of an indivisible good at a price of up to 1 unit of transferrable utility, and a large number of homogeneous sellers, who each want to sell 1 unit of the good for some positive price in terms of transferrable utility.[2] That is, if a seller and buyer exchange the indivisible good for p units of transferrable utility, the former receives an instantaneous payoff of p and the latter an instantaneous payoff of $1 - p$. The friction in the model is that it takes time to trade, and instantaneous utility is discounted at rate r (the same for all agents).

Normalize the total population to 1, and let the measure of buyers be N and the measure of sellers be $1 - N$. For simplicity, assume that agents meet according to a constant returns to scale technology and that meetings are random. Hence, the rate at which a buyer meets sellers is proportional to the number of sellers, $\alpha(1 - N)$, and the rate at which a seller meets buyers is proportional to the number of buyers, αN. If a seller and buyer meet, they bargain over p according to a procedure to be described momentarily. If they come to an agreement, they trade and leave the market for good, to be replaced by a new pair in order to maintain the steady state.

Denote by V_b and V_s the value functions for unmatched buyers and sellers looking for partners with whom to trade. It can be verified that any equilibrium in this model involves a price that is the same in all trades and constant over time, and hence V_b and V_s will also be constant over time (see the exercises). Given a constant price P prevailing in the market, the usual dynamic programming equations for the value functions are

$$\begin{aligned} rV_b &= \alpha(1-N)(1-P-V_b) && (7.1) \\ rV_s &= \alpha N(P-V_s), && (7.2) \end{aligned}$$

which can easily be solved for $V_b = V_b(P)$ and $V_s = V_s(P)$. What we need to do now is describe the bargaining game to determine the price p that a particular buyer-seller pair would negotiate given that P is the price prevailing on the market, say $p = p(P)$, and then look for a fixed point.

The bargaining game is a version of the one introduced by Rubinstein (1982). First, nature determines one party at random to propose a value of

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[2] Rubinstein and Wolinsky refer to transferrable utility as "money;" we do not, because we will present explicit models with fiat money below.

p, which his partner can either accept or reject. If he accepts, trade occurs. If he rejects, he can choose to permanently abandon his partner to look for a new partner, or to try and stay with his current partner. Then his partner can choose to permanently abandon him or to try to stay together. If both try to stay together, they must wait an interval of time of length $\Delta > 0$ before going back to the first stage, where nature again chooses someone at random to propose a value of p. However, they continue to meet new partners during the interval Δ, according to the same arrival rates as when they are unmatched, and when a new partner comes along the agent is forced to begin negotiating with the new agent and abandon the incumbent.

This bargaining game has a unique subgame perfect equilibrium, and it has the property that the first offer is always accepted. The price is given by p_b or p_s depending on who gets to make the first offer, the buyer or seller. We are only interested here in the limiting case where $\Delta \to 0$, in which case p_b and p_s converge to the same limit.[3] This limit is conveniently given by

$$p = \arg\max\ (p - V_s)(1 - p - V_b) \tag{7.3}$$

subject to $1 - p \geq V_b$ and $p \geq V_s$, which is the (symmetric) Nash bargaining solution with threat points given by the outside options V_b and V_s.[4]

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[3] We focus on the limit as $\Delta \to 0$ because we are interested in frictions inherent in the process of finding a trading partner, not those that involve negoatiating a deal once a partner has been located. The latter friction can be interesting in some contexts, but not here, where the unique equilibrium involves settlement as soon as the parties meet. Letting $\Delta \to 0$ allows us to avoid the nuisance of the price depending on who gets to make the first offer.

[4] See the Appendix for details of these standard arguments. In a variation of the above game where agents never meet new partners during the interval between rejection and counteroffer, there is still a unique subgame perfect equilibrium, but now as $\Delta \to 0$ the limiting price is given by

$$p = \arg\max\ p(1 - p)$$

subject to $1 - p \geq V_b$ and $p \geq V_s$, which is the Nash solution with 0 threat points (see the Appendix). Other cases could also be imagined, such as the case where the seller may meet other buyers but not vice-versa. Wolinsky (1987) endogenizes these meeting rates by allowing agents to choose search intensity.

The maximization problem in (7.3) implies $p = \frac{1}{2}(1 + V_s - V_b)$, and inserting $V_j = V_j(P)$ we can determine the function $p = p(P)$. In equilibrium, $p = P$; that is, taking as given the price prevailing in the market, bilateral bargaining between any pair of agents generates the same price. It is easy to solve explicitly for the unique equilibrium price,

$$p = \frac{r + \alpha N}{\alpha + 2r}. \tag{7.4}$$

A natural question is whether the limiting case where either $r \to 0$ or $\alpha \to \infty$ (that is, where the frictions vanish) corresponds to the prediction of a frictionless "Walrasian" model. Taking the limit in (7.4) yields $p = N$, and, in their article, Rubinstein and Wolinsky claim that this is *not* the Walrasian price. Their argument is that a simple (static) supply and demand analysis predicts the Walrasian price is given by

$$\hat{p} = \begin{cases} 0 & if \quad N < \frac{1}{2} \\ [0,1] & if \quad N = \frac{1}{2} \\ 1 & if \quad N > \frac{1}{2} \end{cases} \tag{7.5}$$

Hence, they conclude that the frictionless model is not the limit of the model with frictions.

It is not obvious, however, that a static supply and demand analysis is the relevant Walrasian benchmark. The intuition behind the static model with, for example, $N > \frac{1}{2}$ is as follows. If there are more buyers than sellers then competition must force the price to 1, because if $p = p' < 1$ there would be a buyer not getting the good who would be willing to pay a higher price $p'' \in (p', 1]$. But in the Rubinstein-Wolinsky environment, $N > \frac{1}{2}$ does not really mean that there are more buyers than sellers; there are really the same number − an infinite number − of buyers and sellers.

This is relevant because in a dynamic world a buyer who is not getting the good today is not necessarily willing to pay any price $p'' \leq 1$, because he can always wait for new sellers to show up in the future. Waiting, which is not possible in a static world, means that a seller cannot completely exploit buyers in a dynamic world just because he is on the short side of the market at any point in time. Just what is an appropriate frictionless benchmark for comparison? We address this issue in the next section by considering Gale's (1987) extension of the model.

Exercises:

1. Solve for and graph the function $p = p(P)$.

2. Show that equilibrium must involve the same price in all trades. Hint: Let P be distributed according to and arbitrary $F(P)$, write down the value functions, and consider the bargaining problem.

3. Show that there are no nonstationary equilibria. Hint: For the dynamical system derived from the general versions of Bellman's equations,

$$\begin{aligned} rV_b &= \alpha(1-N)(1-P-V_b) + \dot{V}_b \\ rV_s &= \alpha N(P - V_s) + \dot{V}_s, \end{aligned}$$

   show that all orbits other than the steady state described in the text aree unbounded.

4. Consider a version of the model in which agents do not exit the market, but continue to buy or sell repeatedly; i.e., after a successful purchase a buyer reenters the market to look for another seller, and vice-versa. Show that $p = \frac{1}{2}$.

5. Consider a version of the model in which after a successful trade the buyer becomes a seller and vice-versa. Find the equilibrium price, being careful to worry about constraints. Answer:

$$p = \begin{cases} 0 & if \quad N < \frac{1-r}{2} \\ 1 & if \quad N > \frac{1-r}{2} \\ \frac{r+2N-1}{2r} & if \quad N = \frac{1-r}{2} \end{cases}$$

   Compare the limiting outcome as $r \to 0$ to the Walrasian price predicted by a static supply and demand analysis, given in (7.5).

6. Consider a model where agents can choose to be either buyers or sellers. Derive the price and value functions in each case, for various versions of the model described in the text and in the previous exercises.

7. Redo the analysis in this section with two extensions: assume the seller has a cost c rather than 0, so that his payoff in the event of a trade is $p - c$; and use the generalized Nash solution, where the buyer has bargaining power θ.

7.3 The Gale Model

Gale (1987) generalizes Rubinstein and Wolinsky (1985) by including heterogeneous agents. To illustrate the implications of his analysis, we begin with the case where sellers are identical and each has a constant marginal cost $c \geq 0$ of providing up to one unit of output, but now potential buyers are indexed by u, where u is the utility from consuming 1 unit of the good. The value of u varies across buyers according to the distribution function $F(u)$ with support $[\underline{u}, \overline{u}]$, where $\overline{u} \geq \underline{u} > 0$. We assume $\overline{u} > c$, which is obviously needed for the market to be active at all. Otherwise, the model is the same as the model in the previous section.[5]

The price that a pair negotiates will generally depend on the buyer's valuation, so we write $p = p(u)$. Moreover, if a seller meets a buyer with a very low u, they may not trade at all. In what follows, let R be the *reservation value* of u; that is, a potential buyer trades if and only if $u \geq R$. Let the value function for a buyer of type u and for a seller be denoted $V_b(u)$ and V_s. For now we fix the number of buyers and sellers in the market at N_b and N_s, by assuming that every time an agent trades he is replaced by an identical agent. We do not normalize $N_b + N_s = 1$, as in the previous section,

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[5] In his article, Gale (1987) also considers a version of the model in which there are only finite numbers of buyers and sellers, which implies that the market eventually closes and a steady state analysis is not possible. The reader is referred to the article for details.

because shortly we will endogenize N_b and N_s.[6]

Assuming that the arrival rates are proportional to the fraction of agents on the other side of the market, we can, with no loss in generality, write the rate at which sellers meet buyers as β and the rate at which buyers meet sellers as $1-\beta$, where $\beta \equiv \frac{N_b}{N_b+N_s}$. In a steady state, Bellman's equation for a seller is given by

$$rV_s = \beta E \max[p(u) - c - V_s, 0] = \beta \int_R [p(u) - c - V_s] dF(u) \tag{7.6}$$

where R satisfies $p(R) = c + V_s$. Bellman's equation for a buyer with $u \geq R$ is given by

$$rV_b(u) = (1-\beta)[u - p(u) - V_b(u)], \tag{7.7}$$

while for a (potential) buyer with $u < R$ we have $V_b(u) = 0$.

The bargaining solution is taken to be the generalized Nash solution, where θ is the bargaining power of the buyer,[7]

$$p(u) = \arg\max[u - p - V_b(u)]^{\theta}[p - c - V_s]^{1-\theta}.$$

This maximization problem implies

$$p(u) = \theta(c + V_s) + (1-\theta)[u - V_b(u)]. \tag{7.8}$$

Substituting (7.8) into (7.7) and setting $u = R$, we see that $p(R) = R$. Therefore, (7.8) implies $V_b(R) = 0$, and (7.7) implies $V_s = R - c$.

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[6] Here the agent that replaces a buyer who leaves the market must have the same value of u, rather than a random draw from $F(u)$, since otherwise agents with low u will clog up the matching process. This will not be an issue when we endogenize entry below.

[7] To conserve space, we provide fewer details here than in the previous section concerning the mapping from prices prevailing in the market to the price determined by a particular buyer-seller pair. Also, note that Gale (1987) actually uses a bargaining model where each party has an equal probability of getting to make a take-it-or-leave-it offer. This gives essentially the same result as our model if we set $\theta = \frac{1}{2}$. He also talks about varying the relative discount rates of buyers and sellers, the effects of which we can capture by varying θ.

Our goal is to solve for the reservation buyer R and the price and value functions. To this end, substitute (7.8) into (7.7) and rearrange to get

$$V_b(u) = \frac{\theta(1-\beta)}{r+\theta(1-\beta)}(u-c-V_s)$$

for al $u > R$. Then insert this into (7.6), use $V_s = R - c$, and simplify to get

$$R - c = \frac{\beta(1-\theta)}{r+\theta(1-\beta)} \int_R (u-R)dF(u). \tag{7.9}$$

In equilibrium, the reservation buyer is given by the unique $R \in [c, \overline{u}]$ satisfying (7.9). Given R, we can solve explicitly for the value functions and for

$$p(u) = \frac{r(1-\theta)u + \theta(r+1-\beta)R}{r+\theta(1-\beta)}. \tag{7.10}$$

Hence, in equilibrium, $p(u)$ is linear in u (in fact, it is a weighted average of u and R). Several aspects of this solution are worth comment. First, note that R is increasing in c and β and decreasing in r and θ. It is possible for some parameter values that $R < \underline{u}$, so that all potential buyers are able to trade; it is also possible for other parameter values that $R > \underline{u}$, so that some potential buyers are not able to trade. In the latter case, for some potential buyers, there is simply no price they would be willing to pay that a seller would prefer to waiting for a higher valuation buyer.

For buyers who actually do trade, there is generally a distribution of prices. However, if we give all the bargaining power to the buyer, $\theta = 1$, then $p(u) = c$ for all u. This means that all buyers pay the same price, which is equal to the sellers' cost. At the opposite extreme, if we give all of the bargaining power to the seller, $\theta = 0$, then $p(u) = u$ for all $u \geq R$. In either case, we can still have $R < \underline{u}$ or $R > \underline{u}$, depending on other parameters.

For any θ, in the limit as $r \to 0$, we have $p(u) = R$ for all $u \geq R$. Hence, as the frictions vanish the price distribution collapses to a point, consistent with frictionless competitive theory, where all trades involve the same price. It is convenient to rewrite (7.9) in the limit as $r \to 0$ as

$$\theta(1-\beta)(R-c) = \beta(1-\theta) \int_R (u-R)dF(u). \tag{7.11}$$

If $\theta = \frac{1}{2}$, (7.11) is a version of Gale's (1987, Proposition 11) result that the total surplus of the sellers equals the total surplus of the buyers; see below for the more general case.[8]

As in the previous section, we would like to ask how the limit price R that emerges as the frictions vanish compares to the prediction of a frictionless Walrasian model. Generalizing Rubinstein and Wolinsky (1985), one approach is to define Walrasian demand and supply using the steady state numbers of agents in the market. Here per capita demand is given by $\beta[1 - F(p)]$, since this is the measure of agents willing to buy 1 unit at price p; and per capita supply is 0 for $p < c$ and $1 - \beta$ for $p > c$. As shown in Figure 1, the price that equates these is $\hat{p} = c$ if $\beta[1 - F(c)] < 1 - \beta$, and the solution to $\beta[1 - F(\hat{p})] = 1 - \beta$ if $\beta[1 - F(c)] > 1 - \beta$. Except for sheer coincidence, $\hat{p} \neq R$. In particular, as we lower θ from 1 to 0, R varies from c to $\overline{u}$, while $\hat{p}$ does not depend on θ.

Gale (1987) suggests an alternative notion of a frictionless model to which we should compare the limit of the model with frictions. Rather than consider supply and demand based on the steady state stock of agents in the market, he defines a *flow equilibrium* to be a situation where the number of buyers entering the market equals the number of sellers entering the market at each point in time. In the model of Rubinstein and Wolinsky (1985), this condition is always satisfied because by assumption every pair that trades is replaced by an identical pair of new agents – the flow equilibrium condition is satisfied by any price (or any price distribution, for that matter). To make the issue nontrivial, Gale introduces an endogenous entry, or participation, decision into the analysis.

Let k_b and k_s be birth rates of potential buyers and sellers, who may or may not choose to enter the market. We impose here that $k_b > k_s$ (but see the exercises). In order for a steady state to exist, we must keep potential

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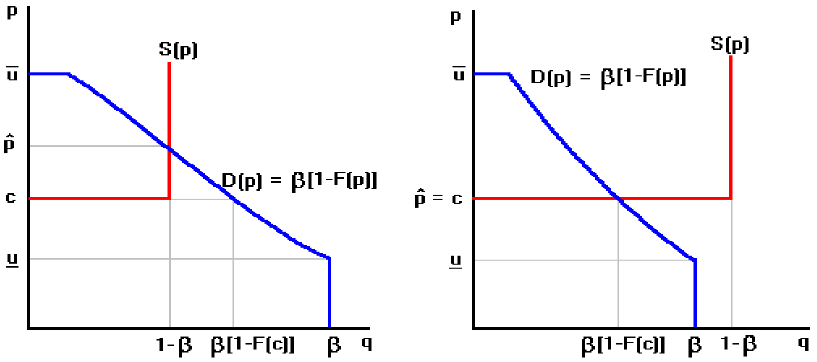
[8] When $\theta = 0$ and $r \to 0$, we have two ostensibly contradictory results: $p(u) = u$ for all $u \geq R$ and $p(u) = R$ for all $u \geq R$. These are reconciled by observing that when $\theta = 0$ and $r \to 0$ we have $R \to \overline{u}$ (that is, when sellers have all the bargaining power and do not discount, they only sell to buyers with $u = \overline{u}$).



Case 1: $1-\beta < \beta[1-F(c)]$

Case 2: $1-\beta > \beta[1-F(c)]$

Figure 7.1: Walrasian Equilibrium Defined on Stocks

buyers with $u < R$ out of the market, since otherwise they will pile up and clog the matching process. This is accomplished by adding an arbitrarily small but positive entry cost e.[9] Hence, the rate at which buyers actually enter is $k_b[1 - F(R)]$, if we assume F is continuous so that the set of agents who are indifferent to entry has measure 0. Since sellers are homogeneous, we simply assume that they all enter, which is a maximizing strategy as long as $V_s \geq 0$.

In this model every meeting results in trade, because $u \geq R$ for every buyer in the market. Thus the number of trades per unit time is given by the number of sellers multiplied by their arrival rate $N_s\beta$ (or, equivalently, number of buyers times their arrival rate). A steady state flow equilibrium simply requires $k_b[1 - F(R)] = k_s = N_s\beta$, the first equality defining flow equilibrium and the second steady state. The interpretation of the role of R is that it produces the right amount of entry. In fact, the flow equilibrium value of $R = R^*$ is determined exclusively by the condition $k_b[1 - F(R)] = k_s$,

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[9] Formally, we can consider equilibrium for $e > 0$ and then let $e \to 0$.

which implies $R^* > \underline{u}$ under our hypothesis $k_b > k_s$.

A version of (7.9) continues to hold, as long as we replace $F(u)$ by $F(u \mid u \geq R)$ to accommodate the fact that only buyers with $u \geq R$ are in the market:

$$R - c = \frac{\beta(1-\theta)}{r + \theta(1-\beta)} \int_R (u - R) dF(u \mid u \geq R). \tag{7.12}$$

Given R^*, (7.12) now serves only to determine $\beta = \beta^*$; that is, the fraction of buyers in the market must adjust so that the flow equilibrium value of R^* is consistent with the search and bargaining model. Given β^*, the steady state conditions determine N_s^* and N_b^*. The price distribution is given by (7.10) with $R = R^*$ and $\beta = \beta^*$.

In the limit as $r \to 0$, the price distribution again collapses to $p(u) = R^*$ for all u. Gale says that the limiting equilibrium is "Walrasian with respect to [the distribution of potential entrants F]" because it entails a single price R^*, and this price equates entry by buyers and sellers (i.e., it is a flow equilibrium). The limit price is not, except by sheer coincidence, Walrasian in the sense that it equates supply and demand defined on the stock of agents in the market at any point in time $-$ that is, R^* need not equal the $\hat{p}$ described in Figure 1. Nonetheless, since the limiting equilibrium is Walrasian according to Gale's criterion, there is a well-defined sense in which one can say that a market with search and bargaining does converge to a Walrasian model as the frictions vanish.

To close this section we present the more general version of the model, with heterogeneity on both sides of the market. Assume now that u is distributed across buyers according to $F_b(u)$ with support $[\underline{u}, \overline{u}]$, and c is distributed across sellers according to $F_s(c)$ with support $[\underline{c}, \overline{c}]$. The price will, in general, depend on both the buyer and seller, $p = p(u, c)$. In a random match, we assume there is a positive probability that $u > c$, which is obviously necessary for the market to be active at all.

Bellman's equations are given by

$$rV_s(c) \quad = \quad \beta E_u \max[p(u, c) - c - V_s(c), 0] \tag{7.13}$$

$$rV_b(u) \quad = \quad (1-\beta)E_c \max[u - p(u,c) - V_b(u), 0], \tag{7.14}$$

where E_j is the expectation with respect to j. Given that buyer u and seller c trade, the bargaining solution is

$$p(u,c) = \theta[c + V_s(c)] + (1-\theta)[u - V_b(u)], \tag{7.15}$$

which generalizes (7.8). Inserting (7.15) into (7.13) and (7.14) and simplifying, we have

$$\begin{aligned} rV_s(c) &= \beta(1-\theta)E_u\Psi(u,c) \\ rV_b(u) &= (1-\beta)\theta E_c\Psi(u,c), \end{aligned}$$

where

$$\Psi(u,c) \equiv \max[u - c - V_b(u) - V_s(c), 0]. \tag{7.16}$$

For any $r > 0$, seller c trades with a buyer if and only if $u \geq R_u(c)$, and buyer u trades with a seller if and only if $c \leq R_c(u)$, where these reservation values $R_u(c)$ and $R_c(u)$ are defined by

$$\begin{aligned} R_u(c) - c - V_b[R_u(c)] - V_s(c) &= 0 \\ u - R_c(u) - V_b(u) - V_s[R_c(u)] &= 0. \end{aligned}$$

Notice that the reservation buyer for a seller depends on the seller's c, and vice-versa. Let the probability that seller c trades when he meets a buyer be denoted $\pi_s(c) = 1 - F_b[R_u(c)]$, and let the probability that buyer u trades when he meets a seller be denoted $\pi_b(u) = F_s[R_c(u)]$. Then we can rearrange (7.13) and (7.14) as

$$\begin{aligned} V_s(c) &= \frac{\beta(1-\theta)\pi_s(c)}{r + \beta(1-\theta)\pi_s(c)} \left\{E_u[u - V_b(u) \mid u \geq R_u(c)] - c\right\} \\ V_b(u) &= \frac{(1-\beta)\theta\pi_b(u)}{r + (1-\beta)\theta\pi_b(u)} \left\{u - E_c[c + V_s(c) \mid c \leq R_c(u)]\right\}. \end{aligned}$$

Although it is difficult to say much about these in general for $r > 0$, in the limit as $r \to 0$ we have

$$\begin{aligned} V_s(c) &= E_u[u - V_b(u) \mid u \geq R_u(c)] - c \\ V_b(u) &= u - E_c[c + V_s(c) \mid c \leq R_c(u)], \end{aligned}$$

as long as $\pi_s(c)$ and $\pi_b(u)$ are strictly positive. Hence, in the limit, $V_s(c)+c$ does not depend on c and $u - V_b(u)$ does not depend on u. Letting $R = V_s(c) - c$, we can write $V_s(c) = \max\{R-c, 0\}$ and $V_b(u) = \max\{u-R, 0\}$. In other words, in the limit, there is a single price, $p(u,c) = R$ for all (u,c). All sellers with $c \leq R$ and all buyers with $u \geq R$ trade, which means $R_c(u) = R_u(c) = R$.

Following the argument in Gale (1987), we can determine R as follows. First, for any $r > 0$, rewrite (7.13) and (7.14) as

$$\begin{aligned} rV_s(c) &= \beta(1-\theta)\int \Psi(u,c)dF_b(u) \\ rV_b(u) &= (1-\beta)\theta\int \Psi(u,c)dF_s(c) \end{aligned}$$

where $\Psi(u,c)$ was defined in (7.16). Then integrate these over c and u, respectively, to get

$$\begin{aligned} r\int V_s(c)dF_s(c) &= \beta(1-\theta)\int\int \Psi(u,c)dF_b(u)dF_s(c) \\ r\int V_b(u)dF_b(u) &= (1-\beta)\theta\int\int \Psi(u,c)dF_s(c)dF_b(u). \end{aligned}$$

Hence, for all $r > 0$,

$$(1-\beta)\theta\int V_s(c)dF_s(c) = \beta(1-\theta)\int V_b(u)dF_b(u).$$

Letting $r \to 0$, we have

$$(1-\beta)\theta\int_{\underline{c}}^{R}(R-c)dF_s(c) = \beta(1-\theta)\int_{R}^{\overline{u}}(u-R)dF_b(u), \tag{7.17}$$

using the result that in the limit $V_s(c) = \max\{R-c, 0\}$ and $V_b(u) = \max\{u-R, 0\}$. Notice how (7.17) generalizes (7.11). In particular, if $\theta = \frac{1}{2}$ then R equates the total surplus of the sellers and the total surplus of the buyers in the market.

If entry is exogenous, in the sense that every pair that trades is necessarily replaced by an identical pair, this completes the description of equilibrium. If, alternatively, there are exogenous birth rates k_s and k_s, but agents only enter the market if they are able to trade, then in the limit as $r \to 0$ Gale's notion of a flow equilibrium is $k_s F_s(R) = k_b[1 - F_b(R)]$ (assuming F_j is continuous). This determines the limit price R, and then β is determined by (7.17) with $F_s(c)$ replaced by $F_s(c \mid c \leq R)$ and $F_b(u)$ replaced by $F_b(u \mid u \geq R)$. Finally, the steady state conditions determine the numbers of traders on each side of the market. This completes the description of Gale's model.

Exercises:

1. What happens in the model with F_s degenerate (heterogeneous buyers and homogeneous sellers) if $k_b < k_s$? What happens in the general model if F_b and F_s are not necessarily continuous? Hint: Do not forget about mixed strategies.

2. Aalyze the model with F_b degenerate (heterogeneous sellers and homogeneous buyers).

3. Redo the analysis assuming that with probability θ the buyer gets to make a take-it-or-leave-it offer to the seller, and with probability $1 - \theta$ the seller gets to make a take-it-or-leave-it offer to the buyer. Note: If a take-it-or-leave-it offer is rejected, the agents necessarily part company and reenter the search process.

4. Redo the analysis assuming a general meeting technology; that is, the total number of meetings between buyers and sellers per unit time is given by $m(N_b, N_s)$.

7.4 Middlemen

In this section, we consider the model of *middlemen* developed by Rubinstein and Wolinsky (1987). As they point out, "Despite the important role played by intermediation in most markets, it is largely ignored by the standard theoretical literature." It is natural to study intermediation in models with frictions. In this framework, intermediation can emerge as an equilibrium because intermediaries may be able to economize on search.[10]

There is a fixed distribution of agents into three groups: buyers, sellers, and (potential) middlemen. As in the Rubinstein-Wolinsky (1985) model, buyers are each interested in purchasing 1 unit of an indivisible good for up to 1 unit of transferable utility, and sellers each can produce 1 unit at 0 cost. That is, a buyer gets instantaneous utility $1-p$ and a seller gets instantaneous utility p from trading at price p. A seller may either sell directly to a buyer, or to a middleman who will then sell to a buyer. Once a buyer or seller trades, he leaves the market and is replaced by a new agent of the same type.

Middlemen remain in the market forever. Also, they can hold at most 1 unit of the good in inventory. Let Ω equal 1 or 0, depending on whether or not middlemen buy from sellers. Of course, in a steady state, middlemen sell to buyers if and only if they buy from sellers. The goal is to determine Ω in equilibrium (this is the sense in which the model generates an endogenous role for middlemen).

Let (N_b, N_s, N_0, N_1) denote the measure of agents who are buyers, sellers, middlemen with no good in inventory, and middlemen with 1 good in inventory. Rubinstein and Wolinsky focus on the special case where $N_b = N_s$, which implies the rate at which buyers meet sellers equals the rate at which sellers meet buyers, and is denoted by α. They also look only at the case where the rate at which sellers meet middlemen with 0 goods equals the rate at which buyers meet middlemen with 1 good, denoted by β, and the rate at which middlemen with 0 goods meet sellers equals the rate at which middlemen with 1 good meet buyers, denoted by γ. This requires that $N_0 = N_1$.

Let V_j denote the value function of a type j agent, where $j = b$, s, 0 or

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[10] In a related paper, Bhattacharya and Haggerty (1986) use a version of Diamond's (1982) search model to analyze intermediaries. See also Yavas (1992) and Winkler (1989). A search model with private information where middlemen are agents who (endogenously) invest in a quality verification technology is studied by Li (1995). Townsend's (1975) develops a model of intermediation build not on search, but on the idea that intermediaries are able to economize on a fixed cost associated with bilateral exchange.

1. Also, let (p_{bs}, p_{bm}, p_{ms}) be the prices in terms of transferable utility that a buyer pays a seller, a buyer pays a middleman, and a middleman pays a seller. Then in steady state the value functions satisfy the usual equations,

$$\begin{aligned} rV_b &= \alpha(1 - p_{bs} - V_b) + \beta\Omega(1 - p_{b1} - V_b) \\ rV_s &= \alpha(p_{bs} - V_s) + \beta\Omega(p_{0s} - V_s) \\ rV_0 &= \gamma\Omega(V_1 - V_0 - p_{0s}) \\ rV_1 &= \gamma(p_{b1} + V_0 - V_1), \end{aligned}$$

where, as indicated above, $\Omega = 1$ if middlemen buy from sellers and $\Omega = 0$ otherwise.

The bargaining problem between a buyer and a seller is assumed to be resolved according to the symmetric Nash solution

$$p_{bs} = \arg\max\ (1 - p - V_b)(p - V_s).$$

This implies $p_{bs} = \frac{1}{2}(1+V_s-V_b)$. An analogous bargaining problem between a buyer and a middleman yields $p_{bm} = \frac{1}{2}(1 + V_s + V_1 - V_0)$ and, if a middleman and a seller trade, an analogous bargaining problem between them yields $p_{ms} = \frac{1}{2}(V_1 - V_0 + V_s)$. A middleman and a seller trade if the surplus is positive; that is

$$\Omega = 1 \Leftrightarrow V_1 - V_0 + V_s > 0. \tag{7.18}$$

In order to construct equilibria, we conjecture a value of Ω, then solve for p_{ij} and V_j, and check (7.18). For $\Omega = 1$, the solutions for the value functions are

$$\begin{aligned} DV_b &= 2\alpha(r + \beta + \gamma) + \beta(2r + \gamma + \beta) \\ DV_s &= 2\alpha\gamma + \gamma\beta + 2\alpha r \\ DV_0 &= \gamma(\gamma - \alpha) \\ DV_1 &= \gamma(2r + \alpha + \beta + \gamma), \end{aligned}$$

where $D = (2\gamma + 2r + \beta)(2\alpha + 2r + \beta)$, and the solutions for the prices are

$$\begin{aligned} p_{bs} &= (r + \gamma)/(2\gamma + 2r + \beta) \\ p_{b1} &= (2\alpha\gamma + 2r^2 + \beta r + 3r\gamma + \alpha r + \gamma\beta)/D \\ p_{0s} &= (r\gamma + 2\alpha\gamma + \gamma\beta + \alpha r)/D. \end{aligned}$$

Given the value functions, (7.18) implies $\Omega = 1$ is an equilibrium if and only if $\gamma > \alpha$.

Similarly, if we set $\Omega = 0$ and then solve, we find $V_1 - V_0 + V_s < 0$ if and only if $\alpha > \gamma$. Hence, equilibrium is generically unique, and middlemen are active in equilibrium if and only if they meet buyers faster than sellers meet buyers ($\gamma > \alpha$). That is, middlemen exist to speed up trade. Notice that $p_{bm} > p_{bs} > p_{ms}$, and middlemen sell for more than they pay; of course, this is how they make a profit. The spread $p_{bm} - p_{ms}$ is decreasing in α, since the opportunity to sell directly disciplines how much a middleman can extract. Also, as $r \to 0$ the spread vanishes, since sellers do not mind waiting to meet a buyer directly.

Exercises:

1. Solve for the value functions and prices when $\Omega = 0$.

2. Evaluate the effects of changes in the parameters α, β, γ, and r.

3. Show that $\alpha = 0$ and $\beta = \gamma$ implies $V_b = V_1$ and $V_s = V_0$. Interpret this result.

4. Analyze the model using the generalized Nash bargaining solution. Also, analyze the model under the Nash bargaining solution with 0 threat points (corresponding to the limiting form of a strategic bargaining game under the assumption that agents never meet other trading partners during the interval between rejection and counteroffer).

5. What changes if all agents stay in the market forever; e.g., after a sale to either a buyer or a middlemen, the seller produces and reenters the search process?

6. Consier the model with arbitrary arrival rates: α_{ij} is the rate at which type i meets type j, for $i, j = b, s, 0, 1$. Note that, in addition to p_{ij} and V_j, you should also determine the steady state number of each type N_j.

7.5 A Monetary Model

Shi (1995) and Trejos and Wright (1995) introduce bargaining into the model of fiat money in Kiyotaki and Wright (1991, 1993) in order to determine the nominal price level endogenously. To review the basic assumptions in the simplest monetary search model, there are $M \in (0,1)$ buyers initially endowed with one unit each of fiat currency, and $1-M$ sellers with the ability to produce nonstorable consumption goods. A key simplifying assumption here is that when buyers spend their money they spend *all of it*. This assumption, although it can be motivated in more than one way, it easiest to interpret if we simply say that fiat money is indivisible.[11]

Agents never consume their own output, and so must trade. Traders meet according to Poisson arrival rates that are proportional to the number of agents on the other side of the market. In a random meeting between two agents, the latter can produce the former's consumption good with probability x, and they can both produce each others' consumption goods with probability 0 (no double coincidence of wants). This implies there is no direct barter, and since consumption goods are nonstorable there is no commodity money. Hence, all trade involves a buyer paying with cash. Later we will generalize the model to allow barter, and, following the approach used in the previous chapter, assume that the probability of a double coincidence is given by x^2.

Let $u(q)$ be the utility of consuming q units of one's consumption good and $c(q)$ the disutility of producing q units of one's production good. We assume $u(0) = c(0)$, $u'(0) > c'(0) = 0$, $u'(q) > 0$, $c'(q) > 0$, $u''(q) \leq 0$, and $c''(q) \geq 0$, for $q > 0$, with at least one of the weak inequalities strict. Also, there is a $\hat{q} > 0$ such that $u(\hat{q}) = c(\hat{q})$. When a buyer meets a seller who can

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[11] This assumption implies that agents always have either 1 or 0 units of cash, rather than some fractional amount that varies across the population and needs to be determined endogenously, at least if we can guarantee that agents do not sell twice in a row in order to acquire a second unit of money. The latter can be guaranteed if we say that you must first consume in order to produce.

produce the right good, they bargain over how much q will be exchanged for the buyer's unit of money, implying a nominal price $p = 1/q$. Once a deal is consummated, the seller becomes a buyer, and vice-versa. All agents remain in the market forever.

If $q = Q$ is taken as given and V_b and V_s denote the value functions, then we have the usual dynamic programming equations

$$\begin{aligned} rV_b &= \alpha(1-M)x[u(Q) + V_s - V_b] && (7.19) \\ rV_s &= \alpha M x[V_b - V_s - c(Q)]. && (7.20) \end{aligned}$$

These can easily be solved for $V_b = V_b(Q)$ and $V_s = V_s(Q)$. Simultaneously, if $V_b(Q)$ and $V_s(Q)$ are taken as given, then q solves a bargaining problem. For now, we adopt the symmetric Nash bargaining solution with 0 threat points,

$$q = \arg\max\ [u(q) + V_s(Q)][V_b(Q) - c(q)], \tag{7.21}$$

subject to $u(q) + V_s \geq V_b$ and $V_b - c(q) \geq V_s$ (other solutions are also considered below).

The bargaining solution (7.21) defines a mapping $q = q(Q)$ from $[0, \hat{q}]$ into itself. That is, if other agents are giving Q units of output for a dollar, then a particular pair bargaining bilaterally will agree to $q = q(Q)$. A steady state equilibrium is a fixed point, $q = q(Q)$, and is called monetary if $q > 0$.

In this model we have to be careful with the constraints on the bargaining problem, because the fiat nature of money implies that there can potentially exist more than one equilibrium, and in some of the equilibria the constraints may bind (something that did not happen in the previous models of this chapter). In fact, with no barter, as we are assuming for now, it will turn out that the constraints are not binding in equilibrium; but as soon as we introduce barter the constraints may bind. Hence, we proceed allowing for this possibility.

We begin by looking at the constraints, which can be written $c(q) \leq D(Q)$ and $u(q) \geq D(Q)$, where $D(Q) = V_b(Q) - V_s(Q)$. The former constraint is satisfied iff $q \leq f(Q)$ and the latter is satisfied iff $q \geq g(Q)$, for increasing functions f and g. As shown in Figure 2, both f and g go through the origin in the (Q, q) plane, and g lies below f and below the 45^o line for all $Q \in [0, \hat{q}]$. Also, g crosses the 45^o at a unique $q_1 \in (0, \hat{q}]$. Hence, our search for equilibria can be constrained to the interval $[0, q_1]$.[12]

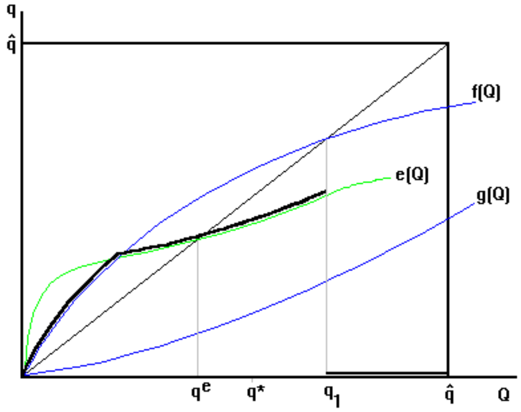

Figure 7.2: Unique Monetary Equilibrium in Model with no Barter

Now consider the first order condition for an interior solution to (7.21), taking $V_b = V_s(Q)$ and $V_s = V_s(Q)$ as given,

$$[V_b(Q) - c(q)]u'(q) - [u(q) + V_s(Q)]c'(q) = 0$$

(note that the second order condition always holds). This defines a function $q = e(Q)$, also shown in the figure. It also goes through the origin and intersects the 45^o line at a unique point q^e. Hence, the correspondence $q = q(Q)$, which is actually a (single-valued) function, can be written as follows: $q(Q) = \min\{e(Q), f(Q)\}$ for all $q \in [0, q^e]$, and $q(Q) = \max\{e(Q), g(Q)\}$ for all $q \in [q^e, q_1]$. For $q > q_1$, it does not really matter how we define $q(Q)$, since it must lie below the 45^o line, and we set it to $q(Q) = 0$.

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[12] Notice that at $Q = q_1$, we have $V_s = 0$. Hence, for all $q > q_1$, sellers would drop out of the economy.

From the above discussion it is clear that for all parameter values $q = q(Q)$ has exactly two fixed points: a nonmonetary equilibrium $q = 0$, and a unique monetary equilibrium $q = q^e > 0$. However, these results are somewhat special. In the generalized version of the model that allows barter, we will see below that for some parameter values there are no monetary equilibria and for other parameter values there multiple monetary equilibria. It is only because we have ruled out barter here that there always exists a unique monetary equilibrium. But it is still worth exploring properties of q^e in this case.

One important property of monetary equilibrium (that continues to hold even if we allow barter) is the following. Let q^* be defined by $u'(q^*) = c'(q^*)$; then it is easy to show $e(q^*) < q^*$ and, therefore, $q^e < q^*$, as seen in Figure 2. This is significant because q^* is what one might expect out of a frictionless model. More precisely, define welfare by $W = MV_b + (1 - M)V_s$, and note that after simplification we have

$$rW = M(1 - M)[u(q) - c(q)]. \tag{7.22}$$

Hence, W is maximized with respect to q at q^*.

The result $q^e < q^*$ says that q^e is too low – or, equivalently, the price level is too high – in equilibrium. This is true even though bargaining is bilaterally efficient in the sense that the agreement is on the Pareto frontier in each exchange, taking as given the value of Q that prevails in other exchanges. The point is that all agents would be better off (in an ex ante sense) if they could get *everyone* to commit to increase q.[13]

The economic intuition for this result is straightforward. If a seller could turn the proceeds from his production into immediate consumption, as in a static or frictionless model, then he would produce until $u'(q) = c'(q)$. But in a monetary exchange the proceeds from production consist of cash that can

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[13] A stronger result is actually true: not only is q^e too low according to the ex ante criterion W, it is also too low according to the ex post criteria V_s and V_b. That is, buyers and even sellers would be better off if q was bigger.

only be spent in the future, when an opportunity to buy comes along. Since he discounts the future, a seller is willing to produce less than the amount that satisfies $u'(q) = c'(q)$. To verify that it is frictions that are driving the result, observe that when $r \to 0$ or $\alpha \to \infty$ we have $q^e \to q^*$ (see the exercises).

Although this tendency for frictions to make $q^e < q^*$ is very natural, there are other effects can offset and even overturn the result. For example, consider using the generalized Nash solution

$$q = \arg\max\, [u(q) + V_s]^{\theta}[V_b - c(q)]^{1-\theta},$$

subject to $c(q) \leq D(Q)$ and $u(q) \geq D(Q)$, where θ is the bargaining power of the buyer. In Figure 2, varying θ does not affect the constraints but does shift the $e(Q)$ function. For all $\theta \in (0,1)$, there is a unique monetary equilibrium q^e, and as θ increases from 0 to 1 the equilibrium value of q^e increases monotonically from 0 to q_1. Since $q_1 > q^*$, we see that it is possible for q^e to exceed q^* when the buyer has sufficient bargaining power.

Returning to the symmetric case $\theta = \frac{1}{2}$, another question to ask is how q^e depends on M, the fraction of the population holding money. One might expect $\partial q^e/\partial M < 0$, since, after all, q is the value of money. It turns out that it is actually possible to have $\partial q^e/\partial M > 0$ for small M (at least if r is also small). The explanation is that when M is near 0 there is very little trade. In this case, increasing M increases the frequency of productive meetings between buyers and sellers, which increases both V_b and V_s. The net effect on the bargaining solution can be a higher q. However, there is some threshold, $\hat{M} < \frac{1}{2}$, such that $\partial q^e/\partial M < 0$ for all $M > \hat{M}$. Hence, we can be sure that the value of money eventually begins to fall as M increases.

We can also ask how M affect welfare, W, as defined in (7.22). It is clear that if a planner can choose both M and q to maximize W, he will choose $M = \frac{1}{2}$ and $q = q^*$. This is because $M = \frac{1}{2}$ maximizes the number of trades and $q = q^*$ maximizes the surplus that results from each trade. But what if the planner can choose only M, with q determined in equilibrium by the private sector?

In this case, the planner chooses M to satisfy the first order condition

$$\frac{\partial rW}{\partial M} = (u-c)(1-2M) + M(1-M)(u'-c')\frac{\partial q^e}{\partial M} = 0.$$

As the second term is negative at $M = \frac{1}{2}$, the solution to the first order condition is $M^o < \frac{1}{2}$. This illustrates the trade-off between providing liquidity to make trade easier, on the one hand, and reducing the value of money, on the other. Reducing the value of money is welfare-reducing here because, as we have already establihed (at least for the case $\theta = \frac{1}{2}$), q is too low in equilibrium.

To close this section we consider an extension of the model in which direct barter is possible.In particular, when a seller meets another seller the probability is x^2 that they have a double coincidence of wants, exactly as in the fixed-price model of fiat money in Kiyotaki and Wright (1993) described in the previous chapter. When this happens, the symmetric Nash bargaining solution implies that each seller will provide the other with q^*, the quantity that solves $u'(q) = c'(q)$. Hence, Bellman's equation for a seller (7.20) becomes

$$rV_s = Mx[V_b - V_s - c(q)] + (1-M)x^2[u(q^*) - c(q^*)],$$

where we have normalized the arrival rate to $\alpha = 1$ to reduce notation. Bellman's equation for a buyer is still given by (7.19) except that now $\alpha = 1$.

Introducing barter shifts the e, f and g functions in Figure 2 as indicated by Figure 3. If $f(Q)$ intersects the 45^o line at all, which it does if r and x are not too big, then it intersects it exactly twice: at $q_0 > 0$ and at $q_1 > q_0$. Moreover, if $f(Q)$ intersects the 45^o line at all, then $e(Q)$ intersects the 45^o line exactly once at $q^e \in (q_0, q_1)$. Hence, if r and x are not too big, as shown in the figure, the function $q = q(Q)$ has exactly three fixed points: $q = 0$, q_0, and q^e. For other values of r and x, $f(Q)$ may lie below the 45^o for all $q > 0$, in which case $q = 0$ is the only fixed point.

More precisely, there exists $\tilde{r} = \tilde{r}(x, M)$ such that $r < \tilde{r}$ implies that there are two monetary equilibria and $r > \tilde{r}$ implies that there are no monetary

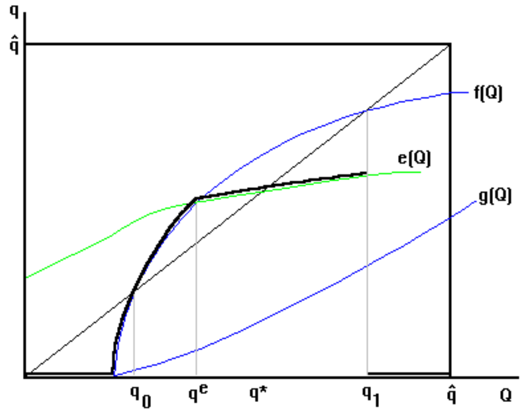

Figure 7.3: Multiple Monetary Equilibria in Model with Barter

equilibria (see Trejos-Wright 1995). At the monetary equilibrium q_0 the bargaining solution is constrained, and at q^e it is interior. Notice also that q^e will be lower in this model than in the model without barter; hence, $q < q^*$ in both of the monetary equilibria. Finally, one can also ask how q depends on parameters in this model, but (as is always the case with multiple equilibria) the answer will depend on whether we consider q_0 or q^e.

Exercises:

1. Verify the properties of the functions e, f and g used in the text in constructing Figures 2 and 3.

2. For model with the bargaining solution given in (7.21), verify that $\partial q/\partial r < 0$ and $\partial q/\partial \alpha > 0$, that $q^e < q^*$, and that $q \to q^*$ as $r \to 0$ or $\alpha \to \infty$. Also verify that $\partial q/\partial M < 0$ if and only if $M > \hat{M}$, where $\hat{M} > 0$ for small r. Also show that $\hat{M} < \frac{1}{2}$ for all r (see Trejos and Wright 1995).

3. For the model with

$$q = \arg\max\, [u(q) + V_s - V_b][V_b - c(q) - V_s],$$

show that $\partial q/\partial r < 0$ and $\partial q/\partial \alpha > 0$, but now we can have $q^e > q^*$, and, in particular, as $r \to 0$ or $\alpha \to \infty$ we have $q^e < q^*$ if and only if $M > \frac{1}{2}$. Also show that $\partial q/\partial M < 0$ for all r and M (see Trejos and Wright 1995).

4. Analyze the existence and uniqueness of equilibrium for the model with the generalized Nash solution, allowing for barter.

5. When multiple monetary equilibria exist, compare q^e and q_0 in terms of welfare.

6. Consider the example $u(q) = q$ and $c(q) = q^2$. In the case of no barter, 0 threat points, and the buyer having bargaining power θ, find the unique monetary equilibrium. Answer:

$$q^e = \frac{(2-\theta)\Phi - \sqrt{\Upsilon}}{4M(r+1-M)(1-\theta)},$$

where $\Upsilon = (2-\theta)^2\Phi^2 - 8(r+M)(r+1-M)M(1-M)\theta(1-\theta)$ and $\Phi = r(1+r) + M(1-M)$.

7.6 A Simplified Monetary Model

Here we present a simplified version of the model in the previous section. It is simplified by assuming that the buyer makes a take-it-or-leave it offer to the seller, which is equivalent to setting $\theta = 1$ in the generalized Nash bargaining solution. Also, it reduces the notation with no loss in generality if we normalize the disutility of production to be $c(q) = q$.[14] We also normalize the arrival rate so that $\alpha x = 1$. Otherwise, the model is as in the previous section.

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[14] We can always normalize $c(q) = q$, as long as we also renormalize $u(q)$. This simply means we have to reinterpret the parties as bargaining over the disutility the seller incurs from a trade, rather than the number of units of output. That is, given $c(q)$ and $u(q)$, it is equivlalent to consider $\tilde{c}(q) = q$ and $\tilde{u}(q) = u(c^{-1}(q))$.

Let $z = \alpha(1-M)x^2[u(q^*) - c(q^*)]$ denote the flow payoff from barter. Given a value Q prevailing in the market, the value functions satisfy

$$\begin{aligned} rV_s &= z + M[V_b - Q - V_s] && (7.23) \\ rV_b &= (1-M)[u(Q) + Vs - V_b]. && (7.24) \end{aligned}$$

These can be solved for V_b and V_s, and also for the difference $D = V_b - V_s$:

$$D = D(Q) = \frac{(1-M)u(Q) + MQ - z}{1+r}.$$

If the buyer can make a take-it-or-leave-it offer, he will make it so that

$$q = V_b - V_s = D(Q), \tag{7.25}$$

as long as $D(Q) \geq 0$. Hence, $q(Q) = \max\{D(Q), 0\}$.

The function $D(Q)$ is exactly the function $f(Q)$ depicted in Figure 2 for the case where $z = 0$ and in Figure 3 for the case where $z > 0$ but not too big, the only difference being that now we have normalized $c(q) = q$. In Figure 2, there is a unique monetary equilibrium at $q = q_1$, and in Figure 3 there are two monetary equilibria at $q_0 > 0$ and $q_1 > q_0$. If z is too big $-$ that is, above some threshold $\tilde{z} = \tilde{z}(r, M)$ $-$ then $D(Q)$ does not intersect the 45^o line at all and no monetary equilibria exist. Alternatively, for a fixed z, if r is above some threshold $\tilde{r} = \tilde{r}(z, M)$ then no monetary equilibria exist.

We can use this model to more easily analyze some results that others have derived using different bargaining solutions. For example, suppose we want to look for nonstationary equilibria, as Trejos and Wright (1995) and Coles and Wright (1995) do for models with symmetric bargaining. The dynamic generalizations of (7.23) and (7.24) are

$$\begin{aligned} rV_s &= z + M[V_b - Q - V_s] + \dot{V}_s \\ rV_b &= (1-M)[u(Q) + V_s - V_b] + \dot{V}_b. \end{aligned}$$

Combining these and using (7.25) yields $\dot{q} = \varphi(q)$, where $\varphi(0) = 0$ and

$$\varphi(q) = z + (r + 1 - M)q - (1-M)u(q)$$

for all $q > 0$.

As shown in Figure 4, for $z > 0$ there are three solutions to $\varphi(q) = 0$, which of course are the three steady states, $q = 0$, q_0, and q_1. For any initial $q > q_1$, we have $q \to \infty$, which cannot be an equilibrium. For any initial $q \in (0, q_1)$, we see that $q \to q_0$. Hence, the set of equilibria includes the steady states and a continuum of paths indexed by their initial condition in $(0, q_1)$ and converging to q_0. If $z = 0$ (the no barter case), then q_0 coalesces with the nonmonetary steady state. In this case the set of equilibria includes the nonmonetary steady state, the unique monetary steady state, and a continuum of paths leading from the latter to the former. Along these dynamic paths, the price level is changing over time due to nothing more than self-fulfilling expectations.[15]

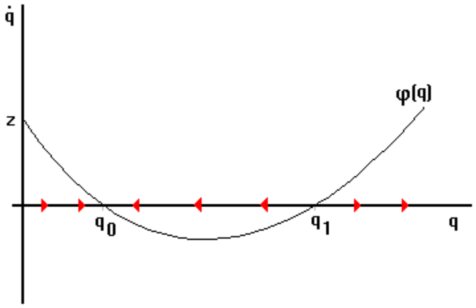

Figure 7.4: Dynmaic Equilibria with z > 0.

As another example of results that are much simpler if buyers make take-it-or-leave-it offers, consider the economy in Shi (1995) with two distinct monies distinguished by color, say, red and blue. Following Shi, we also assume that agents can hold at most 1 unit of one money or the other (not both), and that agents holding one money never trade with agents holding

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[15] The analogy to the overlapping generations model should be apparent; see, e.g., Azariadis (1992).

the other (but see below). Denote the number of buyers with money j by N_j, the value function for a buyer with money j by V_j, and the amount purchased from a seller by a buyer with money j by Q_j. The take-it-or-leave-it assumption immediately implies $Q_j = V_j - V_s$, for $j = R, B$.

The value functions satisfy the ususal equations,

$$\begin{aligned} rV_s &= z \\ rV_R &= N_s[u(Q_R) + V_s - V_R] \\ rV_B &= N_s[u(Q_B) + V_s - V_B], \end{aligned}$$

where $N_s = 1 - N_1 - N_2$ is the number of sellers, and we have used the result that the seller gets no surplus from exchanging goods for money to write $rV_s = z$. It should be clear that for z positive but not too big we have the following results. There exists two monetary equilibria where the monies are treated identically: $Q_R = Q_B = q_1$ and $Q_R = Q_B = q_0$. There also exist two monetary equilibria where the monies have different values, $Q_R = q_0$ and $Q_B = q_1$, as well as $Q_R = q_1$ and $Q_B = q_0$.[16]

These results are based on the result that there are multiple steady state monetary equilibria even with only one money. From this multiplicity, one can construct equilibria where two colors of money have different values. Hence, the results hinge on $z > 0$. A related but different type of result occurs if we set $z = 0$, so that there is a unique monetary equilibrium with one money, but allow agents holding red money to trade with agents holding blue money (although still restricting agents to hold no more than 1 unit of money in total), as in Aiyagari, Wallace and Wright (1995). Of course, allowing agents with money to trade with each other does not matter if $V_R = V_B$; but we now show how to construct equilibria where $V_R > V_B$, in which an agent with red money is able (via a take-it-or-leave-it offer) to get

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[16] The same results are derived by Shi (1995), although they are much simpler here due to the assumption of take-it-or-leave-it offers. In a related exercise, with one money, Shi shows that there also exist dynamic sunspot equilibria, where q fluctuates randomly over time (see the exercises).

an agent with blue money to swap currencies plus give up $q = V_R - V_B > 0$ uits of output.

To this end, first notice that when buyers make take-it-or-leave-it offers and $z = 0$ we have $rV_s = 0$. Then the Bellman equation for a blue money holder is

$$rV_B = N_s[u(V_B) - V_B],$$

where we have inserted $Q_B = V_B$. Observe that (7.6) can be solved for V_B^* independent of V_R. The Bellman equation for a red money holder is

$$rV_R = N_s \max\{u(V_R) - V_R, 0\} + N_B \max\{u(V_R - V_B^*) + V_B^* - V_R, 0\},$$

because an agent with red money has the option to buy from an agent with no money and the option to buy from an agent with blue money. Notice that we have inserted the quantities $q = V_R$ for the amount a buyer with red money gets from a seller, and $q = V_R - V_B$ for the amount a buyer with red money gets from another buyer with blue money.

If the agent with red money does not buy from the agent with blue money then $V_R = V_B$. Hence, in the type of equilibrium we are after, it must be the case that $u(V_R - V_B^*) + V_B^* - V_R > 0$, and all we need to determine is whether an agent with red money buys from agents with no money. That is, we need to determine whether $u(V_R) - V_R > 0$, which depends simply on whether $V_R < \hat{q}$. In any case, the equilibrium value of V_R is the solution to

$$rV_R = N_s \max\{u(V_R) - V_R, 0\} + N_B[u(V_R - V_B^*) + V_B^* - V_R] \qquad (7.26)$$

It may be shows (see the exercises) that (7.26) always has a solution $V_R^* \in (V_B^*, V_B^* + \hat{q})$, and so there always exists an equilibrium of the sort we are after. The equilibrium may involve either $V_R < \hat{q}$ or $V_R > \hat{q}$. That is, agents with red money may either buy from agents with no money, or they may prefer to pass on such opportunities and only buy from agents with bue money. Agents with blue money give them a lower q than agents with no money, but they also give them "change" in the form of the blue money. Not surprisingly, whether agents with red money may buy from agents with no money depends, at least in part, on the rate of time preference, r.

Exercises:

1. In the basic model with take-it-or-leave-it offers, for which there are two monetary equilibria at q_0 and $q_1 > q_0$, prove that the q_1 equilibrium is Pareto superior. Hint: Equilibrium implies $V_b = V_s + q$, and therefore V_b is higher when q is higher, while $V_s = z/r$ is the same in the two equilibria.

2. Prove that in the model with one money, there exist sunspot equilibria where q fluctuates between $\underline{q}$ and $\overline{q}$ according to a Poisson processes, as in Shi (1995).

3. Show that (7.26) has a solution $V_R^* \in (V_B^*, V_B^* + \hat{q})$ that may involve either $V_R < \hat{q}$ or $V_R > \hat{q}$. Argue that it may have multiple solutions, but for small r there is a unique equilibrium with $V_R > V_B$, and it involves $V_R > \hat{q}$, and for big r there is a unique equilibrium with $V_R > V_B$, and it involves $V_R < \hat{q}$.