Chapter 2

Job Search Theory

2.1 Introduction

Here we present the basic job search model, in both discrete and continuous time, and introduce some of the many elaborations and applications that have been discussed in the literature. This focus in this chapter is on decision theory, even though most of the focus in the rest of the book is on equilibrium theory. That is, we are interested for now in the optimization problem of a single agent − such as a worker looking for a job at a good wage − and we make no reference to the problems being solved by other individuals − such as the firms who presumably set the wages − or the conditions that must be satisfied for the decisions of all individuals to be consistent.

It makes sense to understand the optimization problem of a single individual before he is embed into an equilibrium setting. For example, we usually study consumer theory before analyzing market demand before analyzing general equilibrium. Moreover, beginning with a single-agent problem is a good way to learn some of the "tricks of the trade" that will be used extensively in equilibrium modeling. Additionally, our view is that some interesting economic insights can emerge from search theory even when it is not incorporated into a fully articulated equilibrium model.[1]

[1] The early literature on job search was once caricatured by Rothschild (197x) as

So we proceed in this section with what may well be called *one-sided* search models, where workers look for jobs taking as given such things as the distribution of wage offers across firms, without regard to the origin of this distribution. An alternative interpretation, discussed in some of the exercises, concerns the problem of an employer looking for a worker. Many other interpretations are possible, and have been pursued in the literature, including the problem of a buyer looking for a house, an individual looking for a spouse, an investor looking for an opportunity, and so on.

The rest of the chapter is organized as follows. In Section 2, we introduce the basic job search problem in discrete time. In Section 3, we analyze a similar problem in continuous time. In particular, we introduce the notion of Poisson process that will be used throughout the rest of the book. In Section 4, we relax some of the many simplifying assumptions made in the basic model and focus on some reasons for turnover (quits and layoffs). In Section 5, we introduce a restriction on the stochastic structure of the problem and show how it can be used to rule out some rather counterintuitive results that are possible in the unrestricted model. The next chapter will present some extensions and variations on the basic theme.

Since the literature on job search is vast, we cannot hope to survey all of the extensions and applications here, and we are forced to be selective. Some of the things we have left out of the text are outlined in the exercises; others are reviewed in Devine and Kiefer (19xx), who especially emphasize empirical implications and applications.

"partial-partial" equilibrium theory, because it not only considers only one market (the labor market), it only considers one side of that market (workers). It seems silly to us to criticize decision theory for being decision theory.

2.2 Discrete Time Job Search

Consider an individual interested in maximizing the objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(y_t), \tag{2.1}$$

where $\beta \in (0,1)$ is the discount factor, y_t is instantaneous income at date t, $u(y)$ is the instantaneous utility function, and E_0 is the expectation conditional on information available at date 0. For most of what we do it is assumed that $u(y) = y$. One way to interpret this is to say that the individual is risk neutral, so that he does not care about smoothing consumption, and simply consumes y_t each period. In this case (2.1) is expected utility.

A different way to motivate $u(y) = y$, without assuming risk neutrality, is to imagine a world of complete (Arrow-Debreu) markets within which the individual can perfectly insure his consumption against any idiosyncratic income risk. In this scenario, he maximizes expected utility by first maximizing income, and then purchasing any consumption stream he chooses that costs less than income. Hence, it is legitimate study expected income maximizing behavior without regard to consumption decisions.

Still another story that also leads to (2.1) is that there are no markets, so that the agent is forced to consume y_t each period even if he would rather smooth consumption.[2] In this case, there is little loss in generality if we assume that $u(y) = y$, as this essentially amounts to reinterpreting y as utility rather than income; however, in the exercises we sketch some applications with general utility functions. For now, we also maintain the assumption of an infinite horizon. As will be demonstrated, the infinite horizon problem

[2] These two possibilities, where there are complete markets and where there are no markets, are both tractable because they are extreme. Intermediate cases, such as the possibility that risk averse agents can save but cannot buy contingent claims, are much harder because we have to keep track of individual wealth, and, in general equilibrium, of the entire distribution of individual wealth (e.g., see Danforth 19xx for a theoretical analysis of the individual problem, and Hansen and Imrohoroglu 19xx for a numerical analysis of an equilibrium problem).

can be regarded in a well-defined sense as an approximation to a finite but long horizon.

Assume instantaneous income is given by the following specification:

$$y_t = \begin{cases} w & \text{if employed at job } w \\ c & \text{if unemployed} \end{cases}$$

We fix hours of work on the job to unity, for now, so that we can think of w as either the hourly wage or the total (per period) income associated with a job. In fact, it is somewhat restrictive to interpret w as simply the wage; more generally, it is some measure of the desirability of the job, which could be a function of many things (like location, prestige, and so on) other than pecuniary compensation. Nevertheless, we refer to w as the wage in what follows.

The interpretation of unemployment income c is flexible; it may include any income associated with not working, such as the pecuniary value of leisure and home production activities, plus public unemployment insurance (UI) benefits, minus any direct out-of-pocket search costs. To be clear, when we say that an agent is unemployed, we mean that he is not working but is actively searching for work. Alternatively, he may choose to not search, and then we would say he is not in the labor force. Individuals who are not in the labor force pay no search costs, and may or may not receive UI benefits, depending on the institutional structure (see below).

The individual is interested in choosing a *policy* (i.e., a sequence of decision rules) that tells him whether or not to accept any particular job offer. We begin with the case where an unemployed individual samples one i.i.d. (independent and identically distributed) offer each period from a known distribution, called the wage offer distribution, $F(\bar{w}) = \text{Prob}(w \leq \bar{w})$, with finite mean Ew. If an offer is accepted, we assume for now that the agent keeps the job forever; there are no quits or layoffs. If an offer is rejected, the agent remains unemployed at least until the next date, and at no point can he go back and recall a previously rejected offer. Many of these simplifying assumptions are relaxed below. Some, such as no recall, actually turn out to be unrestrictive (see the exercises), while others change the problem in interesting ways.

Let $V(w)$ denote the expected payoff to accepting an offer w at some point in time, referred to as the value of w. It does not depend on when the offer is accepted, given our assumptions of a stationary environment and an infinite horizon. In fact, since jobs are retained forever, $V(w) = w/(1-\beta)$. Let U denote the value of rejecting an offer and remaining unemployed, which also does not depend on time, and does not depend on which wage was rejected since offers are i.i.d. In fact, $U = c + \beta E \max[V(w), U]$, since the value of rejecting an offer equals instantaneous unemployment income plus the discounted expected value of having the option to accept or reject a new offer next period.

Let $J(w) = \max[V(w), U]$ be the value of having offer w in hand. Then $J(w)$ satisfies the following version of *Bellman's equation* of dynamic programming:

$$J(w) = \max\left\{\frac{w}{1-\beta}, c + \beta EJ\right\}. \tag{2.2}$$

As described in the Appendix, dynamic programming provides a method of reducing a complicated problem (in this case, choosing a sequence of decision rules) to a sequence of simpler problems (in this case, at each date choosing whether to accept a single offer).

The nature of the solution can be described as follows. Since $V(w)$ is increasing in w and U is independent of w, there exists a unique R satisfying $V(R) = U$ with the following properties: $w < R$ implies $V(w) < U$ and so w should be rejected, and $w > R$ implies $V(w) > U$ and so w should be accepted. See Figure 1. The value R is referred to as the *reservation wage*, and is the offer at which the individual is indifferent between accepting the offer and remaining unemployed. The optimal search strategy is to accept any offer above the reservation wage.[3]

[3] Generally, a reservation strategy is one with the following property: if w' is acceptable and $w'' > w'$ the w'' is also acceptable. Many, but not all, search problems are solved by reservation strategies, as we shall see.

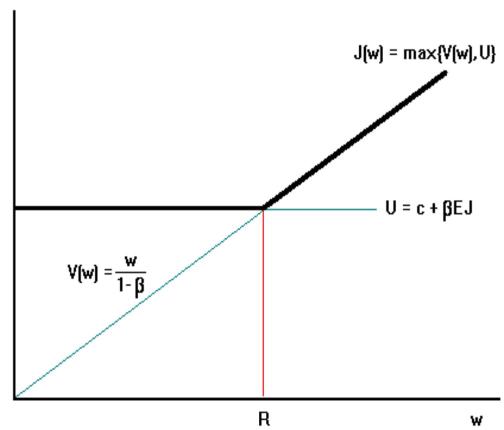

Figure 2.1: Value Function and Reservation Wage

Since $V(R) = R/(1-\beta)$ and $U = c + \beta EJ$, the definition of the reservation wage, $V(R) = U$, is equivalent to

$$R = (1-\beta)c + (1-\beta)\beta EJ. \tag{2.3}$$

This expresses R in terms of the unknown value function, J. However, rewriting (2.2) as

$$J(w) = \begin{cases} \frac{w}{1-\beta} & \text{for } w \geq R \\ \frac{R}{1-\beta} & \text{for } w < R \end{cases}$$

we see that $(1-\beta)EJ = E\max(w, R)$. Combining this with (2.3) we can express the reservation wage as the solution to the following equation:

$$R = (1-\beta)c + \beta \int_0^\infty \max(w, R)dF(w). \tag{2.4}$$

One would like to know if (2.4) has a solution, if it has a unique solution, and if there is a way to find its solution. To this end, define the mapping $T : \mathcal{R} \rightarrow \mathcal{R}$ by

$$T(R) = (1-\beta)c + \beta E\max(w, R), \tag{2.5}$$

so that (2.4) says $R = T(R)$. It turns out that the mapping T has some nice properties due to the fact that it is a *contraction* (see the Appendix for details). In particular, the contraction mapping theorem guarantees that there always exists a solution to $R = T(R)$, that the solution is unique, and the sequence defined by $R_{n+1} = T(R_n)$ converges to the solution as $n \to \infty$ starting from any initial value of R_0.[4]

To illustrate with an example, suppose that w is distributed uniformly on $[0, 1]$. Then (2.5) simplifies to:

$$T(R) = \begin{cases} (1-\beta)\,c + \frac{1}{2}\beta & \text{for } R < 0 \\ (1-\beta)\,c + \frac{1}{2}\beta + \frac{1}{2}\beta R^2 & \text{for } R \in [0,1] \\ (1-\beta)\,c + \beta R & \text{for } R > 1 \end{cases}$$

As illustrated in Figure 2, this function has a unique fixed point $R = T(R)$, and from any initial R_0 the sequence defined by $R_{n+1} = T(R_n)$ converges to R. In this example it is easy to solve for the reservation wage explicitly,

$$R = \frac{1 - \sqrt{(1-\beta)\,(1+\beta-2\beta c)}}{\beta},$$

which is in $(0, 1)$ as long $-\beta/2(1-\beta) < c < 1$.

We now derive a slightly different expression for the reservation wage that will be used repeatedly in the rest of the analysis. First, subtract βU from both ides of $U = c + \beta EJ$ and write the result as

$$(1-\beta)U = c + \beta \int_R^\infty [V(w) - U] dF(w). \tag{2.6}$$

Since U is the value of unemployed search, the left hand side of (2.6) is the *flow* (per period) value associated with this activity; and the right hand side equals unemployment income plus the discounted "option value" of getting another offer, where by option value we mean the gain from switching to employment from unemployment, if positive.

Now, by virtue of the fact that $(1-\beta)U = R$ and $V(w) - U = (w - R)/(1-\beta)$, (2.6) reduces to

$$R = c + \frac{\beta}{1-\beta} \int_R^\infty (w - R) dF(w). \tag{2.7}$$

[4]It will be seen below that the reservation wages from a sequence of finite horizon job search problems is also given by $R_{n+1} = T\,(R_n)$.

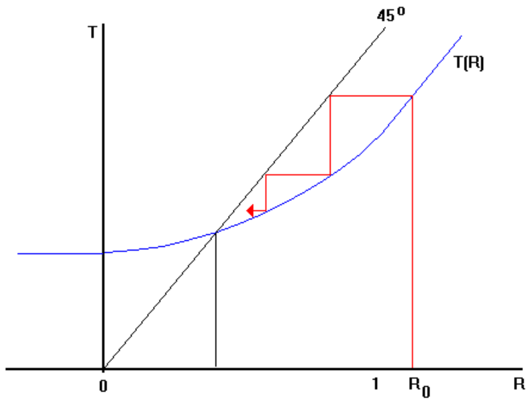

Figure 2.2: The Mapping T and its Fixed Point

Condition (2.7) is sometimes referred to as the *fundamental reservation wage equation*. It equates the utility per period from accepting an offer of exactly R to the utility from rejecting such an offer, which includes c plus the discounted expected improvement in next period's offer.

To simplify the notation in what follows, it will be convenient to define the *surplus function*,

$$\varphi(R) = \int_R^\infty (w - R)dF(w) = \int_R^\infty [1 - F(w)]dw, \tag{2.8}$$

where the second equality follows from integration by parts.[5] Then (2.7) can be written

$$R = c + \frac{\beta}{1-\beta}\varphi(R). \tag{2.9}$$

[5] For any $\bar{w}$, we have

$$\int_R^{\bar{w}} (w - R)dF(w) = (\bar{w} - R)F(\bar{w}) - \int_R^{\bar{w}} F(w)dw = \int_R^{\bar{w}} [F(\bar{w}) - F(w)]dw.$$

Letting $\bar{w} \to \infty$, we have the required result. For future reference, we catalogue the following properties of the surplus function: $\varphi(0) = Ew; \varphi(\infty) = 0$; $\varphi'(R) = -[1 - F(R)] \leq 0$, where the inequality is strict as long as $F(R) < 1$; and $\varphi''(R) = F'(R) \geq 0$, assuming the density F' exists, where the inequality is strict as long as $F'(R) > 0$.

This (or some elaboration) is the form in which the reservation wage equation will often appear in below.[6]

Summarizing the analysis so far, we have shown that the optimal policy involves a stationary reservation strategy: accept any offer above R, where R is given by (2.9). The expected utility from using this strategy for an unemployed worker is $U = R/(1-\beta)$. However, we have so far neglected the possibility that U may be less than the utility generated by dropping out of the labor force (i.e., not actively searching for work). For example, if $c = b - k$, where b denotes unemployment benefits that are available whether or not the agent is actively searching and k denotes costs that are payable only if the agent is actively searching, then the utility of dropping out of the labor force is $b/(1-\beta)$. If $b > R$ the individual will drop out and not search at all.

The probability of an unemployed agent accepting a job at date t is called the *hazard rate*, and denoted H_t. In this (stationary, infinite-horizon) model, $H_t = H = 1 - F(R)$ does not depend on t. The probability of being unemployed for exactly d periods equals the probability of rejecting $d-1$ offers and then accepting, which is $(1-H)^{d-1}H$. If D is the random duration of unemployment then

$$ED = \sum_{d=1}^{\infty} d(1-H)^{d-1}H = \frac{1}{H},$$

by virtue of a well-known formula (see the exercises).

Notice that an increase in c raises ED because it raises R and lowers H. However, it is obvious that an increase in ED does not necessarily mean the individual is worse off. In fact, since the expected utility of an unemployed

[6] An alternative method to get the same result is to first use integration by parts to write (2.6) as

$$R = c + \beta \int_R^{\infty} [V(w) - U] dF(w) = c + \beta \int_R^{\infty} V'(w)[1 - F(w)]dw,$$

and then insert $V'(w) = 1/(1-\beta)$. This method is useful because it is sometimes easier to solve for $V'(w)$ than for $V(w) - U$.

agent is $U = R/(1-\beta)$, any rise in R actually makes him better off. One of the first things that search theory tells us is that individuals, at least to some degree, *choose* whether to be unemployed, and choosing to remain unemployed longer does not mean you are worse off.

Exercises:

1. Derive $\partial R/\partial c$ and $\partial R/\partial \beta$ and interpret the results.

2. Redo the analysis under the assumption that instantaneous utility is given by $u(y)$, where $u' > 0$ and $u'' < 0$. In particular, generalize the mapping defined in (2.5) and show that it is a contraction.

3. Suppose that $w = (1, 2)$ each with probability $\frac{1}{2}$. Set $c = 0$, and use the relation $R = T(R)$ to compute R as a function of β. Answer:

$$R = \begin{cases} \frac{3\beta}{2} & \text{for } \beta \in (0, \frac{2}{3}) \\ \frac{2\beta}{2-\beta} & \text{for } \beta \in (\frac{2}{3}, 1) \end{cases}$$

4. Suppose that $w = (0, 1, 2, 3)$ each with probability $\frac{1}{4}$. Set $c = \frac{5}{4}$ and $\beta = \frac{3}{4}$, and construct the first few terms of the series $R_{n+1} = T(R_n)$ from the initial value $R_0 = c$. Verify that $R_n \to 2$.

5. Suppose that the probability of an offer arriving each period is $\alpha < 1$, so that with probability $1 - \alpha > 0$ an unemployed worker has no choice but to sit out another period. Derive the reservation wage equation. Discuss the effect of α on R and the hazard $H = \alpha[1 - F(R)]$.

6. Relax the assumption that workers cannot quit, but assume that, if they do, they have to sit out one period before a new offer arrives. Verify that the quit option is never exercised. How do things change if we assume a quitter can sample a new offer immediately? Hint: In the latter case there will be *two* reservation wages: one for temporary jobs, R_T, and one for permanent jobs, R_P.

7. Suppose that rejected offers can be recalled. Show that the optimizing strategy is to use the same reservation wage that is used without recall. Hint: Let W denote the maximum offer received until now, and write Bellman's equation as

$$J(W) = \max\left\{\frac{W}{1-\beta}, c + \beta E \max[J(W), J(w)]\right\}.$$

Then note the value function for the case of no recall satisfies this equation, and that the solution to Bellman's equation is unique.

8. Consider a model of a firm searching for a single worker to produce one unit of output per period (forever) that sells for price p. The firm gets to sample one worker per period who demands wage w, where w is an i.i.d. draw from $G(w)$. Show that the profit maximizing strategy is to hire a worker if $w < R$, where R satisfies

$$R = p - \frac{\beta}{1-\beta}\int_0^R (R-w)dG(w).$$

9. Redo the previous exercise assuming that worker demands are not i.i.d., but instead that $w_{t+1} = \gamma w_t + \varepsilon_t$, where $0 < \gamma < 1$, while ε_t is i.i.d. and independent of w_t . Show that the profit maximizing strategy is still to use a reservation hiring policy. Hint: Let $U(w) = \beta E_\varepsilon J(\gamma w + \varepsilon)$ be the value of rejecting w and let $V(w) = (p-w)/(1-\beta)$ be the value of accepting w. Show that U lies below V for small w and above V for large w, and that $J(w) = \max[V(w), U(w)]$ is decreasing and convex, which guarantees that U and V cross exactly once. One way to show $J(w)$ is convex is to show that it is the fixed point of a contraction mapping that takes convex functions into convex functions.

10. Derive the formula used to compute the expected duration of unemployment,

$$\sum_{d=1}^{\infty} d(1-H)^{d-1}H = \frac{1}{H}.$$

Hint: Differentiate with respect to H the identity

$$\sum_{d=1}^{\infty} \text{Prob}(D = d) = \sum_{d=1}^{\infty} H(1-H)^d = 1.$$

11. Suppose that employed and unemployed workers both get new offers next period, conditional on their current offer, z, from the distributions $F_1(w \mid z)$ and $F_0(w \mid z)$, respectively. Assume that $F_j(w \mid z)$ is stochastically increasing in z; that is, $E_j[\psi(w) \mid z] = \int \psi(w) dF_j(w \mid z)$ is increasing in z for any increasing function ψ. Then show that a reservation wage strategy is optimal as long as $F_0(w \mid z) - F_1(w \mid z)$ is increasing in w. Hint: First argue that $J(w) = \max[V(w), U(w)]$ is increasing in w. Then observe that

$$V(z) - U(z) = z - b + \beta\psi(z)$$

where

$$\psi(z) = \int_0^{\infty} J'(w)[F_0(w \mid z) - F_1(w \mid z)]dw.$$

Note that $\psi(z)$ increasing is sufficient (but not necessary) to guarantee that $V(z) - U(z)$ is increasing.

2.3 Continuous Time Job Search

In this section we develop a continuous time search model. There are applications in which continuous time has advantages (such as in some nonstationary models). However, at least at first, some people find the discrete time analysis more transparent. Therefore we begin by developing the continuous time model as the limit of discrete models where the length of time between periods vanishes. We then derive the Bellman's equations directly using simple probability theory. We also show in this section how to endogenize the offer arrival rate by making search intensity a choice variable.

We begin by extending the discrete time model to allow a random number n of offers to arrive each period. Let $\text{Prob}(n) = a(n)$, for $n = 0, 1, \ldots$, and, given $n > 0$, let $G(w, n)$ denote the cumulative distribution function of the best offer from the n received that period, $W = \max(w_1, \dots, w_n)$. Clearly, all that matters is W. Proceeding as in the previous section, the generalization of (2.6) is

$$(1-\beta)U = c + \beta \sum_{n=1}^{\infty} a(n) \int_R^{\infty} [V(W) - U] dG(W, n). \tag{2.10}$$

This holds for any (stationary) specification for $a(n)$. For our move to continuous time, we now put some structure on this object.

A very natural and useful assumption is that n is generated by a stationary *Poisson process*. By this we mean the following: Let the probability of n offers arriving in *any* time interval of length Δ be written $a(n, \Delta)$. Then we assume:

1. $a(1, \Delta) = \alpha\Delta + o(\Delta)$ for some $\alpha > 0$;
2. $\sum_{n=2}^{\infty} a(n, \Delta) = o(\Delta)$;

where $o(\Delta)$ is the standard notation for any function with the property that $\frac{o(\Delta)}{\Delta} \to 0$ as $\Delta \to 0$. Hence, the probability of an arrival is approximately proportional to the length of interval, with the approximation becoming arbitrarily good as $\Delta \to 0$.[7]

One can show that the properties of a Poisson process are satisfied if and only if $a(n, \Delta)$ is given by the Poisson density function with parameter $\alpha\Delta$:

$$a(n, \Delta) = \frac{(\alpha\Delta)^n e^{-\alpha\Delta}}{n!}.$$

The "if" part is obvious; the "only if" part requires solving a simple differential equation, and can be found in any textbook on stochastic processes.

[7] It is sometimes added to the definition of a Poisson process that it is *memoryless* – that is, the numbers of arrivals in any two nonoverlapping intervals are independent. This is already implied by our condition that $a(n, \Delta)$ is stationary.

If we let the random time until the next arrival be denoted τ, then τ has a distribution function given by

$$\Phi(t) = \text{Prob}(\tau \leq t) = 1 - \text{Prob}(\tau > t) = 1 - a(0,t).$$

Hence, the time until the next arrival is an exponential random variable with distribution function $\Phi(t) = 1 - e^{-\alpha t}$ and density $\Phi'(t) = \alpha e^{-\alpha t}$.

The important aspect of the Poisson process for our purposes is that the probability distribution of the time until the next arrival is constant $-$ that is, independent of history. In particular, the mean time until the next arrival is given by

$$E\tau = \int_0^\infty t\alpha e^{-\alpha t} dt = \frac{1}{\alpha}.$$

The parameter α is referred to as the *arrival rate*. We will use the properties of Poisson processes extensively in what follows.[8]

Suppose the length of each period in the discrete time model is given by Δ, and assume offers arrive according to a Poisson process with parameter α. Also, write the payoff to having an income of y for a period as $y\Delta$ and the discount factor as $\beta = e^{-r\Delta}$. Inserting these into (2.10), dividing both sides by Δ, and taking the limit as $\Delta \to 0$, we arrive at

$$rU = c + \alpha \int_R^\infty [V(w) - U] dF(w). \tag{2.11}$$

The value of accepting w is

$$V(w) = \int_0^\infty e^{-rt} w dt = w/r, \tag{2.12}$$

which implies $w = rV(w)$. In particular, $R = rV(R) = rU$. Hence, (2.11) implies

$$R = c + \frac{\alpha}{r} \int_R^\infty (w - R) dF(w) = c + \frac{\alpha}{r} \varphi(R), \tag{2.13}$$

[8] There are a variety of interpretations to the Poisson process generating arrivals in the model. For instance, offers may come in over the phone at a constant rate. Or the worker could randomly sample employers at various locations in some area, only some of whom have vacancies. Or the worker may know exactly where the vacancies are located, but it takes a random amount of time τ to get to a given location where τ is distributed exponentially.

which is the continuous time reservation wage equation.[9]

One can also derive (2.13) directly (without taking limits). Merely to simplify the notation, assume that $c = 0$. Then at any date, which we normalize to $t = 0$, the current value of being unemployed conditional on the next offer arriving at τ is $e^{-r\tau}EJ$. Since the time τ of the next offer has probability density $\alpha e^{-\alpha\tau}$, the expected value of being unemployed not knowing when the next offer will arrive is

$$U = \int_0^\infty \alpha e^{-\alpha\tau} e^{-r\tau} EJ d\tau = \int_0^\infty \alpha e^{-(\alpha+r)\tau} EJ d\tau.$$

Integration implies $U = \alpha EJ/(\alpha + r)$, which simplifies to (2.11), from which (2.13) follows as above.

The instantaneous rate at which a searcher escapes from unemployment in continuous time is also called the hazard rate. Here the hazard is given by $H = \alpha[1 - F(R)]$; sometimes this is described by saying that there are two components involved in getting a job, *choice* and *chance*, the former represented by $1 - F(R)$ and the latter by α. The expected duration of unemployment is $ED = 1/H$ (this follows as soon as we observe that acceptable offers according to a Poisson process with parameter H). An increase in c or a decrease in r raises R and lowers H. An increase in α raises R but has an ambiguous effect on H, since it affects H directly and indirectly through R (more on this below).

[9] A slightly different derivation, which we include because it is often seen in the literature, proceeds as follows. Let $\beta = 1/(1 + r\Delta)$, and write

$$U = \frac{1}{1 + r\Delta} \left\{ c\Delta + \alpha\Delta E \max[V(w), U] + (1 - \alpha\Delta)U + o(\Delta) \right\}.$$

This says that U is the discounted value of $c\Delta$, plus the value of receiving one offer with probability $\alpha\Delta$ and no offers with probability $1 - \alpha\Delta$, subject to approximation error as captured by $o(\Delta)$. Simplification yields

$$r\Delta U = c\Delta + \alpha\Delta E \max[V(w) - U, 0] + o(\Delta).$$

Dividing by Δ and letting $\Delta \to 0$, we get (2.11), from which (2.13) follows.

One simplification that we now relax is the assumption that the arrival rate depends in no way on the behavior of the agent. Making the arrival rate endogenous is important because it indicates that even if individuals were to always accept the first offer they receive there is still a sense in which they are making economic decisions concerning how long to search. In other words, to at least some extent, one controls one's chances as well as one's choices in the job market

We capture this by allowing individuals to choose their search intensity, and letting the arrival rate be $\alpha = \alpha(s)$ where s denotes the amount of resources devoted to search. Assume $\alpha' > 0$ and $\alpha'' < 0$. Also, let net unemployment income be $c = b - sk$, where s is search intensity and k is the per intensity unit search cost. Linearity of total search cost in s is merely a normalization $-$ we could alternatively write total search cost as $k(s)$ and normalize the arrival rate to be proportional to s.

Since the arrival rate is endogenous, the flow value of unemployed search is now

$$rU = \max_s \left\{ b - sk + \alpha(s) E \max[V(w) - U, 0] \right\}.$$

Let s^* be optimal intensity. Since the above arguments hold for any values of s and α, there is a reservation wage that is fully characterized by

$$R = b - s^* k + \frac{\alpha\left(s^*\right)}{r} \varphi(R). \tag{2.14}$$

Moreover, maximizing the value of unemployed search in (2.14) implies

$$\alpha'(s^*)\varphi(R) \leq rk, \quad = \text{if } s^* > 0. \tag{2.15}$$

Hence, the optimal search strategy is fully characterized by two conditions: the reservation wage equation (2.14) and the first order condition for intensity (2.15).

Exercises:

1. Derive $\partial R/\partial c$, $\partial R/\partial \alpha$ and $\partial R/\partial r$, and interpret your results.

2. Redo the analysis under the assumption that instantaneous utility is given by $u(y)$, where $u' > 0$ and $u'' < 0$.

3. Consider the *Pareto distribution*: $F(w) = 1 - (w_0/w)^\gamma$ for $w > w_0$, where $w_0 > 0$ and $\gamma > 1$ (the distribution can actually be defined for any $\gamma > 0$, but Ew does not exist unless $\gamma > 1$). Derive the reservation wage equation. Answer:

$$R = c + \alpha w_0^\gamma R^{1-\gamma}/r(\gamma - 1).$$

4. Suppose that an unemployed worker follows the strategy of accepting any wage above some arbitrary cut off z (where z is not necessarily the utility maximizing reservation wage R described above). Derive his expected value of search using this strategy as a function of z, and use calculus techniques to show the maximizing value of z is indeed the unique solution to the reservation wage equation, $z^* = R$. Does the same approach work in the discrete time model?

5. In the model with endogenous intensity, how do s and R depend on the parameters r, b and k? Derive the hazard rate. How does it depend on these parameters?

6. Suppose hours worked, h, can be chosen by the worker, and utility is $u(y, h)$ with $u_1 > 0$ and $u_2 < 0$. Further, $y = wh$ if employed and $y = c$ if unemployed. Characterize the optimal strategy.

7. Suppose that offers arrive in continuous time according to a more general process, in which the next offer w is not necessarily independent of the time since the last offer was received. Argue that we cannot expect a reservation wage strategy to be optimal in general. (Zukerman 1988 shows that a reservation wage strategy will be optimal under certain reasonable assumptions).

2.4 Turnover: Layoffs and Quits

The models analyzed up to this point have the property that a worker, once he accepts, stays at a job forever. In this section we discuss some reasons why this might not happen. One thing we do is to allow jobs to be terminated at some exogenous rate, and then ask how this affects the reservation wage. We also endogenize quits by allowing workers to search while unemployed. This extension of the basic model is important since it is a fact that many job-to-job transitions occur with no intervening period of unemployment. We also consider models with quits to unemployment due to changes in the job while employed, and due to learning about the job while employed.[10]

We begin with exogenous layoffs. In particular, suppose that in the context of discrete time there is an exogenous probability λ of an employed worker being permanently laid off each period. Let $V(w)$ be the value of accepting w, and $U = c + \beta E \max[U, V(w)]$ the value of rejecting in favor of continued search. Assume for the time being that an agent who is laid off must sit out one period before being able to sample a new offer, so that a layoff leaves him in exactly the same position as when he rejects an offer. Also, assume that a worker may quit, but then he must also sit out one period before sampling a new offer.

By virtue of stationarity, we know that an offer acceptable in one period is also acceptable in the next period, and so there will actually be no quits under these assumptions (but see the exercises). Hence, the value of any acceptable job satisfies

$$V(w) = w + \beta[\lambda U + (1 - \lambda)V(w)],$$

since it pays w this period, and next period a layoff occurs with probability λ. Notice that

$$V'(w) = \frac{1}{1 - (1 - \lambda)\beta} > 0.$$

[10] Another model with quits due to changes in market labor conditions (that is, changes in the offer distribution) over time will be presented in the section on dynamics.

Since V is increasing and U is constant in w, there is a unique R, given by $V(R) = U$, such that w is acceptable if it exceeds R.

As always, we can write

$$\begin{aligned}(1-\beta)U &= c + \beta \int_R^\infty [V(w) - U]dF(w) \\ &= c + \beta \int_R^\infty V'(w)[1 - F(w)]dw\end{aligned}$$

after integration by parts. Inserting $R = (1-\beta)U$ and $V'(w)$, we arrive at the reservation wage equation

$$R = c + \frac{\beta}{1 - (1-\lambda)\beta}\varphi(R), \tag{2.16}$$

which generalizes (2.9). Notice that $\partial R/\partial\lambda < 0$, which means that when jobs become less secure workers become less demanding and not so willing to hold out for a high-wage job.

Layoffs can also be incorporated into the continuous time model by assuming that exogenous separations arrive according to a Poisson process with parameter λ. Because it is useful for future applications, we also incorporate the possibility that the individual may "die" (or otherwise permanently exit the market) according to an independent Poisson process with parameter δ, after which he gets 0 utility.

Generalizing methods in the previous section, for discrete periods of length Δ we can write

$$\begin{aligned}U &= \frac{1-\delta\Delta}{1+r\Delta}\left[c\Delta + \alpha\Delta EJ + (1-\alpha\Delta)U + o(\Delta)\right] \\ V(w) &= \frac{1-\delta\Delta}{1+r\Delta}\left[w\Delta + \lambda\Delta U + (1-\lambda\Delta)V(w) + o(\Delta)\right],\end{aligned}$$

where the right hand sides are multiplied by $1 - \delta\Delta$ because this is the probability that the individual will live to see the next period, subject to approximation error as captured by $o(\Delta)$. Rearranging and letting $\Delta \to 0$, we can rewrite these as

$$(r+\delta)U = c+\alpha\int_R^\infty [V(w)-U]dF(w) \quad (2.17)$$

$$(r+\delta)V(w) = w+\lambda[U-V(w)], \quad (2.18)$$

which should be compared to (2.11) and (2.12).

Notice that

$$V'(w) = \frac{1}{r+\delta+\lambda} > 0.$$

Hence, there is a reservation wage that satisfies $V(R) = U$. Inserting $w = R$ into (2.18), we see that $(r+\delta)V(R) = R$. Combining this with (2.17) and integrating by parts, we get

$$R = c+\alpha\int_R^\infty V'(w)[1-F(w)]dw.$$

Inserting $V'(w)$, we arrive at

$$R = c+\frac{\alpha}{r+\delta+\lambda}\varphi(R) \quad (2.19)$$

which generalizes (2.13). Notice that both the layoff rate λ and the death rate δ affect the results only by changing the effective discount rate from r to $r+\delta+\lambda$.

The models described above generate turnover by layoffs and by deaths, neither of which occurs voluntarily. We now proceed to discuss the model with on-the-job search. This extension to the basic theory was first used by Burdett (19xx) as a way to generate job-to-job transitions, and explain how tenure at a particular job affects certain variables such as the average wage and quit rate. The continuous time presentation here follows Mortensen and Neuman (19xx). To reduce notation, we ignore the death rate δ for the rest of this section (by interpreting it as part of the subject rate of time preference r).

Suppose that offers arrive according to a Poisson process with rate α_0 while unemployed, and according to a Poisson process with rate α_1 while employed. In either case, an offer is a random draw from the same wage distribution $F(w)$. For simplicity, assume F is has bounded support, and normalize the maximum wage to 1. While employed, layoffs occur at rate λ. There is no cost to search, for now, which implies that individuals search at a fixed intensity regardless of their employment status or wage. We will shortly consider the endogenous choice of intensity when search is costly, but it is useful to first illustrate the technique in the case where α_0 and α_1 are fixed exogenously.

The flow value of unemployment satisfies

$$rU = c + \alpha_0 \int_R^1 [V(w) - U] dF(w). \tag{2.20}$$

The flow value of becoming employed at wage w now satisfies

$$rV(w) = w + \alpha_1 \int_w^1 [V(z) - V(w)] dF(z) + \lambda[U - V(w)] \tag{2.21}$$

since new offers arrive at rate α_1 and an employed worker accepts any new offer exceeding his current wage (with no cost to changing jobs). Notice that

$$V'(w) = \frac{1}{r + \lambda + \alpha_1[1 - F(w)]} > 0.$$

Evaluating (2.21) at $w = R$ and combing it with (2.20), we have

$$R = c + (\alpha_0 - \alpha_1) \int_R^1 [V(z) - V(R)] dF(z).$$

From this expression it is obvious that $\alpha_0 = \alpha_1$ implies $R = c$, $\alpha_0 > \alpha_1$ implies $R > c$, and $\alpha_0 < \alpha_1$ implies $R < c$. This says that when the arrival rates are the same the individual will accept any offer above c, when offers arrive more frequently while unemployed the individual will hold out for an offer strictly greater than c, and when offers arrive more frequently while employed the individual will work for less than c. In the last case, they are willing to sacrifice current income to increase future prospects, as might be realistic for at least some occupations (such as those where internships are common).

The usual technique allows us to derive the reservation wage equation: integrate by parts and then insert $V'(w)$ to get

$$R = c + (\alpha_0 - \alpha_1) \int_R^1 \left[\frac{1 - F(z)}{r + \lambda + \alpha_1 [1 - F(z)]} \right] dz.$$

This is the natural extension of simpler models, although we cannot decompose the integral into a constant times the surplus function in this case. Obviously when $\alpha_1 = 0$ the expression reduces to the special case discussed above with no on-the-job search.

We now reintroduce endogenous intensity. Let the arrival rates be $\alpha_0 = \alpha(s_0)$ and $\alpha_1 = \alpha(s_1)$, where s_0 and s_1 are the resources devoted to search while unemployed and employed, respectively, where $\alpha' > 0$ and $\alpha'' < 0$. Generally, we would expect search intensity to depend on the wage at which a worker is employed, so that s_1 is a function of w. Also, let the cost of search while unemployed be $k_0 s_0$, and the cost of search while employed be $k_1 s_1$. Then

$$\begin{aligned} rU &= \max_{s_0} \{ b - k_0 s_0 + \alpha(s_0) \varphi(R) \} \\ rV(w) &= \max_{s_1} \{ w - k_1 s_1 + \alpha(s_1) \varphi(w) + \lambda [U - V(w)] \} \end{aligned}$$

which are the same as (2.20) and (2.21), except that intensity is endogenous and we have replaced the integrals with surplus functions.

The solutions to the maximization problems in these two equations are characterized by

$$\begin{aligned} \alpha'(s_0)\varphi(R) &\leq k_0, \quad = \text{if } s_0 > 0 \\ \alpha'(s_1)\varphi(w) &\leq k_1, \quad = \text{if } s_1 > 0. \end{aligned}$$

These first order conditions imply that if $k_0 < k_1$ then $s_0 > s_1$, and more search effort is expended by the unemployed than by any employed worker. Furthermore, as long as s_1 is positive, it is decreasing in w. Hence, less effort is expended in search while working at a higher wage (indeed, there can be a $\bar{w}$ such $s_1 = 0$ for all $w \geq \bar{w}$, and once a sufficiently good job is achieved search activity ceases). This says that workers with higher wages are less likely to quit, and therefore more likely to be in their current job for a longer time. The model thus implies wages are positively related and quit rates negatively related to tenure on the job.

There are events other than layoffs or new offers that can occur during an employment spell. For instance, suppose that according to a Poisson process with parameter γ the wage on the job changes from w to a new draw from the conditional distribution $F(z \mid w)$. For simplicity, consider a case with no on-the-job search and no layoffs. Then

$$rV(w) = w + \gamma \int_0^1 \max[V(z) - V(w), U - V(w)]dF(z \mid w),$$

which incorporates the fact the when the wage changes the worker has to decide whether to stay at the job or quit to unemployment. Notice that a wage change is different from a new offer because a new offer can be rejected in favor of the current wage.

Assume that $F(z \mid w_2)$ first order stochastically dominates $F(z \mid w_1)$ whenever $w_2 > w_1$; this simply says that a high wage is not a signal of future low wages. Then $V(w)$ is increasing and hence there is a wage R (the same for employed and unemployed agents) such that $w > R$ implies w is acceptable. Stochastic dominance also implies that the quit rate is decreasing in w.

We derive the reservation wage equation for this model only in the special case where $F(w \mid z) = F(w)$; that is, the new wage is independent of the old wage. Then the usual techniques imply

$$R = c + \frac{\alpha - \gamma}{r + \gamma}\varphi(R)$$

Notice that $R > c$ as long as the arrival rate while unemployed exceeds the rate at which the wage changes while employed, $\alpha > \gamma$. Otherwise, we would have a situation where agents are willing to accept wages below c because wage changes arrive faster while employed than offers arrive while unemployed.

To close this section, we present one more reason for turnover: *learning.* Jovanovic (19xx) introduced such a model to rationalize some of the same observations that the on-the-job search model was designed to explain. He considers the case where workers have to learn about how good they are at any job based on productivity observations that are a function of both true productivity and random shocks. Models of this class have been put to use in labor economics, industrial organization, and finance, among other fields. Here, we present only a very simple version where all learning takes place in one period.

In particular, assume that an offer is a signal σ, where σ is drawn from the distribution function $G(\sigma)$, depending on both the true wage w and some noise denoted here by p. As an example, suppose $\sigma = pw$ is the nominal wage and p is the price level, assumed random and independent of the true real wage w. We generally assume that higher signals are "good news" in the sense that $F(w \mid \sigma_2)$ first order stochastically dominates $F(w \mid \sigma_1)$ whenever $\sigma_2 > \sigma_1$. We work with the discrete time model, so that we may say that the true value of w is revealed exactly one period after accepting and offer. However, agents cannot hold onto an offer until w is revealed without actually accepting the job and working one period.

The value of unemployed search is given by

$$U = c + \beta \max \int_0^\infty \{E[V(w) \mid \sigma], U\} dG(\sigma),$$

since the acceptance decision is based on the signal σ and not the true wage w. The value of employment at a known w is given by

$$V(w) = w + \beta \max[V(w), U],$$

since one period after accepting the true value of w is observed and the worker decides whether to stay or quit. If he quits the payoff is U, under the assumption that he must wait a period for the next offer. Since $V(w)$ is increasing, the stochastic dominance assumption implies that $E[V(w) \mid \sigma]$ is increasing in σ. Hence, there is a reservation signal R_σ such that offers should be accepted if $\sigma > R_\sigma$. Once w is revealed, the worker stays if $w > R_w$, where, as usual, $V(R_w) = U$.

Uncertainty has a real effect here because workers can be confused into accepting jobs they would prefer to reject, and vice-versa. The situation is not symmetric, however, as long as the employed can quit easier than the unemployed can recall a previously rejected offer. For example, in the case where σ is the nominal wage pw, higher values of p shift up the distribution of σ to the right. Hence, the probability of accepting a job is higher when the price level is higher, although the probability is also higher that the worker will subsequently quit once p and hence w become known. This example is pursued in Wright (1986) as an interpretation of the macroeconomic observations referred to as the "Phillips curve."

More generally, Jovanovic-style models have the implication that reservation signals will increase with tenure. This is because at the beginning of an employment spell, when there is a lot of uncertainty, there is the possibility of things getting better and so individuals are not so demanding. Of course, there is also the possibility of things getting worse, but since the worker can always quit, the situation is not symmetric. Therefore, the more that is known about a situation the more demanding individuals tend to be. At the same time, individuals with a long tenure at a job have already learned a lot and so they are less likely to quit. Furthermore, given that they are still there, they are more likely to be earning higher wages. Hence, this model also predicts quit rates fall and wages rise with tenure.

Exercises:

1. Analyze the discrete time model under the assumption that a laid off worker can sample a new offer immediately, although if he quits, he still must sit out one period.

2. As in Burdett and Mortensen (1980), consider a discrete time model where an offer is a *pair* (w, λ) drawn from the joint distribution $F(w, \lambda)$; that is, the layoff rate as well as the wage rate can differ across jobs.

Assume concave utility over instantaneous income, $u(y)$. Let $R = R(\lambda)$ be the reservation wage for a job with a given layoff rate λ. Show that as long as both a laid off worker and a quitter must sit out a period before a new offer arrives, $R'(\lambda) = 0$, and reconcile this with the result $\partial R/\partial \lambda < 0$ in the text. Does $R'(\lambda) = 0$ mean that a worker does not care about job security? Hint: Draw some indifference curves in (λ, w) space.

3. What happens in the previous question if laid off workers can sample a new offer immediately while quitters must sit out a period? How about other combinations? See Wright (1987) for details.

4. As in Hey and McKenna (19xx), assume that there is no cost to search and offers arrive at the exogenous rate α whether employed or not, but there is a fixed cost γ to moving from one job to another. Describe the reservation wage for moving for a worker currently employed at wage w, $R(w)$. Does the difference $R(w) - w$ increase or decrease with w? Interpret your results.

5. Suppose that the probability of a layoff is not exogenous, but depends on effort while on-the-job: $\lambda = \lambda(e)$, where $\lambda' < 0$. In particular,

$$V(w) = \max_e \{u(w, e) + \lambda(e)\beta U + [1 - \lambda(e)]V(w)\}$$

where $u_1 > 0$ and $u_2 < 0$. Derive the reservation wage equation and the first order condition for the choice of effort as a function of the wage. Find $e'(w)$ in the general case, and in the special case where $u_{12} = 0$.

6. For the discrete time model with layoffs, derive a mapping $T(R)$ with the property that the reservation wage satisfies $R = T(R)$, and $R_{n+1} = T(R_n)$ converges to R. Answer:

$$T(R) = \frac{1 - (1 - \lambda)\beta}{1 + \lambda\beta} c + \frac{\beta}{1 + \lambda\beta} E \max(w, R).$$

7. As in Fallick (1989), suppose individuals can search in N sectors simultaneously. Offers arrive while unemployed according to independent Poisson processes with arrival rates α_j, which may differ with j. An offer from sector j is a draw from $F_j(w)$. All jobs in sector j are all have layoff rate λ_j. Show there is a reservation wage R such that an offer (from any sector) is accepted if $w \geq R$. Now endogenize search intensity in each sector, s_j , by letting $\alpha_j = \alpha_j(s_j)$. How does s_j vary across sectors with different layoff rates or search costs? A different question is, how do s_j and R vary with a change in λ_j or k_j?

8. Consider taxing wages according to the schedule $T(w) = t_0 + t_1 w$. Derive the reservation wage equation and the effects of increases in t_0 and t_1. Now impose the same tax schedule on unemployment income c as well as wage income and recalculate these effects.

9. As in Wright and Loberg (1987), parameterize the UI tax as follows:

$$T(w) = \begin{cases} tw & \text{for } w < w_0 \\ tw_0 & \text{for } w \geq w_0 \end{cases}.$$

Derive the reservation wage equation. Answer: if $R > w_0$ then

$$R = tw_0 + c + \frac{\alpha}{\lambda + r}\varphi(R);$$

and if $R < w_0$ then

$$R = c + \frac{\alpha}{\lambda + r}\int_R^{w_0}(w - R)dF(w) + \frac{\alpha}{\lambda + r}\int_{w_0}^{\infty}\left(\frac{w - tw_0}{1 - t}\right)dF(w).$$

Derive the effect of changes in t and w_0 on the before- and after-tax reservation wages in each case. Derive the effects of an increase in w_0 combined with a reduction in t that leaves $t_0 = tw_0$ unchanged.

10. As in Hey and Mavromaras (1981), assume that unemployment insurance depends on the wage received before the last layoff, so that $c = c(\bar{w})$, with $c' > 0$, where $\bar{w}$ is the most recent wage. Derive the

reservation wage equation for $R = R(\bar{w})$ and show $R' > 0$. Consider the following realistic parameterization of UI: for $\bar{w}$ between w_0 and w_1, $c(\bar{w}) = \rho\bar{w}$, where ρ is the replacement ratio; for $\bar{w} < w_0$, $c = \rho w_0$; for $\bar{w} > w_1$, $c = \rho w_1$. What is the effect of a change in ρ?

2.5 Log-Concavity and Comparative Statics

In this section we consider the effects of some additional changes in the exogenous parameters of the model, including changes in the offer distribution $F(w)$. Certain effects turn out to be ambiguous, in general, and some intuitively plausible results are not necessarily true. However, we provide a simple condition on $F(w)$ that serves to rule out the counterintuitive cases. This condition will come into play repeatedly in the analysis to follow.

A function $f(w)$ is said to be *log-concave* if its natural logarithm, $\log f(w)$, is a concave function; that is, assuming differentiability, if

$$f''(w)f(w) - f'(w)^2 \leq 0. \tag{2.22}$$

A random variable is said to be log-concave if its density function is log-concave. Many well-know distributions, including the uniform, normal and exponential, satisfy this restriction.[11]

To show how log-concavity is used in search theory, we need to define the (left) truncated mean function,

$$\mu(R) = E[w \mid w \geq R] = \int_R^\infty \frac{wdF(w)}{1 - F(R)}.$$

[11] See Barlow and Proschan (1975) or Karlin (1982) for discussions. Although many well-know distributions are log-concave, examples that are not include the Pareto, log-normal and t distributions. Some interesting results about log-concave distributions include: all of their moments exist, they have non-decreasing hazard functions, and they are unimodal (thus, log-concavity is sometimes referred to as strong unimodality). A result we will use later is that log-concavity is preserved through integration (this follows from Prekopa's theorem; see Dharmadhikari and Joag-Dev 1987). Thus, if a density function $f(w)$ is log-concave then so is the distribution function $F(w)$ and the survivor function $1 - F(w)$. Furthermore, if $1 - F(w)$ is log-concave then so is the surplus function $\varphi(R)$.

Assuming differentiability, we have

$$\mu'(R) = \int_R^\infty \frac{(w-R)F'(R)dF(w)}{[1-F(R)]^2}. \tag{2.23}$$

Clearly $\mu' \geq 0$, and log-concavity implies $\mu' \leq 1$ (see Goldberger 1983). As in Burdett (1981), we can use the result that log-concavity implies $\mu' \leq 1$ to rule out some pathological results that may occur in an unrestricted search model.[12]

Consider the basic continuous time model with reservation wage equation given by (2.13). Then

$$\frac{\partial R}{\partial \alpha} = \frac{\varphi}{r + \alpha(1-F)} > 0,$$

where F and φ are evaluated at R when the argument is suppressed. However, if the hazard rate is $H = \alpha(1-F)$, then the sign of

$$\frac{\partial H}{\partial \alpha} = 1 - F - \alpha F' \frac{\partial R}{\partial \alpha} = 1 - F - \frac{\alpha F' \varphi}{r + \alpha(1-F)}$$

is ambiguous, because the reservation wage may increase by enough to cause a net decrease in H. Hence, we cannot conclude that the hazard rises with an increase in job availability.

However, if we use (2.23) the previous expression can be reduced to

$$\frac{\partial H}{\partial \alpha} = 1 - F - \frac{\alpha(1-F)^2 \mu'}{r + \alpha(1-F)} = (1-F)\left(1 - \frac{H\mu'}{r+H}\right).$$

If $\mu' \leq 1$ then $\partial H/\partial\alpha > 0$; but if $\mu' > 1$ then $\partial H/\partial\alpha$ can be negative (see the exercises). Since log-concavity implies $\mu' \leq 1$, it provides a simple restriction on F that guarantees an increase in the frequency of offers increases H and lowers the expected duration of unemployment.

[12]See also Burdett and Ondrich (1985), Flinn and Heckman (1983), Vroman (1985), Wright and Loberg (1987), Boldrin et al. (1993) and Burdett and Wright (1994) for applications of log-concavity in search models.

Burdett (1981) also investigated the impact of a translation in the offer distribution, which is a shift to the right in F that increases the mean but leaves all higher moments about the mean the same. It is not hard to show that this increases R but by less than the amount of the translation; thus, H unambiguously increases. What is surprising is that the average accepted wage, $\mu(R) = E[w \mid w \geq R]$, may actually fall when the wage distribution shifts to the right.[13] However, log-concavity rules this out.

We present the results not in terms of a translation of F, but (equivalently) in terms of changes in a lump sum wage subsidy T. Thus, the net wage is given by $w_n = w + T$. We can define both a before- and after-subsidy (net) reservation wage, R and $R_n = R + T$, where the usual methods imply

$$R - T = c + \frac{\alpha}{r}\varphi(R)$$

Differentiation yields $\partial R/\partial T = -r/(r+H)$ and $\partial R_n/\partial T = H/(r+H)$, and so the reservation wage goes down with an increase in T, but by less than the change in T, so that the net reservation wage goes up. Clearly, the hazard H is increasing in T.

Notice that $\mu = E[w \mid w \geq R]$ is decreasing in T, since R is. The more interesting question is what happens to $\mu_n = E[w_n \mid w \geq R] = \mu + T$, the expected net wage at which the agent becomes employed. The answer is

$$\frac{\partial \mu_n}{\partial T} = 1 + \mu' \frac{\partial R}{\partial T} = 1 - \frac{r\mu'}{r+H}.$$

Hence $\mu' \leq 1$ implies $\partial \mu_n/\partial T > 0$, while if $\mu' > 1$ then $\partial \mu_n/\partial T$ can be negative. Log-concavity therefore guarantees that an increase in T raises the net expected wage. In other words, log-concavity guarantees that a lump sum subsidy cannot reduce the reservation wage by enough to imply that workers end up working for a lower net wage on average.

To close this section, we consider the effect of an increase in the riskiness of $F(w)$. As in Rothschild and Stiglitz (1970), the relevant notion of risk is

[13] To see how this might happen, hold the truncation point R fixed and imagine a density with a large part of its mass just to the left of R; a translation that shifts this mass just past R can lower $\mu(R)$.

that of a mean preserving spread: given two distribution functions F and G with support normalized to lie in $[0,1]$, we say F is *more risky* than G if

$$\int_0^{\bar{w}} [F(w) - G(w)]dw \geq 0 \tag{2.24}$$

for all $\bar{w} \in [0,1]$, with strict equality if $\bar{w} = 1$. Condition (2.24) says that F has more weight in the tails than G. The requirement that the integral is 0 at $\bar{w} = 1$ guarantees equal means (as can be seen using integration by parts).[14]

There is a version of the above condition that is useful for comparative statics. Let $\{F(x,\sigma)\}$ be a family of distribution functions, with support in $[0,1]$ and equal means, indexed by their degree of riskiness; that is, $\sigma_2 > \sigma_1$ implies

$$\int_0^{\bar{w}} [F(w,\sigma_2) - F(w,\sigma_1)]dw \geq 0.$$

Divide this expression by $\sigma_2 - \sigma_1 > 0$ and take the limit as $\sigma_2 \to \sigma_2$ to arrive at

$$\int_0^{\bar{w}} F_2(w,\sigma)dw \geq 0. \tag{2.25}$$

where $F_2 = \partial F/\partial\sigma$. Notice in an additional that $F_2(0,\sigma) = F_2(1,\sigma) = 0$ for all σ, because F has support in $[0,1]$. Given (2.25), it is a simple matter to show that an increase in risk increases the reservation wage and expected return to search, as you are asked to do in the exercises.

Exercises:

1. Consider the Pareto distribution $F(w) = 1 - (w_0/w)^\gamma$ for $w > w_0$, where $w_0 > 0$ and $\gamma > 1$. Without loss in generality, set $w_0 = 1$. Show that this distribution is not log-concave but log-convex. Construct an

[14] In order words, F is a means preserving spread of G if and only if they have equal means and G second order stochastically dominates F. Hence, F is more risky than G if and only if $\int u(w)dF(w) \leq \int u(w)dG(w)$ for all concave functions u. An alternative notion of risk, of course, is variance; but as is well-known, it is not true that all risk averse individuals prefer lower variance.

example where $\partial H/\partial\alpha < 0$. Hint: The reservation wage equation is $R = c + \alpha R^{1-\gamma}/r(\gamma-1)$. If $\gamma = 2$, this can be solved for:

$$\begin{aligned} R &= \frac{1}{2}\left[c + \sqrt{c^2 + 4\alpha/r}\right] \\ H &= 4\alpha\left[c + \sqrt{c^2 + 4\alpha/r}\right]. \end{aligned}$$

2. Suppose the instantaneous utility function $u(y)$ is concave instead of linear. Derive the reservation wage equation

$$u(R+T) = u(c) + \frac{\alpha}{r}\int_R^{\infty}[u(w+T) - u(R+T)]dF(w),$$

where T is a lump sum subsidy, and show that log-concavity still implies $\partial H/\partial\alpha > 0$ and $\partial\mu_n/\partial T > 0$.

3. Consider an employer searching for a worker to produce one unit of output per period that sells at price p. Let w be the random wage demanded by a potential worker and assume workers arrive according to a Poisson process with parameter α. Derive the employer's reservation wage equation

$$R = p + \frac{\alpha}{r}\int_0^R (w - R)dF(w).$$

Characterize the dependence of R, $H = \alpha F(R)$, and $E[w|w \le R]$ on α and T, where T is a lump sum tax (or a translation parameter). Hint: The right truncated mean function is given by

$$\rho(R) = E[w \mid w \le R] = \int_{-\infty}^{R}\frac{wdF(w)}{F(R)}$$

and log-concavity implies $\rho' \le 1$.

4. The function $h(w) = F'(w)/[1 - F(w)]$ is called the *hazard function* for a random variable with distribution $F(w)$. Prove that $h(w)$ increasing is sufficient for $\mu' \le 1$, where $\mu(R) = E[w \mid w \ge R]$, and that it is necessary and sufficient when $h(w)$ is monotone; see Mortensen (1984).

5. Analyze a proportional wage subsidy (or, equivalently, a scale transformation of the offer distribution) by writing the net wage as $w_n = (1 + t)w$. Prove that $\partial R/\partial t$ depends on the sign of c, but that $\partial R_n/\partial t$ is unambiguously positive. Show that log-concavity guarantees $\partial \mu_n/\partial t > 0$, where $\mu_n = E[w_n \mid w \geq R]$.

6. Prove that a translation of the offer distribution, $w_n = w + T$, shifts the mean but leaves all moments about the mean unchanged. What does a scale transformation do?

7. Show that an increase in risk increases the reservation wage and expected return to search and interpret these results.

8. As in Lippman and McCall (1976), consider two alternatives of the standard discrete time search model: first, each period exactly n independent offers arrive; and second, each period a random number m of independent offers arrive, where $Em = n$. Show that the reservation wage and expected return to searching are higher in the first case. Hint: The distribution of the best offer in the first case is $F_1(w) = F(w)^n$ and in the second case is $F_2(w) = \sum_n \text{Prob}(m = n)F(w)^n$. Use the fact that $F(w)^n$ is convex in n and Jensen's inequality to show that F_1 first order stochastically dominates F_2.

2.6 Dynamic Search Problems

In this section, we consider some aspects of search models that are not set in such a simple stationary environment. Three types of models are considered: those with a finite horizon; those were the wage distribution changes over time; and those where the arrival rate changes over time. Most of the interest in these issues arises in the context of equilibrium models to be described later, but it is worth introducing the techniques in the context of a single-agent decision problem.

We first consider a finite horizon N, after which the individual gets zero payoff. One interpretation is that the worker literally has a finite lifetime; the other is that he has some finite amount of wealth from which to finance search activity, after which he is forced to permanently withdraw from the market. For simplicity, assume $F(w)$ is nondegenerate on $[0, 1]$. Also, assume an offer w is a lump sum payment, and not a wage rate per period. This eliminates the effect of the fact that a wage accepted later in life can be earned for fewer periods and allows us to focus on the two other effects: the fact that accepting later involves waiting longer to get w and the fact that there are fewer periods left to sample offers as time passes.[15]

For $t = 0, 1, \dots N$, let $J_t(w)$ denote the value of an offer w and R_t the reservation wage at date t. Clearly, we have $J_N(w) = \max(w, c)$ and $R_N = c$, since there is no sense holding out for something better than c in the last period. We therefore write $J_N(w) = \max(w, R_N)$. For any $t < N$, we have

$$J_t(w) = \max(w, c + \beta E J_{t+1})$$

and $R_t = c + \beta E J_{t+1}$. Hence, we have

$$R_t = c + \beta E \max(w, R_{t+1}) = T(R_{t+1}),$$

where T is the precisely the mapping in (2.5).

The difference equation $R_t = T(R_{t+1})$ together with the terminal condition $R_N = c$ defines the sequence of reservation wages. Since T is increasing, we see that R_t exceeds R_{t+1} for all t, and the reservation wage gets bigger as we move further from the horizon N. Moreover, since T is a contraction, R_t converges to $R = T(R)$ as t grows. In other words, we conclude that as the horizon gets longer the reservation wage converges monotonically to the solution to the infinite horizon model, and this is the sense in which the infinite horizon problem can be thought of as an approximation to a long but finite problem.

[15] We also consider only the case of no recall; see Lippman and McCall (1976a) for an analysis of the case with recall.

The next issue is to consider job search when the distribution $F(w)$ is changing over time.[16] For simplicity, assume that there are two possible distributions, F_1 and F_2, where F_2 first order stochastically dominates F_1; that is, $F_2(w) \leq F_1(w)$ for all w. The distributions change over time according to a Markov process, with π_{ij} denoting the probability of F_j next period given F_i this period. We assume $\pi_{22} > \pi_{12}$, so that the probability of staying with the better distribution is greater than the probability of moving to it. As long as the distribution remains fixed, each offer is an i.i.d. draw from F_i.

There is no on-the-job search and no learning (everyone knows the current distribution). Still, quits may occur because an offer that was acceptable given F_1 may not be acceptable once we switch to F_2. In other words, when prospects outside the current job improve a worker may quit to unemployed search. The value function when the distribution is F_i is given by $J_i(w) = \max[V_i(w), U_i]$. Assuming that an offer at any date is drawn from the distribution at the previous date (that is, first a new offer arrives and then the distribution of future offers changes), we have

$$
\begin{aligned}
U_i &= c + \beta \sum_j \pi_{ij} E_i J_j \\
V_i(w) &= w + \beta \sum_j \pi_{ij} J_j(w).
\end{aligned}
$$

It is not hard to show that $J_i(w)$ is increasing while $V_i(w)$ is strictly increasing in w, and that $J_i(w)$, $V_i(w)$ and U_i are all nondecreasing in i (see the exercises). Hence, for each i there is a reservation wage R_i satisfying $V_i(R_i) = U_i$. Given F_i, an offer w is acceptable when $w \geq R_i$, and a quit occurs when employed at w but the distribution switches to F_j and the reservation wage switches to $R_j > w$. We wish to show that $R_2 \geq R_1$, so that agents are more demanding given F_2, and quits only occur when F_1 switches to F_2. Suppose by way of contradiction that $R_1 > R_2$. Then once R_1 is accepted quits never occur, and so $V_1(R_1) = V_2(R_1) > V_2(R_2)$ (where the

[16] This issue was first studied by Lippman and McCall (1976b), although the presentation here follows Lippman and Mamer (1989).

inequality follows from the fact that V_i is strictly increasing in w). Using $V_i(R_i) = U_i$ we have $U_1 > U_2$, contradicting the result than U_i is increasing in i.

We conclude that R_2 is greater than R_1. Hence, when the labor market improves, in the sense that the wage offer distribution stochastically increases, there will be some workers who find their current wages below their new reservation wages and therefore quit. This not only provides another way to get quits into the model, it does so in a way that is consistent with the observation that quits are procyclical. In this model, workers are more likely to quit during good times because this is when their opportunites are better.

To complete this section we consider an infinite horizon problem in which the offer arrival rate potentially varies over time. Let time proceed in discrete periods of length Δ and let α_t denote the arrival rate at t. To avoid the complication of layoffs or quits, assume that an offer is a lump sum payment w (or, equivalently, w is the present value of working forever and both quits and layoffs are simply ruled out).

At any date t, $V_t(w) = w$ and $J_t(w) = \max(w, R_t)$. The value of unemployed search at t satisfies

$$U_t = \frac{1}{1 + r\Delta} \left[c\Delta + \alpha_{t+\Delta} E J_{t+\Delta} + (1 - \alpha_{t+\Delta}\Delta) U_{t+\Delta} + o(\Delta) \right],$$

or, if we substitute $R_t = U_t$ and simplify,

$$rR_t = c + \alpha_{t+\Delta} E \max(w - R_{t+\Delta}, 0) + \frac{R_{t+\Delta} - R_t}{\Delta} + \frac{o(\Delta)}{\Delta}.$$

This is a difference equation in R_t. Dividing by Δ and taking the limit as $\Delta \to 0$, we arrive at a differential equation

$$\dot{R}_t = rR - c - \alpha_t E \max(w - R_t, 0),$$

where $\dot{R}$ denotes the time derivative.

The solution to a dynamic search problem is a reservation wage path that follows a difference or differential equation of the above form. Additionally, there is typically a transversality condition that must be imposed, since there are many solutions to such difference or differential equations. Note the similarity between the solution to the infinite and finite horizon problems in discrete time: both generate a R_t as a function of R_{t+1}. The role of the transversality condition in the infinite horizon problem is analogous to that of the terminal condition $R_N = c$ in the finite horizon problem. We will have more to say about these issues later in the context of equilibrium models.

Exercises

1. In a continuous time model with a finite horizon, show that $R_t \to c$ as the horizon gets shorter and that $R_t \to R$ (the reservation wage for the infinite horizon problem) as the horizon gets longer.

2. As in Mortensen (1977) and Burdett (1979), consider a model that is stationary except for the fact that UI benefits expire after a duration of unemployment equal to N. Show that the reservation wage R decreases until UI expires, and is constant thereafter.

3. In the model where the offer distribution shifts between F_1 and F_2, show that $J_i(w)$ is increasing while $V_i(w)$ is strictly increasing in w, and that $J_i(w)$, $V_i(w)$ and U_i are all nondecreasing in i. Hint: Define a mapping Φ from one function $\psi_i(w)$ of i and w into another by:

$$\Phi[\psi_i(w)] = \max\left\{c + \beta \sum_j \pi_{ij} E_i \psi_i, w + \beta \sum_j \pi_{ij} \psi_i(w)\right\}$$

   Show Φ is a contraction with fixed point $J_i(w)$, and that it takes nondecreasing functions of w into nondecreasing functions of w. This implies $J_i(w)$ is nondecreasing in w and hence V_i is strictly increasing in w. Similarly, show that Φ takes functions that are nondecreasing in i into functions that are nondecreasing in i, and hence $J_i(w)$, $V_i(w)$ and U_i are all nondecreasing in i.