1 The Neoclassical Growth Model

The neoclassical growth model, which originated with the work of Solow and Swan, consists of the following relationships: a production function, $y_t = f(h_t, k_t)$, where y_t is output, h_t is labor, and k_t is capital at date $t = 0, 1, 2 \ldots$, plus a law of motion for the capital stock, $k_{t+1} = (1-\delta)k_t + \sigma y_t$, where $\delta \in (0,1)$ is the depreciation rate and $\sigma \in (0,1)$ is the savings rate. We assume that f is homogeneous of degree 1, increasing, concave, and twice continuously differentiable. We assume for now that h_t is constant and normalize $h_t = 1$ (this assumption is not very accurate, but we will soon endogenize h_t by letting agents choose how much time to spend working and how much to spend in other activities). Let $F(k) = f(1,k)$. Then $F' > 0$, $F'' < 0$. We further assume that $F(0) = 0$, $F'(0) = \infty$ and $F'(\infty) = 0$. The initial capital stock, k_0, is given exogenously.

Note that behavior is exogenous here: the representative individual in this economy simply works a fixed number of hours $h_t = 1$, saves or invests $i_t = \sigma y_t$, and consumes $c_t = (1-\sigma)y_t$, each period. Substituting the production function into the law of motion for capital yields a first order difference equation in k_t,

$$k_{t+1} = (1-\delta)k_t + \sigma F(k_t) \equiv g(k_t). \tag{1}$$

Together with the initial condition k_0, (1) completely determines the entire time path of the capital stock. Given this path we can compute the paths for y_t, c_t, etc. A *steady state* of the system is a solution to $k = g(k)$.[1]

[1] One often sees the model in continuous time, in which case the law of motion for capital is written $\dot{k}_t = \sigma F(k_t) - \delta k_t$ (see below for an explicit derivation of the continuous time as the limit of discrete time models). Also, one often sees the model augmented to include population growth, with the number of agents at t given by $N_t = e^{nt}$. In this case, one needs to distinguish between total and per capital variables. Thus, let $Y_t = f(N_t, K_t) = e^{nt}F(K_t/N_t)$ be the production function; then in per capita terms we have $y_t = F(k_t)$. Also, let $\dot{K}_t = \sigma Y_t - \delta K_t$ be the law of motion for capital; then dividing

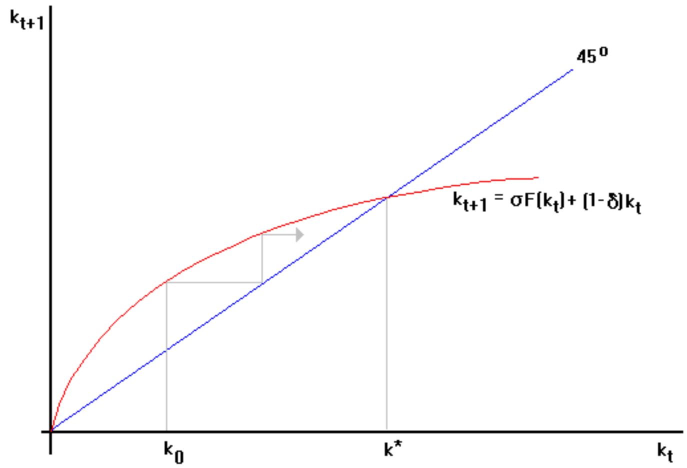

Figure 1: Neolclassical Growth Model

We graph $g(k)$ versus k in Figure 1. Our assumptions imply that, as shown, $g(0) = 0$, $g'(0) > 1$, and there is a unique $k^* > 0$ such that $k^* = g(k^*)$. Hence, the model has two steady states, $k = 0$ and $k = k^*$. Moreover, for all $k_0 > 0$, $k_t \to k^*$ (monotonically). Hence, as $t \to \infty$, $y_t \to y^*$, $c_t \to c^*$, etc. At k^*, we have $\sigma F(k^*) = \delta k^*$, which implies that savings just offsets depreciation and the capital-output ratio is $\frac{k}{y} = \frac{\sigma}{\delta}$, and also that $c^* = y^* - \delta k^*$. Clearly, k^* is increasing in $\frac{\sigma}{\delta}$. Moreover, c^* is first increasing and then decreasing in σ. The saving rate that maximizes steady state consumption can easily be shown to satisfy $F'(k^*) = \delta$; this is Phelps' so-called "golden rule" of capital accumulation.

Observe that the basic neoclassical growth model does not actually exhibit long run growth: for all $k_0 > 0$, as t increases, k_t converges to k^*.

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by N_t and using $\dot{K}/N = \dot{k} + nk$, we have $\dot{k} = \sigma y_t - (n + \delta)k_t$. Holding everything else constant, population growth obviously reduces the per capital capital stock. We will typically abstract from population growth in order to conserve on notation, but notice that one can reinterpret a model with population growth as one without but with a different δ.

Asymtoptically, growth ceases. This is due to the assumption that the marginal product of capital becomes small as the stock increases – that is, the assumption that $F'(k) \to 0$ as $k \to \infty$. Suppose we relax this assumption, say, by assuming a linear technology, $F(k) = Ak$ for some constant $A > 0$. Then the time path of k_t will satisfy

$$k_{t+1} = (1 - \delta)k_t + \sigma A k_t = B k_t,$$

where $B \equiv 1 - \delta + \sigma A$.

This implies that $k_t = B^t k_0$ for all t. If $B > 1$ then this economy grows forever at rate $B - 1 = \sigma A - \delta$, while if $B < 1$ then $k_t \to 0$. In any case, output, consumption and investment all grow at the same rate as k_t. We conclude that to get long run growth, in this setup, the savings rate σ and productivity A must be to be high relative to depreciation δ.

We could go beyond a linear technology and assume that $F(k)$ exhibits increasing returns – that is, $F'' > 0$. Suppose $g'(0) < 1$, which is true if and only if $F'(0) < \frac{1-\delta}{\sigma}$, as shown in Figure 2. Then there is a unique nondegenerate steady state k^* that is unstable: for $k_0 < k^*$, $k_t \to 0$, and for $k_0 > k^*$, k_t exhibits explosive growth. If $F(k)$ first exhibits increasing returns and then decreasing returns, then as shown in Figure 3 there are two nondegenerate steady states, one stable and one unstable. These examples show how a low initial capital stock can forever doom an economy to a low growth rate or a low level of output. Moreover, small changes in initial conditions can make a big difference for the asymtoptic behavior of the economy.

An alternative way to generate persistent growth is to return to a concave technology and add exogenous technical progress. For instance, suppose that $y_t = f(z_t h_t, k_t)$ where $z_t = \gamma^t$ represents deterministic technical change. Notice that technical change here is labor augmenting; this is necessary to get a *balanced growth path* along which capital and output grow at the same rate when the labor input is fixed for any production function other than

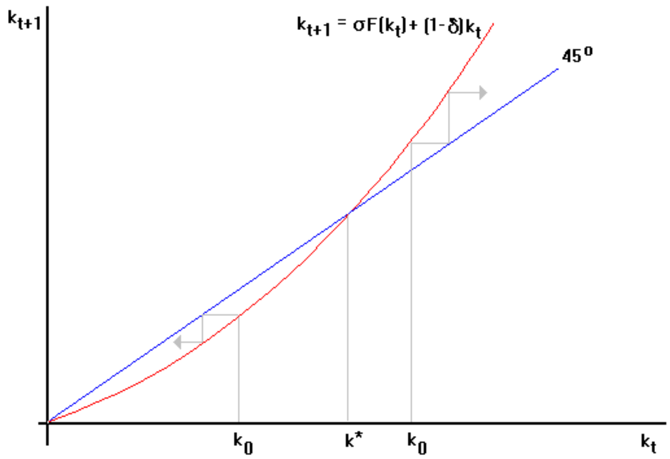

Figure 2: Growth with Increasing Returns

Cobb-Douglas (in the Cobb-Douglas case we can interpret technical change as labor- or capital-augmenting or neutral, since the z_t term can be factored

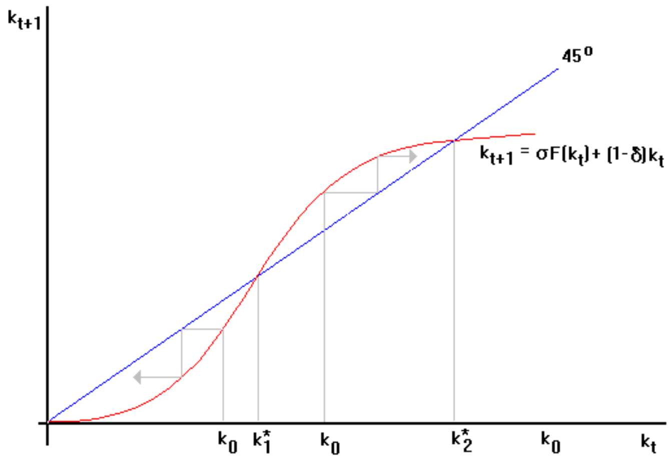

Figure 3: Increasing then Decreasing Returns

out).[2] Then

[2] We verify that balanced growth requires labor-augmenting technical progress. In general, we have

$$Y = f(e^{\gamma_K t}K, e^{\gamma_L t}L) = e^{\gamma_K t}K\phi\left[e^{(\gamma_L-\gamma_K)t}\tfrac{L}{K}\right],$$

where γ_K and γ_L are the rates of capital- and labor-augmenting technological progress and $\phi(\omega) = f(1,\omega)$. Let the growth rate of L be n and of K be $\gamma = \sigma\frac{Y}{K} - \delta$. Then $\frac{L}{K} = Ae^{(n-\gamma)t}$, and

$$\tfrac{Y}{K} = e^{\gamma_K t}\phi\left[e^{(\gamma_L-\gamma_K+n-\gamma)t}\right].$$

Now γ constant implies $\frac{Y}{K}$ constant, and so either $\gamma_k = 0$ and $\gamma = \gamma_L + n$; or $\gamma_K \neq 0$ but the change in $e^{\gamma_K t}$ exactly offsets the change in $\phi\left[e^{(\gamma_L-\gamma_K+n-\gamma)t}\right]$ over time. In the latter case, if we differentiate and rearrange $\frac{d}{dt}\frac{Y}{K} = 0$, we have

$$\frac{\omega\phi'(\omega)}{\phi(\omega)} = \frac{-\gamma_K}{\gamma_L - \gamma_K + n - \gamma} = \text{constant}.$$

This can be integrated to yield $\phi(\omega) = A\omega^\alpha$, where A and α are constants. This means that

$$Y = Ke^{\gamma_K t}\phi\left[e^{(\gamma_L-\gamma_K)t}\tfrac{L}{K}\right] = A(e^{\gamma_K t}K)^{1-\alpha}(e^{\gamma_L t}L)^\alpha;$$

or, in other words, the production function is Cobb-Douglas.

$$k_{t+1} = (1-\delta)k_t + \sigma f(\gamma^t, k_t).$$

Let $\tilde{k}_t = k_t \gamma^{-t}$. Then the previous expression is equivalent to

$$\gamma^{t+1}\tilde{k}_{t+1} = (1-\delta)\gamma^t \tilde{k}_t + \sigma f(\gamma^t, \gamma^t \tilde{k}_t).$$

Dividing by γ^{t+1} and using homogeneity of f, we can reduce this to

$$\tilde{k}_{t+1} = (1-\tilde{\delta})\tilde{k}_t + \tilde{\sigma} f(1, \tilde{k}_t) = g(\tilde{k}_t),$$

where $1-\tilde{\delta} = \frac{1-\delta}{\gamma}$ and $\tilde{\sigma} = \frac{\sigma}{\gamma}$. Hence, after a suitable transformation of variables, the model with exogenous technical change looks exactly like the standard growth model. For any $\tilde{k}_0 > 0$, $\tilde{k}_t \to \tilde{k}^*$ as $t \to \infty$, where

$$\frac{\tilde{k}^*}{F(\tilde{k}^*)} = \frac{\tilde{\sigma}}{\tilde{\delta}} = \frac{\sigma}{\delta + \gamma - 1}.$$

Inverting our transformation, $\tilde{k}_t = \tilde{k}^*$ translates into $k_t = \gamma^t \tilde{k}^*$. This means that, asymptotically, k_t grows at the same rate as the labor augmenting technology progress. Obviously, output (and therefore consumption and investment) asymptotically also grow at the same rate. This outcome, referred to as a balanced growth path, is the same type of behavior displayed by the model with $y = Ak$, and is to a fair approximation consistent with the behavior displayed by actual economies.

2 The Optimal Growth Model

In the model described above, there are no economic decisions being made: individuals simply save a constant fraction σ of output. Cass and Koopmans alternatively assume that individuals have preferences over consumption sequences, and think about making choices in a purposeful manner. To be more precise, imagine that all consumers have identical preferences and that their decisions are coordinated by a benevolent "social planner." We will consider below the relation between this formulation and a decentralized competitive equilibrium.

The planner's problem is

$$\begin{aligned} \max \quad V &= \sum_{t=0}^{\infty} \beta^t U(c_t) \\ \text{st} \quad k_{t+1} &= F(k_t) + (1-\delta)k_t - c_t, \end{aligned}$$

and k_0 given. We impose $U' > 0$, $U'' < 0$, and we impose all of the assumptions from the basic neoclassical growth model on F. If we substitute the constraint into the objective function, the planner's problem is

$$\max \quad V = \sum_{t=0}^{\infty} \beta^t U[F(k_t) + (1-\delta)k_t - k_{t+1}].$$

The element of choice is the capital stock sequence $\{k_t\}$, with the initial value k_0 given.

This problem fits nicely into a deterministic dynamic programming framework. Let us take the state variable to be k_t and the control variable k_{t+1}. Then the law of motion is simply that the value of next period's state variable is given by this period's control variable.[3] Bellman's equation is

$$V(k_t) = \max_{k_{t+1}} \{U[F(k_t) + (1-\delta)k_t - k_{t+1}] + \beta V(k_{t+1})\}. \tag{2}$$

Standard assumptions guarantee that there exists a unique solution V to (2) in the space of continuous, bounded functions. It may be shown that V is twice differentiable, strictly increasing, and strictly concave under standard assumptions on f and U (see Stokey et. al. 1989 and Santos 1991).

[3] One can alternatively take the control variable to be either consumption c_t or investment i_t, because of the identities $k_{t+1} = F(k_t) + (1-\delta)k_t - c_t = (1-\delta)k_t - i_t$.

We will assume for the time being that the solution to the maximization problem on the right hand side of Bellman's equation is interior $-$ that is, $0 < k_{t+1} < F(k_t) + (1-\delta)k_t$; this can also be guaranteed under standard assumptions on f and U. Then the maximizing choice k_{t+1} is characterized by the first order condition

$$U'[F(k_t) + (1-\delta)k_t - k_{t+1}] = \beta V'(k_{t+1}). \tag{3}$$

Let $k_{t+1} = \alpha(k_t)$ be the solution to the first order condition, as a function of k_t, which is unique because V is strictly concave. In other words, the function $\alpha(k)$ is defined by

$$U'[F(k) + (1-\delta)k - \alpha(k)] = \beta V'[\alpha(k)]. \tag{4}$$

The optimal policy for this dynamic programming problem is given by the time-invariant decision rule $k_{t+1} = \alpha(k_t)$. This decision rule is a first order difference equation which, together with the initial condition k_0, completely characterizes the optimal sequence of capital stocks, $\{k_t\}$. One thing that is immediate is that $\alpha(0) = 0$. By the Implicit Function Theorem, we also know that $\alpha(k)$ is differentiable. Differentiating and using the result that V is twice differentiable and concave, we have

$$\alpha'(k) = \frac{[F'(k) + 1 - \delta]U''(c)}{U''(c) + \beta V''[\alpha(k)]} > 0. \tag{5}$$

Hence, α is strictly increasing.

We now derive an alterative way of representing the solution. The idea is to eliminate $V'(k_{t+1})$ from (3). To this end, first differentiate (2) to get

$$V'(k_t) = U'[c(k_t, k_{t+1})][F'(k_t) + 1 - \delta], \tag{6}$$

where to save space we have inserted $c_t = c(k_t, k_{t+1}) = F(k_t) + (1-\delta)k_t - k_{t+1}$. Since (6) is valid for every date, we can update the subscripts from t to $t+1$ and substitute the result into (3) to arrive at what is called the *Euler equation*

$$U'[c(k_t, k_{t+1})] = \beta U'[c(k_2, k_{t+2})][F'(k_{t+1}) + 1 - \delta]. \tag{7}$$

Hence, the marginal rate of substitution between consumption at t and $t+1$, given by $\mu = \frac{u'(c_t)}{\beta u'(c_{t+1})}$ equals the marginal rate of transformation, which is given by $F'(k_{t+1}) + 1 - \delta$ because this is how much extra output results next period from an additional unit of savings this period.

Notice that (7) is a second order difference equation in k_t. An optimal $\{k_t\}$ sequence must satisfy the Euler equation; but, since it is a second order equation with only one initial condition k_0, there are many $\{k_t\}$ sequences that satisfy (7) and not all of them are optimal. In other words, the Euler equation is a necessary condition. It can be shown that a sequence $\{k_t\}$ satisfying (7) is optimal is it additionally satisfies the *transversality condition*

$$\lim_{t\to\infty} \beta^t U'(c_t)[F'(k_t) + 1 - \delta]k_t = 0; \tag{8}$$

this is a direct application of Theorem 4.15 in Stokey et. al. (1989). Alternatively, using (7) to eliminate $F'(k_t) + 1 - \delta$, we can write (8) as

$$\lim_{t\to\infty} \beta^{t-1} U'(c_{t-1})k_t = 0; \tag{9}$$

In order to understand the transversality condition, it is useful to interprert the planner's problem as the limit of a sequence of finite horizon problems.[4] Thus, consider choosing $\{k_1, \ldots, k_{T+1}\}$ to solve

$$\max \quad V = \sum_{t=0}^{T} \beta^t U[F(k_t) + (1-\delta)k_t - k_{t+1}].$$

Standard techniques imply the Kuhn-Tucker conditions,

$$\frac{\partial V}{\partial k_t} = -\beta^{t-1}U'(c_{t-1}) + \beta^t U'(c_t)[F'(k_t) + 1 - \delta] \leq 0, \quad = \text{ if } k_t > 0,$$

for $t = 1, \ldots, T$, and in the last period,

$$\frac{\partial V}{\partial k_{T+1}} = -\beta^T u'(c_T) \leq 0, \quad = \text{ if } k_{T+1} > 0.$$

[4]Standard dynamic programming results guarantee that the value functions and decision rules from the sequence of finite horizon problems will converge to the value function and decision rule from the infinite horizon problem.

Assumptions can be made to guarantee $k_t > 0$, so that the Kuhn-Tucker conditions hold with equality for $t = 1, \ldots, T$ (which yield Euler equations). But, as long as $U'(c_T) > 0$, we will never have $k_{T+1} > 0$. If we write the Kuhn-Tucker condition for the last period as $k_{T+1}\partial V/\partial k_{T+1} = k_{T+1}\beta^T U'(c_T) = 0$, then taking the limit as $T \to \infty$ yields (9). Hence, the Euler equations and transversality condition characterize the infinite horizon problem.

The initial condition k_0 togeher with the decision rule $k_{t+1} = \alpha(k_t)$ completely determine the sequence k_t. A steady state is value of k such that $k = \alpha(k)$; hence, in steady state capital, output, consumption and investment are all time-invariant. From (7) the steady state capital stock is either $k = 0$ or $k = k^*$ where k^* satisfies $F'(k^*) = \delta - 1 + 1/\beta$. The usual curvature conditions on F guarantee that there exists a unique such k^*. Letting the rate of time preference ρ be defined by $\beta = 1/(1+\rho)$, we can write the steady state condition compactly as $F'(k^*) = \rho + \delta$. Notice that k^* does not depend on the instantaneous utility function u at all; the only aspect of preferences that matters is the rate of time preference. Also notice that the steady state of the optimal growth problem differs from the solution to the problem of maximizing steady state consumption, which was given by the "golden rule" $F'(k^*) = \delta$, as long as $\rho > 0$. Intuitively, if agents discount then they do not want to maximize steady state consumption, and would rather take a little more today at the expense of the long run.

We now consider behavior away from the steady state. It is not too hard

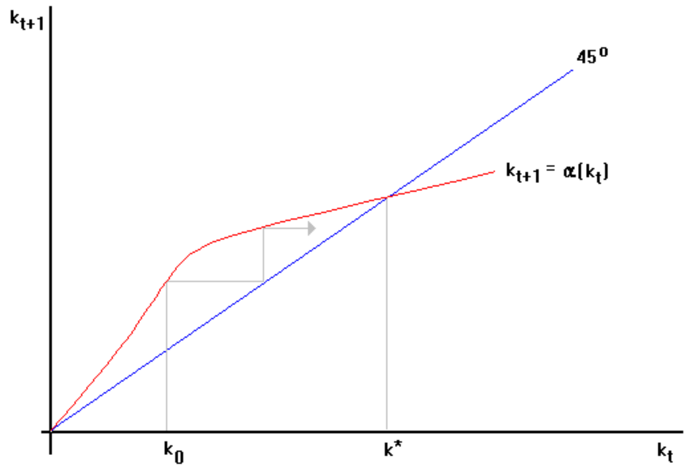

Figure 4: Optimal Growth Model

to show that the slope of the decision rule at the steady state is less than 1.[5] Hence, as shown in Figure 4, $\alpha(k)$ cuts the 45^o line from above at k^*. This means that for any $k_0 > 0$, k_t converges monotonically to k^* as $t \to \infty$.

Observe that the qualitative behavior of k_t is the same in the optimal

[5] One way to proceed is as follows. First, insert $c_t = F(k_t) + (1-\delta)k_t - \alpha(k_t)$ into the expression for $V'(k_t)$ in (6) and differentiate with respect to k_t to get

$$V'' = (F' + 1 - \delta)^2 U'' + U'F'' - (F' + 1 - \delta)^2 U'' \alpha'.$$

At $k = k^*$ this becomes

$$V'' = \frac{U''}{\beta^2} + U'F'' - \frac{U''}{\beta}\alpha'$$

Inserting this into with (5) and rearranging, we see that the slope of the decision rule at the steady state, call it $a = \alpha'(k^*)$, satisfies

$$\beta a^2 - \left(\frac{\beta^2 U' F''}{U''} + 1 + \beta\right) a + 1 = 0.$$

This quadratic has two real roots, $a_+ > 1$ and $a_- \in (0,1)$. One can easily show that $\alpha' = a_+$ implies V is convex (simply solve for the larger root, insert into V'', and rearrange); hence, we know $\alpha' = a_-$, and this establishes the claim.

growth model as in the simple neoclassical model with the savings rate fixed at some exogenous value σ (compare Figures 1 and 4). Nonetheless, it is extremely important to derive the decision rule $\alpha(k)$ endogenously for the following reason. Suppose we are interested in the effects of a change in some exogenous parameter (often, in applications, a policy parameter). How do we know how this change affects σ? We do not. The only way to know how behavior changes is to derive it from first principles – in this case, from optimization.

We now present an example where one can actually derive the solution analytically (while typically one is forced to use numerical techniques). As in Long and Plosser (1983), assume $U(c) = \log(c)$, $F(k) = k^{\theta}$, and $\delta = 1$. We will find the optimal policy and value function by iterating on Bellman's equation. For the first step, set $V_0(k) = 0$, and write

$$V_1(k_t) = \max\{\log(k_t^{\theta} - k_{t+1}) + 0\}.$$

This can be interpreted as Bellman's equation for a finite horizon problem at the terminal date (1 period before the end of time). The solution to the maximization problem is the decision rule $k_{t+1} = \alpha_1(k_t) = 0$, since no one saves at the terminal date. This implies

$$V_1(k_t) = \log(k_t^{\theta}) = \theta \log(k_t).$$

For the second step, write Bellman's equation as

$$\begin{aligned} V_2(k_t) &= \max\{\log(k_t^{\theta} - k_{t+1}) + \beta V_1(k_{t+1})\} \\ &= \max\{\log(k_t^{\theta} - k_{t+1}) + \beta\theta \log(k_{t+1})\} \end{aligned}$$

This implies the decision rule $k_{t+1} = \alpha_2(k_t) = \gamma_2 k_t^{\theta}$, where $\gamma_2 = \frac{\theta\beta}{1+\theta\beta}$. Similarly, for the third step, write

$$V_3(k_t) = \max\{\log(k_t^{\theta} - k_{t+1}) + \beta\theta(1 + \beta\theta) \log(k_{t+1}) + D\},$$

where D is a constant. This implies the decision rule $k_{t+1} = \alpha_3(k_t) = \gamma_3 k_t^\theta$, where $\gamma_3 = \frac{\theta\beta(1+\theta\beta)}{1+\theta\beta+\theta^2\beta^2}$. Then continue in this manner, where at each step n one can interpret the value function V_n and policy function α_n as those arising in a finite horizon problem with n periods left to go.

At step n, the policy function is $k_{t+1} = \alpha_n(k_t) = \gamma_n k_t^\theta$, where γ_n is given by

$$\gamma_n = \frac{\theta\beta(1 + \theta\beta + \ldots + \theta^{n-1}\beta^{n-1})}{1 + \theta\beta + \ldots + \theta^n\beta^n}.$$

As $n \to \infty$, we see that $\gamma_n \to \theta\beta$. Hence, the decision rule converges to $k_{t+1} = \alpha(k_t) = \theta\beta k_t^\theta$. The value function at step n is given by

$$V_n(k_t) = \theta(1 + \theta\beta + \ldots + \theta^{n-1}\beta^{n-1}) \log(k_t) + D_n,$$

where D_n is a constant. Hence, it converges to

$$V(k_t) = \frac{\theta}{1 - \theta\beta} \log(k_t) + D_0,$$

where again D_0 is a constant. One can confirm that this value function does indeed satisfy Bellman's equation for the infinite horizon problem by substitution into (2).

The Euler equation in this example can be written

$$\beta\theta k_{t+1}^{\theta-1}(k_t^\theta - k_{t-1}) = k_{t+1}^\theta - k_{t+2}.$$

The decision rule derived above, $k_{t+1} = \alpha(k_t) = \theta\beta k_t^\theta$, is easily confirmed to satisfy this equation. There are other solutions to the Euler equation, but this is the unique solution that also satisfies the transversality condition. Finally, in this example we can show directly from the decision rule $k_{t+1} = \theta\beta k_t^\theta$ that $k \to k^*$ for any initial condition $k_0 > 0$. If we take the logarithm of both sides, then $\log(k_{t+1}) = \log(\theta\beta) + \theta \log(k_t)$, and therefore $\log(k_t) \to \frac{1}{1-\theta} \log(\theta\beta)$. Hence, $k^* = (\theta\beta)^{\frac{1}{1-\theta}}$.

Now consider a different example with constant relative risk aversion (CRRA) preferences, $u(c) = (c^{1-\alpha} - 1)/(1 - \alpha)$ where $\alpha > 0$, and a linear technology, $y_t = Ak_t$.[6] Also, let $B = A + 1 - \delta$, so that we can write $c_t = Bk_t - k_{t+1}$. For reasons to be explained below, we need to assume that $B < \beta^{1/(\alpha-1)}$. Bellman's equation is

$$V(k_t) = \max\{u(Bk_t - k_{t+1}) + \beta V(k_{t+1})\}.$$

The first order condition is $u'(Bk_t - k_{t+1}) = \beta V'(k_{t+1})$. Inserting V' yeilds the Euler equation, $u'(Bk_t - k_{t+1}) = \beta B u'(Bk_{t+1} - k_{t+2})$, or, for out CRRA utilty function,

$$(Bk_t - k_{t+1})^{-\alpha} = \beta B(Bk_{t+1} - k_{t+2})^{-\alpha}.$$

The necessary and sufficient condition for a path $\{k_t\}$ to be optimal is that it satisfies the Euler equation and the transversality condition. We now show that this is true for the path that, starting from the given initial condition k_0, satisfies $k_{t+1} = \gamma k_t$ with $\gamma = (\beta B)^{1/\alpha}$. This will imply that no matter what value of k_0 we start with the economy starts and continues growing at at fixed rate $\gamma - 1$, which depends on preferences and technology but not on the initial conditions. To proceed, insert $k_{t+1} = \gamma k_t$ into the Euler equation to yield

$$(B - \gamma)^{-\alpha} = \beta B(B - \gamma)^{-\alpha}\gamma^{-\alpha}.$$

This holds if and only if $\gamma = (\beta B)^{1/\alpha}$, as claimed.[7] This implies $k_{t+1} = (\beta B)^{1/\alpha} k_t$, and $c_t = (B - \gamma)k_t$. Notice that $c_t > 0$, and also $V < \infty$, under the right parameter assumptions – e.g., if $\alpha < 1$ then we assume $B < \beta^{1/(\alpha-1)}$.

[6] In this model it is desirable to think of k as including a broader measure of capital than physical capital (e.g., human capital, organizational capital, etc.), since we know that physical capital's share in national income is less than 1.

[7] Note that we cannot say it holds at $\gamma = B$ because this implies $c_t = 0$ for all t and $u'(0)$ is not defined.

Moreover, this same assumption implies $\beta^T u'(c_t)k_{T+1} \to 0$ as $T \to \infty$, and so the the transversality condition holds. This verifies the above claim.

In partiuclar, from any k_0, we have $k_t = k_0\gamma^t$ with $\gamma = (\beta B)^{1/\alpha}$. Hence, $B < 1/\beta$ implies that $k_t \to 0$, and $1/\beta < B < \beta^{1/(\alpha-1)}$ implies that k_t grows without bound, although V is bounded because of discounting. Notice that if two economies start with different values of k_0 then this difference will persist indefinitely: initial conditions matter forever. Also, if two economies have different values of $(\beta B)^{1/\alpha}$ then they will grow forever at different rates, with the difference between their income levels becoming ever larger over time.

To close this setion, we mention that some growth theorists work in continuous rather than discrete time, and therefore many results in the literature are stated in a different language than the one adopted here. We therefore sketch the basic continuous time model, where the planner's problem is

$$\max \quad \int_0^\infty e^{-\rho t} u(c_t)dt,$$

subject to $\dot{k}_t = f(k_t) - \delta k_t - c_t$ and k_0 given. There exist mathematical techniques for solving such a problem directly (see, e.g., Hadley and Kemp 1971); but rather than develop these techniques, we will derive the continuous time results as the limit of discrete time results.

To proceed, let time periods be of length $\Delta > 0$, so that the individual consumes at dates $t = 0, \Delta, 2\Delta, \ldots$ and his utility function is

$$\sum_t e^{-\rho t} \Delta u(c_t).$$

Since the period length is Δ, we write the discount factor as $\beta = \beta(\Delta) = e^{-\rho\Delta}$, and since consumption is fixed at c_t between t and $t + \Delta$ the period utility is $\Delta u(c_t)$. Notice that as $\Delta \to 0$ this summation converges to the integral. Also, since production and depreciation as well as consumption take place over the period, the law of motion for capital becomes

$$k_{t+\Delta} = k_t + [f(k_t) - \delta k_t - c_t]\,\Delta.$$

As $\Delta \to 0$, this yields the law of motion for the continuous time model. For any fixed $\Delta > 0$, consumption satisfies the identity

$$c_t = \frac{k_t - k_{t+\Delta}}{\Delta} + f(k_t) - \delta k_t.$$

It is easy to characterize the solution to the discrete time problem. Inserting the consumption identity into the objective function and differentiating with respect to the control variable $k_{t+\Delta}$, we get the Euler equation

$$-u'(c_t) + e^{-\rho\Delta} u'(c_{t+\Delta})[1 + \Delta f'(k_{t+\Delta}) - \Delta\delta] = 0.$$

This can be rearranged as

$$\frac{u'(c_{t+\Delta}) - u'(c_t)}{\Delta} + u'(c_t)\frac{1 - e^{\rho\Delta}}{\Delta} + u'(c_{t+\Delta})[f'(k_{t+\Delta}) - \delta] = 0.$$

In the limit as $\Delta \to 0$, we get the continuous time Euler equation

$$u''(c_t)\dot{c}_t = u'(c_t)[\rho + \delta - f'(k_{t+\Delta})].$$

Together with the usual transverality condition, this characterizes the solution.

Hence, the continuous time model is described by the two dimensional dynamical system

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} f(k) - \delta k - c \\ A(c)[\rho + \delta - f'(k)] \end{bmatrix},$$

where $A(c) = u'(c)/u''(c)$. Given k_0, any solution to this system that satisfies the transversality condition is an equilibrium. To study solutions, first note that $\dot{k} = 0$ if and only if $c = f(k) - \delta k$ and $\dot{c} = 0$ if and only if $f'(k) = \rho + \delta$, as Figure 5 illustrates in the (k, c) plane. It is easy to show that there is a unique nondegenerate steady state and it is a saddle point. Thus, for any $k_0 > 0$, there is a unique c_0 on the saddle path such that starting from (k_0, c_0), (k_t, c_t) goes to the steady state monotonically; any other c_0 results in a path that violates the transversality condition. The saddle path is the optimal consumption rule, $c_t = c(k_t)$.

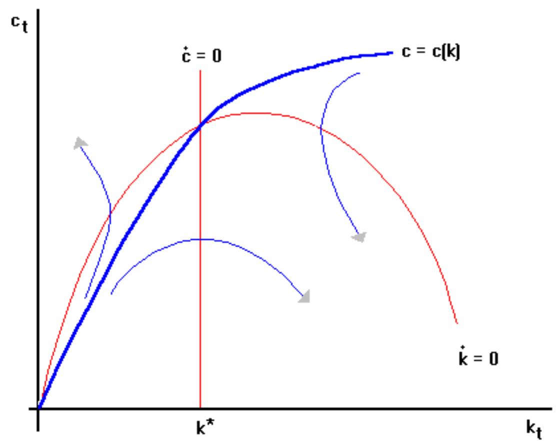

Figure 5: Continuous Time Model

3 Generalizations

The model described in the previous section is not well-suited to the study of many macroeconomic issues (like business cycles) for two reasons: the labor input is assumed to be constant, and the model is deterministic. In this section we remedy both of these deficiencies.

The first thing we do is to allow variations in hours worked by generalizing the instantaneous utility function to $u(c_t, 1 - h_t)$, where h_t is the labor input of the representative agent and, normalizing the time endowment to unity,

$1 - h_t$ is leisure. Assume u is increasing and concave. Given the production function $y_t = f(h_t, k_t)$, the social planner's problem is

$$\max \quad V = \sum_{t=0}^{\infty} \beta^t u[f(h_t, k_t) + (1-\delta)k_t - k_{t+1}, 1 - h_t].$$

We can set this up as a dynamic program with state variable k_t and control vector $a_t = (h_t, k_{t+1})$. Again, the law of motion is simply that next period's state variable equals this period's control variable, $k_{t+1} = a_{2t}$.

Bellman's equation is given by

$$V(k_t) = \max_{a_t}\{u[f(h_t, k_t) + (1-\delta)k_t - k_{t+1}, 1 - h_t] + \beta V(k_{t+1})\}. \tag{10}$$

The maximization problem in (10) implies a pair of first order conditions derived by differentiating with respect to h_t and k_{t+1} (assuming an interior solution),

$$u_2[c(k_t, a_t), 1 - h_t] = u_1[c(k_t, a_t), 1 - h_t] f_1(h_t, k_t) \tag{11}$$

$$u_1[c(k_t, a_t), 1 - h_t] = \beta V'(k_{t+1}), \tag{12}$$

where current consumption is a function of the state and control variables, as given by $c_t = c(k_t, a_t) = f(h_t, k_t) + (1-\delta)k_t - k_{t+1}$. The solutions to (11) and (12), which we write as $h_t = h(k_t)$ and $k_{t+1} = k(k_t)$, are the optimal decision rules.

Note that (11) has the standard interpretation that the marginal rate of substitution between c and h equals the marginal product of labor f_1. As in the previous section, the Euler equation can be derived by calculating V' from (10) and combining the result with (12) to get

$$u_1[c(k_t, a_t), 1 - h_t] = \beta u_1[c(k_{t+1}, a_{t+1}), 1 - h_{t+1}][f_2(h_{t+1}, k_{t+1}) + 1 - \delta], \tag{13}$$

which generalizes (7). Steady state conditions can be derived as in the model with h_t fixed. In particular, (13) implies that in steady state

$$f_2(h^*, k^*) = \delta + \rho. \tag{14}$$

For example, consider the Cobb-Douglas production function, $y = k^{\theta} h^{1-\theta}$. Then (14) implies

$$\frac{k}{y} = \frac{\theta}{\rho + \delta}, \tag{15}$$

and the steady state capital-output ratio is determined exclusively by the rate of time preference and two technology parameters, independent of u. If we additionally assume that $u(c.1-h) = \log(c) + A\log(1-h)$, then (11) implies[8]

$$\frac{Ac}{1-h} = (1-\theta)k^{\theta}h^{-\theta} = (1-\theta)\frac{y}{h}.$$

Since steady state requires $c = y - \delta k$, this simplifies to

$$h = \frac{1-\theta}{1-\theta+A-A\delta k/y}. \tag{16}$$

Inserting (15) into (16) yields h^* as a function of parameters. Then (15) can be used to determine k^* and the rest of the steady state variables.

Reasonable parameter values are $\theta = 0.3$, $\delta = 0.1$, and $\rho = 0.05$, where the rates of depreciation and time preference are interpreted as annual rates. Given these numbers, (15) implies $\frac{k}{y} = 2$, which is not far from what we see in the annual U.S. data. This further implies that the steady state investment-output ratio is $\frac{i}{y} = 0.2$, which is reasonable. Given any value for A, we can then determine h; alternatively, in calibration exercises, A is typically set so that in steady state individuals work the right number of hours − which is about $\frac{1}{3}$ of discretionary time for a typical household in the U.S. according to time-use studies.

[8] We could more generally assume

$$u(c, 1-h) = \frac{\left[c^b(1-h)^{1-b}\right]^{1-\alpha} - 1}{1-\alpha},$$

where the limiting case of $\alpha \to 1$ reduces to the log-linear utility function in the text, and get exactly the same result (with $A = \frac{1-b}{b}$). That is, the risk aversion parameter α does not affect the steady state at all.

Next we add stochastic technology shocks to the model, as in Brock and Mirman, by writing the production function as $y_t = z_t f(k_t, h_t)$. In general, we assume $z_{t+1} = z(z_t, \varepsilon_t)$, where ε_t is an i.i.d. sequence of random variables. For now we assume that the z_t process is stationary; for example, we could assume

$$\log(z_{t+1}) = \lambda \log(z_t) + \varepsilon_t. \tag{17}$$

The planner's problem becomes

$$\max \quad U = E \sum_{t=0}^{\infty} \beta^t u[z_t f(h_t, k_t) + (1-\delta)k_t - k_{t+1}, 1 - h_t].$$

This defines a dynamic programming problem with state $x_t = (k_t, z_t)$ and control $a_t = (h_t, k_{t+1})$. The law of motion is described by $x_{1t+1} = a_{2t}$ and $x_{2t+1} = z(x_{2t}, \varepsilon_t)$.

Bellman's equation is

$$V(x_t) = \max_{a_t} \{u[c(x_t, a_t), 1 - h_t] + \beta EV(x_{t+1})\},$$

where $c_t = c(x_t, a_t) = z_t f(h_t, k_t) + (1-\delta)k_t - k_{t+1}$. The first order conditions are

$$\begin{aligned} u_2[c(x_t, a_t), 1 - h_t] &= u[c(x_t, a_t), 1 - h_t] z_t f_t(h_t, k_t) \\ u_1[c(x_t, a_t), 1 - h_t] &= \beta E V_1(k_{t+1}, z_{t+1}) \end{aligned}$$

The solution to these equations, $h_t = h(x_t)$ and $k_{t+1} = k(x_t)$, yields the optimal decision rules as functions of the state $x_t = (k_t, z_t)$. Calculating V_1 from Bellman's equation and combining it with the first order condition for k_{t+1}, we get the Euler equation

$$u_1[c(x_t, a_t), 1 - h_t] = \beta E u_1[c(x_{t+1}, a_{t+1}), 1 - h_{t+1}][z_{t+1} f_2(h_{t+1}, k_{t+1}) + 1 - \delta].$$

Consider an example with $f(h, k) = h^{1-\theta} k^{\theta}$, $\delta = 1$, and $u(c, h) = U(c) = \log(c)$. This example does not have leisure in the utility function, so we can set $h_t = 1$; but it is still different from the case in the previous section because of the uncertainty. Nevertheless, one can show that the optimal investment policy has a similar form, $k_{t+1} = k(k_t, z_t) = \theta\beta z_t k_t^\theta$, which implies

$$\log(k_{t+1}) = \log(\theta\beta) + \theta\log(k_t) + \log(z_t).$$

Given (17), if we assume that ε_t is normal with mean 0 and standard deviation σ_ε, then $\log(k_t)$ has a long run distribution that is also normal with mean $\frac{1}{1-\theta}\log(\theta\beta)$.[9]

To close this section, we temporarily direct attention away from optimization issues and describe how the stochastic growth model can be used to measure technological progress. If we try to account for changes in output by changes in inputs, the residual can be interpreted – or rather, *defined* – to be technical change. Taking the derivative of $y_t = z_t f(h_t, k_t)$ with respect to t (ignoring the distinction between discrete and continuous time for the moment), we have

$$\dot{y} = \dot{z}f + zf_1\dot{h} + zf_2\dot{k},$$

where the function f is evaluated at arguments as of date t. Dividing by y and simplifying, we have

$$\frac{\dot{y}}{y} = \frac{\dot{z}}{z} + \frac{zf_1h}{y}\frac{\dot{h}}{h} + \frac{zf_2k}{y}\frac{\dot{k}}{k}.$$

Under the assumptions of constant returns to scale and competition, labor and capital will be paid their marginal product, and factor payments will exhaust output; that is, the wage and rental rate on labor and capital will be $w = zf_1$ and $r = zf_2$, and $zf_1h + zf_2k = y$. This implies that $s_k = zf_2k/y$ is

[9] Consider the stochstic process $x_{t+1} = \bar{x} + \theta x_t + \varepsilon_t$, where ε_t is i.i.d. normal with mean 0 and standard deviation σ_ε. Conditional on x_0, x_1 is normal with mean $E_0x_1 = \bar{x} + \theta x_0$; x_2 is normal (as the sum of two normals) with mean $E_0x_2 = \bar{x} + \theta E_0x_1 = \bar{x} + \theta\bar{x} + \theta^2 x_0$, and so on. If $\theta \in (-1, 1)$ then $E_0x_t \to \frac{\bar{x}}{1-\theta}$ as $t \to \infty$. The long run standard deviation can be similarly computed as $\sqrt{\frac{\sigma_\varepsilon^2}{1-\theta^2}}$.

capital's share of income and $s_h = 1 - s_k = zf_1 h/y$ is labor's share of income at each point in time (note that in general these shares vary with t). The national income and product accounts provide data on these shares, and on the growth rates of y, h and k. Hence, we can measure of the growth rate of z as

$$\frac{\dot{z}}{z} = \frac{\dot{y}}{y} - s_h \frac{\dot{h}}{h} - s_k \frac{\dot{k}}{k}.$$

The process for z so measured is referred to as the *Solow residual*; it is the change in output for which we cannot account by growth in inputs, and as such can be interpreted as technological change.[10] In the special case where the production function is a Cobb-Douglas function, $f(h,k) = k^\theta h^{1-\theta}$, capital's share is given by $s_k = zf_1 h/y = \theta$ for all t (i.e., it is constant over time), and

$$\frac{\dot{z}}{z} = \frac{\dot{y}}{y} - (1-\theta) \frac{\dot{h}}{h} - \theta \frac{\dot{k}}{k}.$$

See Prescott (1986), for example, for further discussion.

4 Equilibrium and Optimal Growth

In this section, rather than studying optimal growth as determined by the solution to a social planner's problem, we present a competitive equilibrium model. It turns out, however, that these models yield the same predictions, at least in models without distortions. In the next section we consider models with distortions, where the competitive equilibrium allocation is different from that implied by the planning problem.[11]

[10] Of course, in actual data, the measured process for z_t is not stationary, but this can be accomodated (say, by detrending the data before computing the residuals).

[11] The result that competitive equilibria in economies without distortions are equivalent to Pareto optimal allocations is of course found the fundamental theorems of welfare economics. But the notion of equilibrium consider here is that of a *recursive competitive equilibrium*, due to Mehra and Prescott (1980), which is somewhat different from the usual

Suppose the economy contains a large number of constant returns to scale firms. At each date t the representative firm solves the following profit maximization problem,

$$\max \quad \Pi_t = f(h_t, k_t) - w_t h_t - r_t k_t,$$

taking as given the wage rate on labor w_t and the rental rate on capital r_t. There are no intertemporal considerations in this problem, since the choice of (k_t, h_t) in no way affects Π_s for $s > t$. The solution is fully characterized by $f_1\ (h_t, k_t) = w_t$ and $f_2(h_t, k_t) = r_t$. Moreover, constant returns implies that the maximized profits are zero: $\Pi_t = f(h_t, k_t) - f_1(h_t, k_t)h_t - f_2(h_t, k_t)k_t = 0$.

Due to the constant returns assumption, the model makes no predictions regarding the industrial organization of the economy: there may be many small firms, or few large firms, or any combination thereof, as long as they behave competitively. To simplify the presentation we assume there is a $[0, 1]$ continuum of identical firms. Then aggregate labor and capital demands, denoted H_t and K_t, are given by the solution to $f_1\ (H_t, K_t) = w_t$ and $f_2(H_t, K_t) = r_t$. That is, the wage and rental rates will "adjust" to satisfy these equations.

There is also a large number, again normalized to unity, of individuals. For simplicity, they are assumed to be identical in both their preferences and opportunities. The representative individual solves the problem

$$\begin{aligned} \max \quad v &= E \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) \\ \text{st} \quad c_t &= w_t h_t + r_t k_t + (1 - \delta)k_t - k_{t+1}, \end{aligned}$$

equilibrium notion in an infinite horizon model, Debreu's *valuation equilibrium*. Hence, to use the welfare theorems one must first establish that recursive competitive equilibrium are equivalent to valuation equilibria. Our approach here will be to show directly that recursive competitive equilibrium allocations satisfy the same conditions as the solution to the planner's problem described above.

by choosing $\{c_t, h_t, k_t\}$, taking as given stochastic processes for w_t and r_t and his initial capital k_0 (which is assumed to be the same across agents). Notice that he saves in the form of real capital that is rented to firms and returned, net of depreciation, within the period; hence, we can think of the net return on savings as $r_t + 1 - \delta$. Generally, the individual would also receive the profits of the firms of which he is a shareholder, but recall that profit is zero here.[12]

Assume for now that the individual does not value leisure, so that $u(c_t, 1-h_t) = U(c_t)$ and we can set $h_t = 1$ for all t. In order to solve his investment problem, which is truly dynamic, he needs to have some way of forecasting w_t and r_t. Assume that he expects $-$ or *takes as given* $-$ that these depend (in a stationary, deterministic way) on K_t at each date t. In particular, he takes as given that $w_t = w(K_t) = f_1(1, K_t)$ and $r_t = r(K_t) = f_2(1, K_t)$, which we know must be the case in equilibrium because these are the conditions implied by the solution to the firm problem. He also takes as given that the aggregate capital stock evolves according to some law of motion $K_{t+1} = K(K_t)$. Combined with initial condition K_0, this is all he needs in order to forecast current and future wage and rental rates.

These assumptions make the problem a well-posed dynamic programming problem. The individual's state includes his own capital k_t plus the aggregate state K_t, because he needs to know the latter for forecasting, and his control variable is k_{t+1}. Bellman's equation is

$$v\left(k_t, K_t\right) = \max\{U(c_t) + \beta V\left(k_{t+1}, K_{t+1}\right)\},$$

[12] There are no contingent claims or insurance markets, but they could be included with no change in the results since we are only considering homogeneous agent economies with aggregate (as opposed to individual) uncertainty. That is, if there were insurance markets, then in equilibrium these markets would have to have zero trades; explicitly considering these markets allows us to determine the prices at which zero trades satisfy the individual maximization problem, but does not affect the allocation.

subject to the budget equation

$$c_t = w\left(K_t\right) + r\left(K_t\right)k_t + (1-\delta)\,k_t - k_{t+1}.$$

The solution takes the form of a decision rule $k_{t+1} = A(k_t, K_t)$.

A *recursive competitive equilibrium* (RCE) for this economy can be defined as a law of motion for the aggregate capital stock, $K_{t+1} = K(K_t)$, such that the solution to the individual problem is a decision rule $k_{t+1} = A(k_t, K_t)$ with the property that $A(K_t, K_t) = K(K_t)$. Note that we are only considering symmetric equilibria, where $k_t = K_t$, but these will be the only equilibria given that we start every individual with the same k_0 because the decision rule $A(k_t, K_t)$ will be unique. Hence, the definition of RCE simply embodies individual maximization and rational expectations; the comparison to the concept of Nash equilibrium if game theory should be obvious.[13]

To pursue the properties of equilibrium in more detail, first notice that the solution to the consumer problem is characterized by

$$\begin{aligned} U'(c_t) &= \beta v_1(k_{t+1}, K_{t+1}) \\ v_1(k_t, K_t) &= U'(c_t)[r(K_t) + 1 - \delta]. \end{aligned}$$

Moreover, consumption is given by

$$\begin{aligned} c_t &= w\left(K_t\right)h_t + r\left(K_t\right)k_t + (1-\delta)\,k_t - k_{t+1} \\ &= f_1(1,k_t)h_t + f_2(1,k_t)k_t + (1-\delta)k_t - k_{t+1} \\ &= f(1,k_t) + (1-\delta)k_t - k_{t+1}, \end{aligned}$$

where we have used the equilibrium conditions $w = f_1$, $r = f_2$, and $k_t = K_t$, and the result that $f_1 h + f_2 k = f$ because f is homogeneous of degree 1.

[13] We could include more detail in our definition and say that an equilibrium is a list of functions, $[K(\cdot), w(\cdot), r(\cdot), k(\cdot), v(\cdot)]$, such that: (a) the first order conditions for the firm problem are satisfied, $w(K_t) = f_1(1, K_t)$ and$r(K_t) = f_2(1, K_t)$; and (b) the solution to the consumer problem generates the value function $v(k_t, K_t)$ and decision rule $k(k_t, K_t)$, with the property that $k(K_t, K_t) = K(K_t)$. We have opted for brevity in the text.

Hence, equilibrium is characterized by

$$U'[F(k_t) + (1-\delta)k_t - k_{t+1}] = \beta v_1(k_{t+1}, k_{t+1})$$
$$v_1(k_t, k_t) = U'[F(k_t) + (1-\delta)k_t - k_{t+1}][F'(k_t) + 1 - \delta],$$

where we have reintroduced the notation $F(k) = f(1, k)$.

Define $\varphi(k) = v(k, k)$ and $\psi(k) = A(k, k)$ from the equilibrium value function and decision rule. Then an equilibrium is given by a pair of functions (φ, ψ) that satify the functional equations

$$\begin{aligned} U'[F(k) + (1-\delta)k - \psi(k)] &= \beta\varphi'[\psi(k)] \\ \varphi'(k) &= U'[F(k) + (1-\delta)k - \psi(k)][F'(k) + 1 - \delta]. \end{aligned} \tag{18}$$

For comparison, we repeat the conditions for the planner's problem in the optimal growth model,

$$\begin{aligned} U'[F(k) + (1-\delta)k - \alpha(k)] &= \beta V'[\alpha(k)] \\ V'(k) &= U'[F(k) + (1-\delta)k - \alpha(k)][F'(k) + 1 - \delta], \end{aligned} \tag{19}$$

where $V(k)$ and $\alpha(k)$ are the value function and the decision rule for the planner. What one is supposed to notice is that (18) and (19) are identical. Then, since there exists a unique (V, α) that solves (19), there exists a unique solution to (18) and it is given by $\varphi(k) = V(k)$ and $\psi(k) = \alpha(k)$.

The conclusion is that the RCE allocation coincides with the planner's allocation. Since it can be much easier to characterize the latter, we often study it instead of looking directly for an equilibrium, with knowledge that the implied allocations are the same. If one is interested in the wage and rental rates, which are not part of the planner's problem, one can find them from the marginal conditions $w_t = f_1$ and $r_t = f_2$. Of course, in some models the equilibrium is not efficient and so it cannot be found as the solution to a planner's problem; but when it can, we may as well take advantage of the equivalence between equilibrium and optimal growth.

We now briefly show that the equivalence between equilibrium and optimal allocations hold in the model with variable labor input and technology shocks. The representative firm problem is now

$$\max \quad \Pi_t = z_t f(h_t, k_t) - w_t h_t - r_t k_t,$$

taking as given w_t, r_t, and z_t. The solution is fully characterized by $z_t f_t$ $(h_t, k_t) = w_t$ and $z_t f_2(h_t, k_t) = r_t$, and the maximized value of profit is $\Pi_t = 0$ for all t.

The representative individual solves

$$\begin{aligned} \max \quad v &= E\sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \\ \text{st} \quad c_t &= w_t h_t + r_t k_t + (1-\delta)k_t - k_{t+1}, \end{aligned}$$

taking as given his initial capital stock k_0 (the same for all agents), as well as the aggregate decision rules, $K_{t+1} = K(K_t, z_t)$ and $H_t = H(K_t, z_t)$, and the stochastic process $z_{t+1} = z(z_t, \varepsilon_t)$. Together with initial conditions (K_0, z_0), these are all he needs to know to forecast future wage and rental rates, since $w_t = w(K_t, z_t) = z_t f_1[H(K_t, z_t), K_t]$ and $r_t = r(K_t, z_t) = z_t f_2[H(K_t, z_t), K_t]$, follow from the firm problem.

The individual's state includes his own capital plus the aggregate state, $x_t = (k_t, K_t, z_t)$, and his decision variables are $a_t = (h_t, k_{t+1})$. Bellman's equation is

$$v\left(x_t\right) = \max_{a_t}\{u(c_t) + \beta E V x_{t+1})\}$$

subject to

$$c_t = w\left(K_t, z_t\right) h_t + r\left(K_t, z_t\right) k_t + (1+\delta)\, k_t - k_{t+1}.$$

The solution takes the form of a decision rule $a_t = a(x_t)$, or, more explicitly, $k_{t+1} = k(x_t)$ and $h = h(x_t)$. A RCE for this economy can be defined as a pair of functions, $[H(K_t, z_t), K(K_t, z_t)]$, such that the solution to the individual problem is a pair of decision rules $[h(x_t), k(x_t)]$ with the property that $h(K_t, K_t, z_t) = H(K_t, z_t)$ and $k(K_t, K_t, z_t) = K(K_t, z_t)$.

To pursue this, write $X_t = (k_t, z_t, h_t, k_{t+1})$, so that in equilibrium $c_t = c(X_t)$. Then equilibrium satisfies

$$\begin{aligned} u_2[c(X_t), 1 - h_t] &= u_1[c(X_t), 1 - h_t]w(K_t, z_t) \\ u_1[c(X_t), 1 - h_t] &= \beta E v_1(K_{t+1}, K_{t+1}, z_{t+1}) \\ v_1(K_t, K_t, z_t) &= u_1[c(X_t), 1 - h_t][r(K_t, z_t) + 1 - \delta]. \end{aligned}$$

For comparison, the conditions for the planner's problem are:

$$\begin{aligned} u_2[c(X_t), 1 - h_t] &= u_1[c(X_t), 1 - h_t]z_t f_1(h_t, k_t) \\ u_1[c(X_t), 1 - h_t] &= \beta E V_1(K_{t+1}, z_{t+1}) \\ V_1(K_t, z_t) &= u_1[c(X_t), 1 - h_t][z_t f_2(h_t, k_t) + 1 - \delta]. \end{aligned}$$

This parallels the comparison for the simpler model with $h_t = z_t = 1$ for all t. See Mehra and Prescott (1980) for further discussion.

5 Economies with Distortions

Sometimes competitive equilibria do not solve planning problems. As an example, consider a model where the government must consume a constant amount of output g per period. For simplicity, let us make the environment nonstochastic by setting $z_t = 1$ for all t. We will first solve a social planning problem to characterize the optimal outcome, and then compare this to the decentralized equilibrium outcome.

Using the resource constraint $c_t + i_t + g_t = y_t$, we can write the planner's problem as

$$\max \quad V = \sum_{t=0}^{\infty} \beta^t u[f(h_t, k_t) + (1 - \delta)k_t - k_{t+1} - g, 1 - h_t].$$

Notice the way g enters as a pure drain on the output (that is, it does not enter the utility or production function directly). Bellman's equation is

$$V(k_t) = \max_{a_t}\{u[f(h_t, k_t) + (1-\delta)k_t - k_{t+1} - g, 1 - h_t] + \beta V(k_{t+1})\}.$$

The optimal decision rules, $k_{t+1} = k(k_t)$ and $h_t = h(k_t)$, and value function satisfy:

$$\begin{aligned} u_2(c_t, 1-h_t) &= u_1(c_t, 1-h_t) f_1(h_t, k_t) \\ u_1(c_t, 1-h_t) &= \beta V'(k_{t+1}) \\ V'(k_t) &= u_1(c_t, 1-h_t)[f_2(h_t, k_t) + 1 - \delta]. \end{aligned}$$

Now suppose the representative individual solves the problem

$$\max \quad v = \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t)$$

$$\text{st} \quad c_t = w_t h_t (1-\tau_h) + r_t k_t (1-\tau_k) + (1-\delta)k_t - k_{t+1} - T_t,$$

where τ_h and τ_k are constant proportional tax rates on labor and capital income that do not depend on time, and T_t is a lump sum tax that may vary with time. The government budget constraint is

$$g = w_t h_t \tau_h + r_t k_t \tau_k + T_t,$$

so that the budget is balanced each period. The individual takes as given the aggregate decision rules $K_{t+1} = K(K_t)$ and $H_t = H(K_t)$, and therefore the usual first order conditions from the firm problem imply $w_t = w(K_t)$ and $r_t = r(K_t)$. Given this, the government budget constraint implies $T_t = T(K_t)$.

These assumptions make the consumer's decision problem a well-posed dynamic programming problem with state $x_t = (k_t, K_t)$ and decision variables $a_t = (h_t, k_{t+1})$. Bellman's equation is

$$v(k_t, K_t) = \max_{a_t}\{u(c_t, 1-h_t) + \beta E V(k_{t+1}, K_{t+1})\},$$

where $c_t = c(x_t, a_t)$ is given by inserting $w_t = w(K_t)$, $r_t = r(K_t)$ and $T_t = T(K_t)$ into the individual budget equation. The solution consists of a pair of decision rules $h_t = h(k_t, K_t)$ and $k_{t+1} = k(k_t, K_t)$. A RCE is a pair $[K(K_t), H(K_t)]$, such that the solution to the individual problem generates optimal individual decision rules with the property that $k(K_t, K_t, z_t) = K(K_t, z_t)$ and $h(K_t, K_t, z_t) = H(K_t, z_t)$.

The solution to the consumer problem is characterized by

$$u_2(c_t, 1-h_t) = u_1(c_t, 1-h_t) f_1(h_t, k_t)(1-\tau_h) \tag{20}$$

$$u_1(c_t, 1-h_t) = \beta v_1(k_{t+1}, K_{t+1}) \tag{21}$$

$$v_1(k_t, K_t) = u_1(c_t, 1-h_t)[f_2(h_t, k_t)(1-\tau_k) + 1 - \delta], \tag{22}$$

after substitution of the conditions $w = f_1$ and $r = f_2$. Observe that these are *not* the same as the conditions for the planner's problem unless $\tau_h = \tau_k = 0$; that is, unless the government raises all of its revenue via lump sum taxation. Hence, and equilibrium for this economy must be computed directly, and cannot be found by solving the planning problem.

Although finding the equilibrium decision rules can be difficult, it is sometimes not so difficult to characterize a steady state. In steady state, (21) and (22) combine to yield $f_2(h,k)(1-\tau_k) = \rho + \delta$. For example, if $f(h,k) = k^\theta h^{1-\theta}$ then

$$\frac{k}{y} = \frac{\theta(1-\tau_k)}{\rho + \delta}. \tag{23}$$

If we further assume $u(c, 1-h) = \log(c) + A\log(1-h)$ then (20) implies $\frac{Ac}{1-h} = (1-\tau_h) f_1(h,k) = (1-\tau_h)(1-\theta)y/h$. If we use the steady state condition $c = y - g - \delta k$ and set $g = \gamma y$, where γ is exogenous (so that government consumes a constant fraction of output), this becomes

$$\frac{A(y - \gamma y - \delta k)}{1-h} = \frac{(1-\tau_h)(1-\theta)y}{h},$$

which can be solved for

$$h = \frac{(1-\tau_h)(1-\theta)}{(1-\tau_h)(1-\theta) + A(1-\gamma-\delta k/y)}. \tag{24}$$

Substitution of (23) into (24) yields steady state h^* as a function of the parameters. Then (23) can be used to find k^*, y^*, and so on.

Reasonable (annual) parameter values are $\theta = \frac{1}{3}$, $\delta = 0.075$, $\rho = 0.05$, $\tau_k = 0.25$, $\tau_h = 0.25$, and $\gamma = 0.20$.[14] These numbers yield $\frac{k}{y} = 2$. We then set $A = 1.54$ in order to get $h = \frac{1}{3}$. This implies $k = 0.943$ and $y = \frac{k}{2}$. Also, $r = \theta\frac{y}{k} = \frac{1}{6}$, $w = (1-\theta)\frac{y}{h} = k$, and, from the government budget constraint, $T = -0.05$ (i.e., in steady state the incomes taxes raise more than the government consumes, and so they rebate 5% of output as a lump sum transfer).

For the sake of illustration, consider the following experiment: reduce τ_k to 0 and allow any change in the government budget to be made up by adjustments in the lump sum tax T. Then recalculating the steady state implies $h = 0.351$, $k = 1.528$, and $y = 0.573$. The new capital-output ratio is $\frac{k}{y} = \frac{8}{3}$. Also, $r = \frac{1}{8}$, $w = 1.088$, and $T = .0333y$. Thus, the government would have to impose a lump sum tax equal to 3.3% of output (instead of a lump sum rebate of 5% of output) in order to pay for this policy. The net impact of the policy change is to increase output by22%, the capital stock by 62%, employment hours by 5%, and the wage by 15%.

[14] Notice that $\delta = 0.075$ is lower than the value of 0.1 used earlier; in both cases, δ was calibrated to get a capital-output ratio of 2, but with taxes this requires a lower δ (or a lower ρ or a higher θ).