

# Chapter 6

## Search Models of Production and Exchange

### 6.1 Introduction

In this chapter we study a class of equilibrium search models first introduced by Diamond (1982), and since extended by several others. The basic model is a prototype used to illustrate how multiple equilibria can arise. It can also be generalized and applied in many contexts, including monetary economics, where the frictions inherent in search theory lead naturally to a role for money. To keep things simple, in this chapter, we assume the terms of trade are exogenous; that is, we assume that goods are indivisible and therefore every exchange is a one-for-one swap. The next chapter takes up the important issue of endogenizing prices using bargaining theory.

In the basic framework, there is a sector in which agents locate potential production projects randomly over time, where a project is an opportunity to produce one unit of output at a some cost. In the simplest version there is only one good, but it is assumed that producers do not consume their own output and hence must trade. Traders with goods interact in an exchange sector where they meet bilaterally and, upon meeting, they trade, consume

and return to the production sector.<sup>1</sup>

A difference between the class of models in this chapter and the model of the labor market studied earlier is that here agents do not form enduring relationships: a pair simply trades once and then parts company. However, a key ingredient here, just like in the labor market, is the aggregate meeting technology. Suppose that this technology displays increasing returns to scale. Then when there are more agents in the exchange sector trade becomes easier and producers are willing to take on more costly projects, increasing the flow of agents into the trading sector. Hence, there are potentially multiple steady state equilibria. There can also be complicated and interesting dynamic equilibria (as emphasized, e.g., by Diamond and Fudenberg 1989).

We extend this model in a number of ways. For example, it is easy to include more than one type of commodity and agents with different tastes. In the multigood version of the model, exchange itself is difficult, since you not only have to meet another trader, you have to meet one who both has what you want and wants what you have. As emphasized by Kiyotaki and Wright (1991, 1993), it is this feature that allows money to play a genuine medium of exchange role in the model, a role not at all easy to capture in more traditional general equilibrium theory.

One can use the monetary search model to show how fiat currency can have value, to analyze how an endogenous commodity money may arise, to discuss international monetary issues, and to address a variety of other questions in monetary economics. We also present a version of the model that

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<sup>1</sup>Diamond (1982, 1984b) motivates the set up with the following story. The economy consists of *islands*, on some of which there are trees of various heights bearing *coconuts*. Individuals search for an acceptable tree to climb in order to pick a nut. Since there is a “taboo” against consuming one’s own coconut, they then travel to a trading island on which they look for others in a similar position with whom to trade. The important economic decision in this simplest version of the story is, how much would one be willing to pay (how high a tree should one climb) to enter the exchange sector? As surveyed below, generalized versions of the model introduce many other economic decisions and can be applied to a variety of issues.

can be used to examine the relations between specialization, efficiency, the extent of the market, and the use of money, something discussed in detail by Adam Smith. This is another example of an issue that is difficult to conceptualize in a Walrasian model but can be handled easily and naturally using search theory.

## 6.2 The Basic Model

The economy contains a continuum of infinitely lived agents, with a total population normalized to unity. There is only one good, for now, and we assume it is technologically indivisible and comes in units of size one. The good is perfectly storable at zero cost, but only one unit at a time. It can be produced according to the following technology. An agent with no goods in inventory enters a production sector, where potential projects arrive according to a Poisson process with constant arrival rate  $\alpha > 0$ .

Each potential project yields one unit of the good at a nonnegative cost in terms of disutility. This cost is denoted by  $c$ , and is randomly distributed according to the distribution function  $F(c)$  with support  $[\underline{c}, \bar{c}]$ . We typically assume  $0 < \underline{c} < \bar{c} < \infty$ . The value of  $c$  is observed before the individual decides whether or not to undertake the project. The utility of consuming one unit of the good is  $u > 0$ , with one exception: an agent receives no utility from consuming a good he has produced himself. Therefore, after producing, an agent proceeds to an exchange sector where he looks to trade for something that he can consume.<sup>2</sup>

In the exchange sector, agents meet trading partners bilaterally according to a Poisson process with arrival rate  $\beta$ . We assume that  $\beta$  depends on the number of agents in exchange,  $N$ , so that  $\beta = \beta(N)$ . This is based on an

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<sup>2</sup>The assumption that agents do not consume their own produce is necessary in a one-good economy in order to generate gains from trade. As Diamond (1982) says, it is a simple way of capturing “the advantage of specialized production and trade over self-sufficiency” (p. 883). In any case, an explicit multigood generalization of the model is presented below.

underlying meeting technology that gives the number of meetings per unit time as a function of the size of the exchange sector:  $m(N)$ , with  $m'(N) > 0$ . When the set of agents in the exchange sector is of measure zero we have  $\beta(0) = m(0) = 0$ , and as long as  $N > 0$  we have  $\beta(N) = m(N)/N$ . If  $m(N)$  displays increasing returns to scale then  $\beta'(N) > 0$ ; if  $m(N)$  displays decreasing returns then  $\beta'(N) < 0$ ; and if  $m(N)$  displays constant returns then  $\beta(N)$  is constant.

When two traders meet, in this model, they always exchange inventories one-for-one, after which they consume, enjoy utility  $u$ , and proceed back to the production sector. Trade is relatively easy in two senses: first, to a given individual, all other traders' goods are equivalent in terms of the utility that they provide; second, all exchanges involve a one-for-one swap of inventories. The only nontrivial decision an individual has to make is, when should a potential production project be undertaken? In other words, agents have to determine how much they are willing to pay to move from production to exchange. This depends on expected trading opportunities, which depend on the decisions made by other agents, and this generates a potentially interesting equilibrium problem.<sup>3</sup>

We start with an analysis of steady states, in which the fractions of agents in production and exchange and their strategies are time-invariant (dynamics will be considered shortly). Let  $V_0$  be the value of search in the production sector and  $V_1$  be the value of search in the exchange sector. By the standard techniques, Bellman's equation for a producer is

$$rV_0 = \alpha E \max(V_1 - V_0 - c, 0) = \alpha s_0(k), \quad (6.1)$$

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<sup>3</sup>We pause here to mention that Diamond (1982) interprets agents in the production sector as "unemployed" and those in the trading sector as "employed." This is consistent with the idea that employed individuals have *purchasing power* while unemployed individuals do not. However, one could also interpret agents in the production sector as employed in a job that yields purchasing power at stochastic points in time. This matter of interpretation can affect certain conclusions, such as whether there is too much or too little unemployment in equilibrium, but does not really matter for the formal analysis.

where  $k = V_1 - V_0$  is the reservation production cost and

$$s_0(k) \equiv \int_{\underline{c}}^k (k - c) dF(c).$$

Bellman's equation for a trader with a good is

$$rV_1 = \beta(N)(u + V_0 - V_1) = \beta(N)(u - k). \quad (6.2)$$

These have the usual interpretations; for example, (6.1) sets  $rV_0$  equal to the arrival rate of projects times the option value of producing and switching from production to exchange.

Subtracting (6.1) from (6.2) and simplifying, we have

$$S(N, k) \equiv rk + \alpha s_0(k) - \beta(N)(u - k) = 0. \quad (6.3)$$

Given a constant  $N$ , the optimal production strategy for an individual is to produce all projects that cost  $c \leq k$  where  $k$  is the unique solution to (6.3). For  $N$  to actually be constant, the flow of agents from production to exchange  $\alpha F(k)(1 - N)$  must equal the flow back  $m(N)$ ; that is,

$$T(N, k) \equiv \alpha F(k)(1 - N) - m(N) = 0. \quad (6.4)$$

A *steady state equilibrium* is a pair  $(N, k)$  satisfying (6.3) and (6.4).

It is clear that one steady state is given by  $(N, k) = (0, 0)$ , since when  $N = 0$  the arrival rate in the exchange sector is  $\beta(0) = 0$  and therefore no one would ever pay a positive cost to produce. Nondegenerate equilibria are given by nonzero intersections in the  $(N, k)$  plane of the curves  $S(N, k) = 0$  and  $T(N, k) = 0$ , which we call the S-locus and the T-locus, respectively. These curves are shown in Figure 1 for the case of increasing returns in the meeting technology, which means  $\beta' > 0$  for all  $N > 0$ , and therefore the S-locus slopes upward from  $(0, 0)$  to  $(1, \hat{k})$  where  $\hat{k}$  solves (6.3) with  $N = 1$ . The T-locus slopes upward irrespective of returns to scale, and along the T-locus,  $k \leq \underline{c}$  implies  $N = 0$ , and  $k \geq \bar{c}$  implies  $N = \bar{N}$  where  $\bar{N} < 1$  is the steady state number of traders when *all* production opportunities are undertaken.

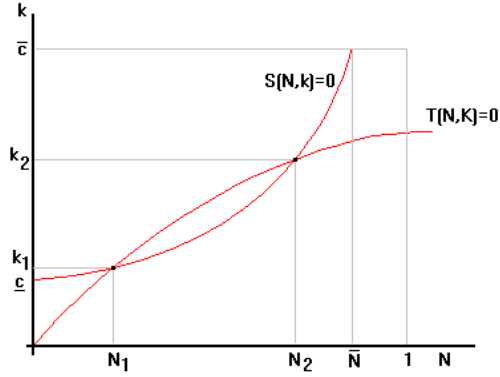


Figure 6.1: Multiple Steady States with Increasing Returns

Given  $\underline{c} > 0$ , the S-locus lies above the T-locus both at  $N = 0$  and at  $N = \bar{N}$ ; hence, if the two curves cross they must cross again. To be precise, if the curves cross for  $N > 0$  they cross an even number of times generically (that is, except for exceptional situations in which the S-locus is tangent to the T-locus at some point or coincident with it over some range). In the exercises you are asked to verify that these curves cross for some parameter values (e.g., low  $r$ ) but not for others. The figure depicts a case where they cross exactly twice, and therefore there exist exactly two nondegenerate steady states, at the points labeled  $(N_1, k_1)$  and  $(N_2, k_2)$ .

The multiplicity of steady state equilibria can be explained intuitively as follows. On the one hand, if agents pessimistically believe that trading opportunities will be poor because  $\beta = \beta(N_1)$  is relatively low, they are not willing to pay a lot to move from production to exchange, and they set  $k = k_1$ . This implies a relatively slow flow into the exchange sector and thereby justifies the pessimistic belief that  $N$  and  $\beta$  will be low. On the other hand, if they optimistically believe trading opportunities will be good because  $\beta = \beta(N_2)$  is relatively high, they are willing to pay more to move from production to exchange, and they set  $k = k_2$ . This implies a relatively fast

flow into exchange sector and justifies the belief that  $N$  and  $\beta$  will be high.<sup>4</sup>

When the meeting technology displays constant returns,  $\beta(N) = \beta_0$  is constant for all  $N > 0$ . In this case  $S(N, k)$  does not vary with  $N$ , and so the S-locus is horizontal at  $k = k_0$  where  $k_0$  solves (6.3) with  $\beta(N) = \beta_0$ . Hence, there cannot exist more than one nondegenerate steady state, and one exists if and only if  $\underline{c}$  is not too big. In the case of decreasing returns, the S-locus is downward sloping and again there cannot exist more than one nondegenerate steady state. We conclude that it is increasing returns in the meeting technology that is at the heart of the multiplicity.

In the exercises you are asked to analyze the welfare properties of the model. To summarize the results, the degenerate outcome is Pareto inferior to any nondegenerate equilibrium, and when multiple nondegenerate equilibria coexist they can be ranked, with higher values of  $(N, k)$  implying greater utility. In the case of constant returns to scale in the meeting technology the unique nondegenerate equilibrium is efficient. However, in the case of increasing returns to scale even the equilibrium with the highest level of  $(N, k)$  is inefficient.

To clarify this last result, if a social planner could choose which production projects were undertaken so as to maximize expected utility for the representative agent, he would choose a higher reservation cost than the equilibrium value of  $k$ . This is due to the “externality” inherent in an increasing returns to scale meeting technology. When producers undertake a project they not only provide themselves with a consumption opportunity, they also provide others with improved consumption opportunities through the increase in the arrival rate  $\beta(N)$ . Since agents do not take the latter effect into account when they choose  $k$ , the equilibrium is not efficient.

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<sup>4</sup>In Diamond’s (1982) terminology, production decisions are affected by the level of employment, or *aggregate demand*, which is measured by the level of  $N$ , since it is only the  $N$  agents in the exchange sector who have purchasing power. If it is believed that demand will be low then there will be less production, which reduces demand and justifies the pessimistic belief. If, on the other hand, it is believed that demand will be high then there will be more production, which increases demand and justifies the optimistic belief.

**Exercises:**

1. Compute the slopes of the S-locus and T-locus explicitly for the cases of constant, increasing and decreasing returns to scale in the meeting function, and verify the properties described in the text.
2. How do the steady state equilibrium values of  $N$  and  $k$  depend on the parameters  $\alpha$ ,  $u$ , and  $r$ , both for the case of constant and increasing returns to scale in the meeting function? Be careful in the case of multiplicity, since the answer may depend on which steady state you choose.
3. Consider the case where  $\beta'(N) > 0$  and  $c$  is constant. Plot the S-locus and T-locus. How many equilibria are there? Hint: Do not forget about mixed strategies.
4. Consider an example where  $c$  is uniform on  $[d, 1]$ . Show that this implies

$$s_0(k) = \begin{cases} 0 & k < d \\ \frac{(k-d)^2}{2(1-d)} & k \in [d, 1] \\ \frac{2k-1-d}{2} & k > d \end{cases}$$

and plot the S-locus.

5. Assume that  $m(N) = N^\theta$  and plot the T-locus on the same graph as the S-locus derived in the previous exercises. Show that these curves cross for some parameter values but not for others.
6. Show that the degenerate equilibrium is Pareto dominated by any of the nondegenerate equilibria, and that if multiple nondegenerate equilibria exist they can be Pareto ranked. Is the steady state with the highest  $N$  and  $k$  efficient? Consider constant, increasing, and decreasing returns in the meeting technology.

7. Design a production cost subsidy, financed by a lump sum tax, to achieve an efficient equilibrium. Try the same thing with a tax on “unemployment ” (see Diamond 1982).
8. Analyze the model where  $m(N)$  displays constant returns, but  $\alpha = \alpha(N)$ .
9. Suppose production and exchange take place on the same island, so that a trader can meet a producer with nothing to trade. How does this affect the outcome? Now suppose that if a trader gives a producer a good they can “stay together” until the producer produces. Analyze the implications for private *credit* (see Diamond 1991).

### 6.3 A Multigood Model

In this section we introduce many goods and agents with different preferences. As in Kiyotaki and Wright (1991), we assume there is one commodity type associated with each agent and that, although agent  $i$  may consume any of the goods, he regards commodity  $i$  as his “favorite” in the following sense:  $u(z)$  is the utility of consuming one unit of good  $j$ , where  $z$  is the some measure of the distance between  $i$  and  $j$ , and  $u(z)$  is a decreasing function. For instance, suppose commodities are described by a characteristic such as color; then agents are labeled by the favorite color and their utility of consuming a good is lower the further is its color from their favorite.

For concreteness, assume that the sets of agent types and commodity types are both distributed uniformly around a circle of circumference 2, and  $z$  measures the distance along the circle between  $i$  and  $j$ . Then for any agent  $i$ , if we draw at random a commodity  $j$ , the distance  $z$  between  $i$  and  $j$  is uniformly distributed on  $[0, 1]$ . This means, for instance, that if an agent decides to consume only goods within distance  $x$  of his favorite, then the probability that a random good will be one of those that he consumes is given by  $prob(z < x) = x$ . The model in the previous section is the special

case where  $u(z)$  is constant, agents consume all goods, and  $x = 1$ .

We continue to assume that the goods are indivisible and can be stored one unit at a time, and that agents with nothing in inventory enter a production process where potential projects arrive according to a Poisson process with arrival rate  $\alpha$ . Each project is characterized by a nonnegative cost of production  $c$  distributed according to  $F(c)$ , and yields one unit of a commodity drawn at random from the set of commodities. For simplicity and to facilitate comparison with the one-good model, we continue to assume that agents derive no utility from the consumption of goods they produce themselves, but this is not strictly necessary in a model with many goods (see below).

The exchange sector is the same as in the one-good model, but for the time being we concentrate on the case of constant returns to scale in the meeting technology, which means that the arrival rate  $\beta$  is constant for all  $N > 0$ . Meeting a potential trading partner no longer necessarily means that a trade takes place, however. For one thing, the other agent may not have a good that you are willing to accept; for another, he may not be willing to accept the good that you have. Exchange requires mutual agreement. Still, if both parties agree to an exchange, they simply swap inventories one-for-one because the goods are indivisible.

In this model we not only have to determine agents' production strategies but also their trading strategies. To make things more interesting, we also add to the model a potentially small but strictly positive transaction cost  $\varepsilon$  in terms of disutility, incurred by an agent each time he accepts any good in trade. As we show below, this will imply that agents are not willing to accept goods they do not consume (i.e., there is no indirect trade). For now, let  $\theta(i)$  denote the expected probability that a random agent is willing to accept good  $i$ ; or course, its value will be determined as part of an equilibrium in what follows.

We continue for the moment to focus on steady states, in which the fractions of agents in production and exchange and their strategies are time-

invariant. We also concentrate exclusively on *symmetric* equilibria, in which neither individuals nor commodities are treated as special simply because of their index  $i$ . In particular, agents expect that the probability a random trader is willing to accept an offer to trade is independent of the type commodity being offered:  $\theta(i) = \theta$  for all  $i$ . We now argue that this expectation will be rational, as follows.

Suppose that potential trading partners accept each commodity with the same probability. Then nothing is gained by trading good  $i$  for good  $j$ , unless good  $j$  is going to be consumed. Hence, no one is willing to pay the transaction cost  $\varepsilon$  to make a trade unless he consumes the good received. In a symmetric equilibrium all traders consume the same fraction of goods, and so all accept the same fraction of goods, and since tastes are distributed uniformly all goods must be accepted by a random trader with the same probability:  $\theta(i) = \theta$  for all  $i$ . Although we do not claim this is the only possible type of equilibrium, we can restrict attention here to equilibria where a good is accepted in exchange if and only if it is immediately consumed.

As above, let  $V_0$  be the value of search in the production sector and  $V_1$  the value of search in the exchange sector. Bellman's equation for a producer is still given (6.1). Bellman's equation for a trader is now given by

$$rV_1 = \beta\theta E \max[u(z) - \varepsilon + V_0 - V_1, 0] = \beta\theta s_1(x), \quad (6.5)$$

where  $x$  is defined by  $u(x) - \varepsilon = V_1 - V_0$ , and

$$s_1(x) \equiv \int_0^x [u(z) - u(x)] dx.$$

The optimal trading strategy is to accept goods within distance  $x$  of one's favorite. Notice that when all agents use the trading strategy derived above we have  $\theta = x$ . Also, since  $u(x) - \varepsilon = V_1 - V_0 = k$ , we can write  $x = x(k)$ . Inserting these results, subtracting (6.1) from (6.5), and simplifying, we have<sup>5</sup>

$$S(N, k) = rk + s_0(k) - \beta x(k) s_1[x(k)] = 0. \quad (6.6)$$

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<sup>5</sup>When we assume constant returns to scale in the meeting technology the function  $S(N, k)$  does not actually depend on  $N$  as long as  $N > 0$ ; but we write it this way in preparation for the more general case.

A steady state again requires that the flow of agents from production to exchange equals the flow back:

$$T(N, k) = \alpha F(k)(1 - N) - \beta N x^2 = 0. \quad (6.7)$$

Comparing this with (6.4), now the rate at which traders flow back to production is the number of meetings,  $m(N) = \beta N$ , multiplied by the probability that both agents in a given meeting agree to trade,  $x^2$ . A steady state equilibrium is a pair  $(N, k)$  satisfying (6.6) and (6.7). As in the previous section, there always exists a degenerate steady state  $(N, k) = (0, 0)$ , and nondegenerate steady states are given by nonzero intersections in the  $(N, k)$  plane of the T-locus and the S-locus.

The T-locus and the S-locus, and hence the set of equilibria, are qualitatively the same as in the previous section. In particular, the case of constant returns to scale the S-locus is horizontal in the  $(N, k)$  plane and there exists a unique nondegenerate equilibrium as long as  $\underline{c}$  is not too big. Although the set of equilibria is qualitatively the same as in the one-good model, it is useful to introduce a variety of goods, for a variety of reasons. One reason is that the welfare implications of the models differ (see the exercises). A related reason is that the multigood model can be used, as we shall see, to analyze some fundamental issues in monetary economics.

**Exercises:**

1. Compute the slopes of the S-locus and T-locus explicitly, for the cases of constant, increasing, and decreasing returns to scale in the meeting function, and describe their behavior in the  $(N, k)$  plane and in the  $(N, x)$  plane.
2. Consider the case where  $c$  is uniform on  $[d, 1]$ . Plot the S-locus. Assume that  $m(N) = N^\theta$  and plot the T-locus on the same graph. For what parameter values do equilibria exist?
3. Show that the degenerate equilibrium is Pareto dominated by any of the nondegenerate equilibria, and that if multiple nondegenerate equilibria

exist then those with higher values of  $N$  and  $k$  imply higher utility. Show that even in the case of a constant returns meeting technology the nondegenerate steady state is not efficient. This result is discussed in Boldrin et al. (1993, Section 5), although the argument is wrong! Say why, and provide a correct argument.

4. For the case of constant returns, derive the effects of a shift in the cost distribution from  $F(c)$  to  $G(c)$ , where  $G$  second order stochastically dominates  $F$ . Derive the effects of changes in  $\alpha$  and  $\beta$  on  $k$ , on the hazard rates,  $H_0 = \alpha F(k)$  and  $H_1 = \beta x^2$ , and on  $N$ . Hint: The effect of a change on  $\alpha$  has ambiguous effects on  $H_0$  and  $N$ , in general, but can be signed if we assume that  $F$  is log-concave (see Boldrin et. al. 1993, Section 6).
5. Analyze the special case of the model with

$$u(z) - \varepsilon = \begin{cases} U & z \leq x \\ 0 & z > x \end{cases}$$

where  $U > 0$  is constant and  $x \in (0, 1)$  is now interpreted as an exogenous parameter.

6. Analyze the special case of the model where  $\alpha \rightarrow \infty$  (i.e., production becomes instantaneous).

## 6.4 Dynamics

Here we generalize the analysis in the previous sections to dynamic equilibria, as do Diamond and Fudenberg (1989) for the one-good model, and Boldrin et al. (1993) for the multigood model. The analysis is necessarily more complicated than the discussion of steady states. Nevertheless, some interesting outcomes emerge as a result.

Using our general techniques for dynamic search problems, equations (6.1) and (6.2) generalize to

$$rV_{0t} = \alpha s_0(k_t) + \dot{V}_{0t} \quad (6.8)$$

$$rV_{1t} = \beta(N_t)(u - k_t) + \dot{V}_{1t}, \quad (6.9)$$

where  $k_t = V_{1t} - V_{0t}$  for all  $t$ . Notice that  $\dot{k} = \dot{V}_1 - \dot{V}_0$  (time subscripts are omitted from now on unless there is risk of confusion). Subtracting (6.8) from (6.9) and simplifying, we arrive at

$$\dot{k} = S(N, k) = rk + \alpha s_0(k_t) - \beta(N)(u - k). \quad (6.10)$$

Given a path for  $N$ , (6.10) is a necessary condition for maximization. It is sufficient as long as  $k$  remains bounded for all  $t$ , which we impose by restricting  $k \leq u$ .

When all agents choose the same path for  $k$ ,  $N$  actually evolves according to

$$\dot{N} = T(N, k) = \alpha F(k)(1 - N) - m(N). \quad (6.11)$$

Given the initial condition  $N_t = N_0$  at  $t = 0$ , a *rational expectations equilibrium* is a solution  $(N, k)$  to the dynamical system defined by (6.11) and (6.10) that remains in the set  $\mathcal{A} = [0, 1] \times [0, u]$ . In other words, an equilibrium is simply a feasible path consistent with individual maximization and rational expectations.

Notice that while the initial value of  $N_0$  is fixed by nature, the initial value of  $k_0$  is not: any  $k_0$  such that the implied path starting from  $(N_0, k_0)$  stays in  $\mathcal{A}$  constitutes an equilibrium. The analysis of equilibria therefore amounts to the analysis of bounded solutions to the dynamical system, given  $N_0$ . The set of points in  $\mathcal{A}$  such that  $\dot{k} = 0$  and the set such that  $\dot{N} = 0$  are exactly the S-locus and T-locus described earlier. However, a steady state is not an equilibrium unless the initial conditions happen to start us at the steady state value of  $N$ . We now consider dynamic equilibria.<sup>6</sup>

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<sup>6</sup>The presentation here will be somewhat informal; see Boldrin et al. (1993) for a more rigorous analysis. See Guichenheimer and Holmes (1983), for example, for more on the mathematics used below, including the Hopf and the Poincaré-Bendixson theorems.

It may be shown that when there is a unique nondegenerate steady state the phase plane in  $(N, k)$  space looks as shown in Figure 2, which is drawn for the case of decreasing returns in the meeting technology. In particular, the unique nondegenerate steady state is a saddle point, and for any  $N_0 > 0$  there is a unique value of  $k_0$  (that puts us on the saddle path) such that  $(N, k)$  does not leave  $\mathcal{A}$ . Hence, for any  $N_0 > 0$  there is a unique equilibrium and it converges to the steady state. In the limiting case of constant returns,  $S(N, k)$  does not depend on  $N$ , and the saddle path is the S-locus itself. In this case,  $k$  is constant along the equilibrium path.

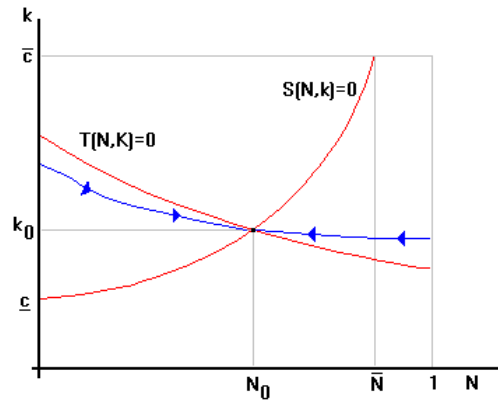


Figure 6.2: Equilibrium with Decreasing Returns

Things get more interesting when there are multiple steady states generated by increasing returns. In this case, the S-locus and T-locus both slope upward and (generically) intersect at an even number of steady states in the interior of  $\mathcal{A}$ . Figure 3 shows the situation when there are exactly two nondegenerate steady states,  $(N_1, k_1)$  and  $(N_2, k_2)$ . A standard local analysis implies that  $(N_2, k_2)$  is a saddle point while  $(N_1, k_1)$  is either a sink or a source depending on parameter values. This suggests that a wide range of dynamic equilibrium behavior may be possible.

Suppose that  $(N_1, k_1)$  is a sink. Then for any initial condition in some neighborhood of  $N_1$ , there is a continuum of initial values for  $k$  such that the system stays in  $\mathcal{A}$ , spiraling in toward  $(N_1, k_1)$ . This means that there is a continuum of equilibrium paths starting from the same  $N$ , indexed by a different initial values of  $k$ . As shown in Figure 3, the stable manifolds of  $(0, 0)$  and  $(N_2, k_2)$  respectively lie below and above  $(N_1, k_1)$ . Hence, given  $N_0$  there is also an equilibrium that starts with a optimistic  $k$  and converges to  $(N_2, k_2)$ , and an equilibrium that starts with a pessimistic  $k$  and converges to  $(0, 0)$ .<sup>7</sup>

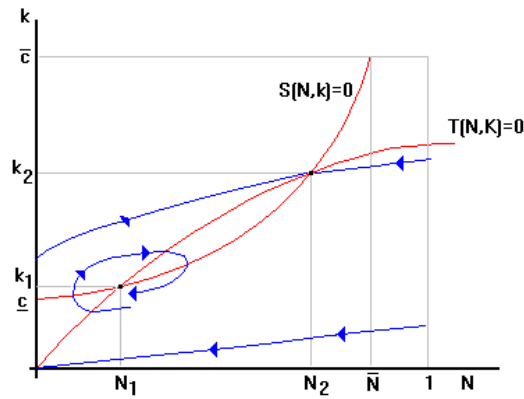


Figure 6.3: Multiple Equilibria with Increasing Returns

Now suppose that as we vary some parameter, such as  $r$ ,  $(N_1, k_1)$  changes continuously from a sink to a source. As argued in Diamond and Fudenberg (1989), the Hopf Bifurcation Theorem then implies that there exists a limit cycle  $\Gamma$  around  $(N_1, k_1)$  for all values of  $r$  in some interval  $(\underline{r}, \bar{r})$ . Alternatively, suppose that we have a situation where the stable manifold of  $(N_2, k_2)$  loops around  $(N_1, k_1)$ , forming a region  $\mathcal{R}$  shown as the shaded region in Figure

<sup>7</sup>There may actually be more than one initial  $k$  that leads us to  $(N_2, k_2)$ , or to  $(0, 0)$ , because the saddle paths to these two steady states may loop around  $(N_1, k_1)$ ; see below.

4, such that orbits starting in  $\mathcal{R}$  cannot escape. Then, as argued in Boldrin et al. (1993), if  $(N_1, k_1)$  is a source the Poincaré-Bendixson Theorem implies there exists a limit cycle  $\Gamma$  around  $(N_1, k_1)$ , and moreover it is stable in the sense that orbits starting near  $\Gamma$  converge to it. See the exercises for explicit examples.

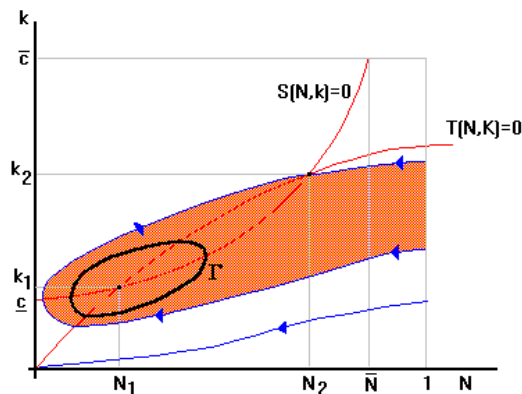


Figure 6.4: Multiple Equilibria, Including Limit Cycle

A limit cycle such as the one in Figure 4 involves recurrent waves of optimism and pessimism, along with corresponding fluctuations in many real variables, including output. The model thus generates persistent “business cycles” even though the fundamental structure is nonstochastic and time-invariant. One does not have to believe that actual business cycles are driven by nothing more than self-fulfilling expectations to find these results interesting. Even if technology shocks like those used in real business cycle theory are granted an important role, once such shocks are integrated into any non-linear structure such as the one described here, it can display dynamics that simpler models have little chance of capturing.

**Exercises:**

Note: A good differential equation solver (we recommend PHASPLANE)

is extremely useful for some of the following exercises.

1. Compute explicitly the determinant and trace of the dynamic system defined by (6.11) and (6.10) and discuss local stability.
2. Consider an example where  $\beta(N) = N^\theta$  and  $c$  is uniform on  $[d, 1]$ . Set  $\alpha = 1$ ,  $\theta = 2$ ,  $d = 0.5$ ,  $u = 1.1$ , and plot the S-locus and T-locus for various values of  $r$ .
3. Verify the following for the example in the previous exercise. At  $r = 0.23$ , the left branch of the stable manifold of  $(N_2, k_2)$  emerges from  $(N_1, k_1)$ , which is a source, and spirals towards  $(N_2, k_2)$ , while the left branch of the unstable manifold of  $(N_2, k_2)$  loops around the outside of the stable manifold. Then, at  $r = 0.24$ , the situation reverses, and the unstable manifold of  $(N_2, k_2)$  heads towards  $(N_1, k_1)$  while the stable manifold loops around the unstable manifold. Note that orbits that start in the region surrounded by the stable manifold cannot escape. Then, since  $(N_1, k_1)$  is a source, there must exist a stable limit cycle around  $(N_1, k_1)$ . As  $r$  increases the cycle gets smaller, and when  $r = 0.27$   $(N_1, k_1)$  becomes a sink, the cycle vanishes, and the unstable manifold of  $(N_2, k_2)$  converges to  $(N_1, k_1)$ .
4. Show that for the many-good version of the model, equilibria can be described by bounded solutions to a dynamical system in either  $(N, k)$  or  $(N, x)$  space. Assume that  $u(x) - \varepsilon = U - x$ ,  $m(N) = N^\theta$ , and  $c$  is uniform on  $[d, 1]$ . Show that in  $(N, x)$  space the system reduces to

$$\begin{bmatrix} \dot{N} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \alpha F(U - x)(1 - N) - N^\theta x^2 \\ -r(U - x) - \alpha s_0(U - x) + \frac{1}{2}N^{\theta-1}x^3 \end{bmatrix}.$$

Set  $\alpha = 1$ ,  $\theta = 2$ ,  $d = 0.5$ , and  $U = 1.05$ , and consider the behavior of the system for values of  $r$  between 0.039 and 0.043. Show that the results for this model are similar to those outlined in previous exercises for the one-good model.

## 6.5 A Monetary Model

Search theory provides a framework within which to model the frictions that can lead to money having a useful role as a *medium of exchange*. By definition, a medium of exchange is an object that is accepted in trade not to be used for consumption or production, but to be traded again later for something else that will be used for such purposes. When a commodity that is also (for some people at some times) a consumption or production good serves as a medium of exchange, it is referred to as *commodity money*. When an object that has no consumption or production value whatsoever serves as a medium of exchange, it is referred to as *fiat money*.

Fiat money may potentially have value as a medium of exchange even though it has no intrinsic value simply because agents believe that it will. The reason sellers surrender goods in exchange for dollars is that they rationally expect that other sellers will do the same for them in the future. For a variety of well-known reasons, it is not easy to construct monetary models in the Walrasian tradition. Hicks' (1935) famous "suggestion" was to "look frictions in the face." While there have certainly been many attempts to pursue this idea within the general equilibrium paradigm, search theory would seem to provide the most natural approach.<sup>8</sup>

In this section we present a very simple model of fiat currency based on search. Later we complicate things and show how to address some substantive issues in monetary economics. We begin by making several rather strong

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<sup>8</sup>See Ostroy and Starr (1990) for a survey of attempts to model money in a more or less traditional general equilibrium framework. Standard perturbations used by some macroeconomists, such as imposing a cash-in-advance constraint, or putting money in the utility or production function, may have merit in certain contexts, but have no hope of answering some of the questions in which we are interested. See the discussions in Kareken and Wallace (1980), Starr (1989), or Helwig (1993), for example. An early model of money that is similar in spirit, if not detail, to the search-theoretic approach presented here is contained in Jones (1976). A search model with fiat money similar in many details to the models in this chapter, except for the crucial distinction that there money enters via a cash-in-advance constraint, is contained in Diamond (1984).

assumptions, and, in particular, until the next chapter we restrict the set of possible exchanges by assuming that all objects are indivisible and therefore must trade one-for-one. This allows us to avoid determining exchange rates so that we can focus more clearly on the exchange process.

The starting point is the multigood model analyzed above. We begin with the simplest possible version in order to illustrate its salient aspects for monetary economics. One simplification comes from concentrating on a constant returns to scale meeting technology and on steady states. We also assume that agents still derive zero utility from their own output.<sup>9</sup> Another simplification comes from focusing on the case where production is both free and instantaneous:  $c = 0$  with probability 1 and  $\alpha = \infty$ . This means that upon consumption an agent immediately is endowed with a new unit of output, and we do not have to worry for now about analyzing production strategies.

The next simplification comes from specifying a utility function of the form

$$u(z) - \varepsilon = \begin{cases} U & z \leq x \\ 0 & z > x \end{cases} \quad (6.12)$$

where  $U > 0$  and  $x \in (0, 1)$  are constants and  $z$  is the distance between the good being consumed and the consumer's favorite (notice we have included the transactions cost  $\varepsilon$  in  $U$ ). This makes one part of the trading strategy obvious: accept and consume any good that is within distance  $x$  of one's favorite, and reject any good that is not, for reasons detailed in the nonmonetary version of the model. Therefore, exactly as in that version, the probability that a random trader accepts any real commodity is  $\text{prob}(z \leq x) = x$ .

These simplifications allow us to focus for now exclusively on one aspect of the problem – the decision to accept fiat money. We make the following as-

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<sup>9</sup>All of the results are essentially the same if we assume that agents are allowed to consume their own output, but they do not always have a perfect match between the type of commodity they want to consume and the type of commodity they are best able to produce (see the exercises).

sumptions concerning money. To guarantee that every trade is a one-for-one swap, we assume that money is indivisible, just like real consumption goods. This is obviously a severe abstraction, since one of the desirable properties of money is its divisibility; but we are able to address some interesting issues even with this restriction. At the initial date a fraction  $M$  of the population are each endowed with one unit of money and the remaining  $1 - M$  are each endowed with real commodities.

In equilibrium, money may or may not be accepted in trade. If it is not, we have to decide what happens to agents endowed with money. One possibility is that they dispose of it and produce a real consumption good. However, it turns out to facilitate the presentation if we instead assume that consumption is a necessary input into production, so that agents endowed with money never consume nor produce if money is not valued. Most aspects of the model are robust to relaxing this restriction (see the exercises).

Since every trade is a one-for-one swap, and since consumption is a necessary input to production, the aggregate distribution of money holdings will remain the same at every date: there are  $M$  agents with money, called money traders or buyers, and  $1 - M$  agents with real commodities, called commodity traders or sellers.<sup>10</sup> If we normalize the arrival rate of traders to  $\beta = 1$ , then since meetings are random, the rate at which any agent meets money traders is  $M$  and the rate at which he meets commodity traders is  $1 - M$ .

Consider a representative agent in this model who believes that  $\Pi$  is the probability that a random commodity trader accepts fiat money. He also knows that  $x$  is the probability that any real commodity is accepted by a random trading partner, whether that partner has money or a real commodity. Let  $V_g$  be his value function when he has a good in inventory and let  $V_m$  be his value function when he has money. Given we already know which real commodities he accepts, we only have to decide what he does

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<sup>10</sup>Without these assumptions we would have to solve for the equilibrium distribution of money and goods holdings across the population; see Diamond and Yellen (1987, 1990) for analyses of this problem in related search models.

when he has a commodity and gets an offer to trade it for cash.

In steady state, Bellman equations are

$$rV_g = (1 - M)x^2U + Mx \max(V_m - V_g, 0) \quad (6.13)$$

$$rV_m = (1 - M)x\Pi(U + V_g - V_m). \quad (6.14)$$

Equation (6.13) equates  $rV_g$  to the sum of two terms. The first term is the rate at which a goods trader meets other goods traders,  $1 - M$ , times the probability both want to trade,  $x^2$ , times the utility of trading and consuming,  $U$ . The second term is the rate at which he meets money traders,  $M$ , times the probability the money trader desires his good,  $x$ , times the option value of accepting money. The second equation has a similar interpretation for a money trader.<sup>11</sup>

The (potentially random) strategy that solves the maximization problem in (6.13) is to accept money with probability  $\pi$ , where  $\pi = 1$  if  $V_m > V_g$ ,  $\pi = 0$  if  $V_m < V_g$ , and  $\pi = [0, 1]$  if  $V_m = V_g$ . Since  $V_m$  and  $V_g$  depend on  $\Pi$ , we write individual's best response correspondence as  $\pi = \pi(\Pi)$ . An equilibrium for the model is a fixed point,  $\Pi = \pi(\Pi)$ . It is easy to show the following:  $\Pi < x$  implies  $V_m < V_g$  which implies  $\pi = 0$ ;  $\Pi > x$  implies  $V_m > V_g$  which implies  $\pi = 1$ ; and  $\Pi = x$  implies  $V_m = V_g$  which implies the best response is any  $\pi$  in  $[0, 1]$ .

This implies that there are exactly three equilibria:  $\Pi = 0$ , which is called a nonmonetary equilibrium since money is never accepted;  $\Pi = 1$ , which is called a pure monetary equilibrium since money is universally accepted; and  $\Pi = x$ , which is called a mixed monetary equilibrium since money is accepted but not universally. The mixed monetary equilibrium can be interpreted either as one in which all agents randomly accept cash with probability  $\Pi =$

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<sup>11</sup>We have assumed here that accepting money involves no transaction cost, unlike accepting real commodities, but this is *not* essential. In fact, money can have a *greater* transaction cost than the cost  $\varepsilon$  of accepting goods (see the exercises). This is important because we do not want to give the impression that money can be valued only if it has intrinsic advantages over goods as an asset. As we will see, it is rather an endogenous property of money – its liquidity – that gives it value.

$x$ , or as one in which the fraction  $x$  of traders always accept money while the fraction  $1 - x$  never accept money.

When  $\Pi = 1$ , using cash has the advantage that once you locate a trader with something that you want you can be sure that he will be willing to trade. By contrast, barter requires not only that you want his good, but also that he wants yours, which is exactly Jevons' (1875) famous "double coincidence of wants" problem with pure barter. Hence, when  $\Pi = 1$  everyone wants to use money in exchange. But when everyone wants to use money then  $\Pi = 1$ . This is the "bootstrap" nature of money.

We can ask several welfare questions of the model. For one thing, we can compare utility across the three equilibria given the exogenous variables. It can be shown (see the exercises) that  $V_g$  is strictly greater in the pure monetary equilibrium than the mixed monetary equilibrium, and the same in mixed monetary and nonmonetary equilibria. Also,  $V_m$  is strictly greater in the pure monetary equilibrium than the mixed monetary equilibrium, and strictly greater in the mixed monetary equilibrium than the nonmonetary equilibrium. This means that all agents are at least weakly better off if money is acceptable than if it is not, and all agents are strictly better off if it is universally acceptable than if it is only partially acceptable.

We can also ask how welfare varies with  $M$ . In the setup under consideration, an increase in the number of agents initially endowed with money requires a decrease in the number initially endowed with real commodities. Nevertheless, it is still possible for money to increase welfare. For simplicity, we concentrate on the pure monetary equilibrium, and measure welfare by

$$W = MV_m + (1 - M)V_g \tag{6.15}$$

(but see the exercises). Simplifying, we discover  $rW = \varphi(\Pi, M)U$ , where<sup>12</sup>

$$\varphi(\Pi, M) = (1 - M)x[M\Pi + (1 - M)x].$$

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<sup>12</sup>Note that  $\varphi$  equals consumption per unit time: it is the rate at which you meet commodity traders who have a good you desire,  $(1 - M)x$ , multiplied by the probability they want to trade, which is the probability you have money and they accept it,  $M\Pi$ , plus the probability you have a good and they accept it,  $(1 - M)x$ .

If  $x > \frac{1}{2}$  then  $W$  is maximized at  $M^o = 0$ , and if  $x < \frac{1}{2}$  then  $W$  is maximized at  $M^o = \frac{1-2x}{2-2x} > 0$ . The result has a simple interpretation. When  $x$  is large, direct barter is not very difficult and the role for a medium of exchange is not very important; in this case it is better to endow all agents with real commodities rather than giving some of them intrinsically worthless fiat money. When  $x$  is small, however, barter is sufficiently difficult that it is desirable to give some agents fiat money even if this means crowding out real commodities.

**Exercises:**

1. Redo the analysis of this section in the following variant of the model where agents are “specialists in production but generalists in consumption” in the following sense. There are  $K$  goods and  $K$  types of agents, where type  $K$  can produce only good  $K$ . At the beginning of time, each agent has a desire to consume one and only one good drawn at random, and maintains this desire until he actually consumes it. Once he consumes the desired good, he randomly draws a taste for another good, so that over time he desires each good a fraction  $1/K$  of the time. Agents are allowed to consume their own output, but of course most of the time they must satisfy their desires through trade.
2. Consider a variant of the previous exercise where type  $i$  *always* desires good  $i + 1$  (modulo  $K$ ).
3. Suppose there is a transaction cost  $\eta$  associated with accepting money, analogous to the transaction cost  $\varepsilon$  associated with accepting commodities. Show that there exist monetary equilibria if and only if  $\eta < \bar{\eta}$ , where  $\bar{\eta}$  is to be determined. Can we have a monetary equilibrium with  $\eta > \varepsilon$ ?
4. Analyze the model with storage costs  $k_g$  and  $k_m$  associated with holding goods and money. Note that if  $k_j < 0$  then it can be interpreted as a rate of return rather than a storage cost. Describe the set of equilibria.

Are there monetary equilibria when money is “dominated in rate of return” in the sense that  $k_g > k_m$ ?

5. As in Burdett et al. (1995), suppose costs  $k_g$  and  $k_m$  are only paid if an agent actively searches for trading partners, or “moves.” If he does not move, but rather “stays,” he avoids this cost but has a lower arrival rate because, while a mover can meet both movers and stayers, a stayer can only meet movers. Let fractions  $\sigma_g$  of agents with goods and  $\sigma_m$  of agents with money move. Then the arrival rate of money traders for a mover with goods is  $M$ , the arrival rate of money traders for a stayer with goods is only  $\sigma_m M$ , and so on. Characterize the set of equilibria.
6. Verify the welfare comparison across the three equilibria asserted in the text. How do the results change if we assume that in a nonmonetary equilibrium agents can freely dispose of money and immediately produce a commodity, rather than as in the text where it is assumed that they can only produce after consuming?
7. Derive the values of  $M$  that maximize  $V_g$  and  $V_m$ , and compare them to the value  $M^o$  that maximizes  $W$ . Notice that both  $V_g$  and  $V_m$  are decreasing in  $M$  at  $M^o$ , even though  $M^o$  maximizes the average of  $V_g$  and  $V_m$ . Explain.
8. As in Aiyagari and Wallace (1995), one can think about policy as follows. Suppose there is a subset of agents, called “government agents,” of measure  $\gamma$ ; the remaining  $1 - \gamma$  private agents are as before. Government agents are identical to private agents in every way but one: they accept money with an *exogenous* probability  $\Pi_g$ . How can the value of  $\Pi_g$  influence the set of equilibria, and how this this depend on  $\gamma$ ?
9. Suppose that there is some random variable  $s$  in the economy called a *sunspot* that switches back and forth between states according to Poisson processes; that is, when  $s = s_i$  it switches to  $s_j$  at rate  $\lambda_{ij}$ . A sunspot equilibrium is a situation where money is accepted with

probability  $\Pi_s$  in state  $s$ . Characterize the set of sunspot equilibria. See Wright (1994).

## 6.6 Generalizing the Monetary Model

Here we extend the model in the previous section so that the search problems in the production and exchange processes are level trivial. We begin with the case where the utility function is given by (6.12) and the cost distribution is still degenerate, but now  $\alpha < \infty$  so that production takes time. This allows us to address the interesting connection between money, specialization, and productivity. We then extend the monetary model further by incorporating general cost distributions and utility functions. This leads us to address concepts like liquidity.

We assume throughout this section that the meeting technology displays constant returns to scale and normalize the arrival rate in the exchange sector to  $\beta = 1$ . Let  $N_p$ ,  $N_g$  and  $N_m$  denote the fractions of the population in steady state who are producers, goods traders and money traders, and let  $\mu = N_m/(N_m + N_g)$  be the fraction of agents in the exchange sector with money. Hence,  $\mu$  is the rate at which a trader meets agents with money and  $1 - \mu$  is the rate at which he meets agents with goods.

If  $\Pi$  is the probability that a random agents accepts money, then a steady state requires that the flow out of production equals the flow in:

$$\alpha N_p = (1 - \mu)x^2 N_g + (1 - \mu)\Pi x N_m. \quad (6.16)$$

The right hand side is the number of goods traders who meet other goods traders and trade, plus the number of money traders who meet goods traders and trade. Given  $N_p + N_g = N_m$  and the exogenous value of  $N_m = M$ , (6.16) can be solved for the steady state distribution and the relevant meeting probability,  $\mu$ . It may be shown (see the exercises) that there is a unique solution  $\mu = \mu(\Pi, M)$ .

Now let  $V_p$ ,  $V_g$  and  $V_m$  denote the value functions. Bellman's equations

are

$$\begin{aligned} rV_p &= \alpha(V_g - V_p) \\ rV_g &= (1 - \mu)x^2(U + V_p - V_g) + \mu x \max(V_m - V_g, 0) \\ rV_m &= (1 - \mu)x\Pi(U + V_p - V_m). \end{aligned}$$

The solution to this problem is the best response correspondence  $\pi = \pi(\Pi, \mu)$ . A equilibrium is defined as a pair  $(\Pi, \mu)$  such that  $\Pi = \pi(\Pi, \mu)$  and  $\mu = \mu(\Pi, M)$ ; i.e., a fixed point of the best response correspondence together with a solution to the steady state conditions.

Despite the fact that  $\mu$  is endogenous in this model, the best response correspondence is exactly the same as in the previous section:  $\pi = 0$  if  $\Pi < x$ ;  $\pi = 1$  if  $\Pi > x$ ; and  $\pi = [0, 1]$  if  $\Pi = x$ . Hence, as in the previous section, there is a nonmonetary equilibrium  $\Pi = 0$ , a pure monetary equilibrium  $\Pi = 1$ , and a mixed monetary equilibrium  $\Pi = x$ . The same intuition applies. Given  $\Pi$ , we can then determine  $\mu(\Pi, M)$ .

In Kiyotaki and Wright (1993), this model is adapted to illustrate how money is related to specialization and productivity.<sup>13</sup> To pursue this, we now assume that  $x$  is a *choice variable for producers*, and not an exogenous preference parameter, in the following sense: a producer can choose the fraction of the population that is willing to buy his output by choosing how specialized to become. The advantage of specialization is that it leads to greater productivity; the cost is that specialized output is more difficult to sell.

This is formalized by assuming that the arrival rate in the production process is given by  $\alpha = \alpha(x)$ , with  $\alpha' < 0$  (a bigger  $x$  means less specialization and therefore less output per unit time). When choosing  $x$ , an individual takes the decisions of other producers as given by  $X$ ; in equilibrium,  $x = X$ .

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<sup>13</sup>This venerable notion goes back to Adam Smith, who describes how specialization enhances efficiency in production, but is limited in application by the “extent of the market.” Once individuals start to specialize they must trade. If trade is difficult then specialization is costly. At the same time, the more specialization there is the more difficult is trade. Smith describes how the use of a medium of exchange, by making trade less difficult, can encourage specialization and thereby enhance productivity.

Bellman's equations become

$$\begin{aligned}
rV_p &= \max_x \alpha(x)[V_g(x) - V_p] \\
rV_g(x) &= (1 - \mu)xX[U + V_p - V_g(x)] \\
&\quad + \mu x \max[V_m - V_g(x), 0] \\
rV_m &= (1 - \mu)X\Pi(U + V_p - V_m).
\end{aligned}$$

Notice  $x$  is a choice variable at the production stage, but once the agent becomes a trader  $x$  is a state variable.

Kiyotaki and Wright (1993) show that there exist three equilibria, a non-monetary, a pure monetary and a mixed monetary equilibrium, each with an associated value of  $X$ . They also show that  $X$  is lowest in the pure monetary equilibrium, which means that a universally accepted money induces the greatest specialization and output. It is because money makes exchange less cumbersome by eliminating the need for a double coincidence that it permits greater specialization, exactly as argued by Adam Smith.

Additionally, Kiyotaki and Wright consider what happens when the frictions in the exchange sector vanish by letting the arrival rate  $\beta \rightarrow \infty$ . It is argued that in the limit  $X = 0$ . That is, agents specialize *completely* by producing goods that almost no one desires, but, since the meeting rate is so high, they still sell them in finite time. One interesting aspect of this result is the following. Because the probability of a barter transaction is of order  $X^2$  while the probability of a monetary transaction is of order  $X$ , as  $\beta \rightarrow \infty$  the ratio of barter to monetary exchange vanishes. In the limit, there is money on one side of (almost) every transaction.

For the rest of this section we return to the case where  $x$  is fixed and proceed to other generalizations of the model. Consider a general utility function  $u(z)$ , where a randomly selected commodity has a value of  $z$  that is uniformly distributed on  $[0, 1]$ , and a general production cost distribution  $F(c)$ . Bellman's equations are

$$rV_p = \alpha s_0(k)$$

$$\begin{aligned}
rV_g &= (1 - \mu)\theta_g \max[u(z) - \varepsilon + V_p - V_g, 0]dz \\
&\quad + \mu\theta_m \max(V_m - V_g, 0) \\
rV_m &= (1 - \mu)\Pi \max[u(z) - \varepsilon + V_p - V_m, 0]dz
\end{aligned}$$

where the agent takes as given that  $\theta_g$  is the probability a random commodity trader is willing to accept a good and that  $\theta_m$  is the probability a random money trader is willing to accept a good.

It is clear what the trading strategies are: commodity traders should accept a good if and only if  $z < x$  where  $u(x) - \varepsilon = V_g - V_p$ ; and money traders should accept a good if and only if  $z < y$  where  $u(y) - \varepsilon = V_m - V_p$ . When all agents follow these strategies we have  $\theta_g = x$  and  $\theta_m = y$ . Now an equilibrium is defined as a list  $(k, x, y, \Pi, \theta_g, \theta_m, \mu)$  such that  $k$ ,  $x$  and  $y$  are the solutions to the above problem,  $\Pi$  is a best response to itself,  $\theta_g = x$  and  $\theta_m = y$ , and  $\mu$  satisfies a generalized version of the steady state condition,  $\mu = \mu(k, x, y, \Pi, M)$ .

Kiyotaki and Wright (1991) prove results for this model that generalize those derived for the simpler models: there exists a nonmonetary equilibrium with  $\Pi = 0$  and  $x \in (0, 1)$ ; there exists a mixed monetary equilibrium and it satisfies  $\Pi = x = y \in (0, 1)$ ; and there exists a pure monetary equilibrium with  $\Pi = 1$  and  $x, y \in (0, 1)$ . In pure monetary equilibrium we have  $V_m \geq V_g$ , with strict inequality except for nongeneric cases. Combining this with the conditions  $u(x) - \varepsilon = V_g - V_p$  and  $u(y) - \varepsilon = V_m - V_p$ , we see that  $y < x$ , and so money traders are more choosy than commodity traders. However, it can also be shown that  $y > x^2$ , which means the probability of trading is greater for a money trader even though he is more choosy.

This is related to the notion of *liquidity*. Define the liquidity of money and of a good by  $\lambda_m = 1/D_m$  and  $\lambda_g = 1/D_g$ , where  $D_j$  is the random duration from now until the purchase of a consumption good starting with

object  $j$ .<sup>14</sup> Kiyotaki and Wright (1991) compute

$$\begin{aligned}\lambda_m &= (1 - \mu)y \\ \lambda_g &= (1 - \mu)[\mu y \Pi + (1 - \mu)x].\end{aligned}$$

When  $\Pi = 1$ ,  $\lambda_m - \lambda_g$  is proportional to  $y - x^2$ . In a pure monetary equilibrium we have  $y > x^2$ , and therefore  $\lambda_m > \lambda_g$ . It is precisely because money is so liquid that everyone accepts it, and it is because everyone accepts it that it is so liquid.

**Exercises:**

1. Show that in the version of the model with  $\alpha < \infty$ ,  $c = 0$  and  $x$  exogenous, steady state condition (6.16) reduces to

$$M = \frac{\alpha\mu}{\alpha + \varphi(\mu, \Pi)},$$

where  $\varphi(\mu, \Pi) = (1 - \mu)x[\mu\Pi + (1 - \mu)x]$ . Argue that for any  $M \in [0, 1]$  there is a unique  $\mu = \mu(\Pi, M)$  satisfying this condition. Derive the analogous condition for the version of the model with a general cost distribution and utility function, and argue for the existence (if not uniqueness) of  $\mu$ .

2. In the model with  $\alpha < \infty$ ,  $c = 0$  and  $x$  exogenous, suppose that the government *taxes* money holdings as follows. Each money trader faces a Poisson arrival rate  $\tau$  of “treasury agents” who confiscate his cash, sending him back to the production process without eating. Also, each commodity trader faces a Poisson arrival rate  $\gamma$  of government agents who offer cash for his good. The government budget constraint is  $\tau M = \gamma(1 - M)$ . Demonstrate that a pure monetary equilibrium exists at least for some parameter values.

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<sup>14</sup>See also Lippman and McCall (1986). Textbooks on stochastic processes (e.g., Heyman and Sobel 1982) refer to an object like  $\lambda_j$  as a “first passage time.”

3. As in Li (1995), consider the model in the previous exercise with the following changes: money traders choose search intensity endogenously; and there is no direct barter. Show that with  $\tau = 0$  a positive tax  $\tau$  can improve welfare.
4. Consider the version of the model with  $\alpha < \infty$  and  $c = 0$  but a general utility function  $u(z)$ . Prove that there exists a pure monetary equilibrium. Is it unique?
5. We have assumed production and trade take place in distinct sectors, and therefore constant returns to scale in the meeting technology implies that a random trading partner will have money with probability  $\mu = N_m/(N_g + N_m)$ . Consider the alternative assumption that production and exchange take place in the same location, so that constant returns implies that a random trading partner will have money with probability  $N_m$ , a good with probability  $N_g$ , and nothing to trade with probability  $N_p$ . For the general version of the model ( $k$ ,  $x$  and  $y$  endogenous), determine the steady state population distribution and write down Bellman's equations. Answer:

$$N_g = \frac{-M\Pi y - \alpha F(k) + \sqrt{\Upsilon}}{2x^2},$$

where  $\Upsilon = [M\Pi y + \alpha F(k)]^2 + 4(1 - M)x^2\alpha F(k)$ .

6. In the model in the previous exercise, suppose that if a trader gives a producer a good they can “stay together” until the producer produces. Analyze the implications for private credit and the interaction between credit and money (see Hendry 1993).

## 6.7 An International Model

Matsuyama, Kiyotaki and Matsui (1993) develop a two-country, two-currency, search model of money. This allows them to answer several questions in international economics that cannot even be raised in the context of traditional

models, including the following: What features of a country make it possible, or likely, for its currency to circulate internationally? When can local currencies survive in the presence of a universally accepted international currency? Do international currencies emerge naturally as economies become integrated? What are the costs and benefits to a country of having its currency serve as an international medium of exchange?<sup>15</sup>

To outline the approach, there are two types of agents,  $i = 1, 2$ , and we say type  $i$  is a “citizen” of country  $i$ . We present only a simplified version of the model here in which each country’s population has measure  $\frac{1}{2}$ , for a world population of 1. Agents meet bilaterally with fellow citizens and with foreigners. Merely to reduce notation, the arrival rate for a type  $i$  agent of fellow citizens is assumed independent of  $i$  and normalized to 1. The arrival rate for a type  $i$  agent of foreigners (which must be independent of  $i$  because of equal populations) is  $\beta \leq 1$ . Note that  $\beta$  measures the degree of *economic integration*.

It is not the case in this model that one physically travels between one country and another, nor does one choose in any other way to interact with foreigners versus fellow citizens. It is simply that you sometimes meet foreigners in your daily routine. Imagine, for example, a town on the border populated by both Mexicans and Americans. Americans trade with fellow Americans and with Mexicans (although perhaps less frequently with Mexicans), and vice-versa. The only issue is, which currencies do the agents use?

The supply of currency  $j$  is  $M_j < \frac{1}{2}$ . As in the simplest monetary model presented above, goods are produced at zero cost immediately after consumption, and the probability an agent desires the good held by a random commodity trader is  $x$ . Also, goods can only be stored by their producer (which means no one ever trades for a good he does not consume). Agents

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<sup>15</sup>One shortcoming in their model, as in all of the models in this chapter, is that every exchange is assumed to be a one-for-one trade and this makes it impossible to talk about the determination of prices and exchange rates. This is resolved in the next chapter.

do not consume their own output but otherwise get the same utility  $u$  from all goods. This implies that agents never trade one money for the other in a pure strategy equilibrium.<sup>16</sup>

To reduce the notation further, set  $x = 1$  and simply rule out direct barter. This means that a money trader accepts any good he is offered, and that a commodity trader must first acquire money before consuming, although we do not yet know *which* money. Let  $\Pi_{ij}$  be the probability that type  $i$  accepts money  $j$ . Let  $p_{ij}$  be the fraction of type  $i$  agents holding object  $j$ , where  $j = 0$  denotes a consumption good and  $j = 1$  or  $2$  denotes money from country  $j$ , and let  $V_{ij}$  be the value functions.

For type 1, Bellman's equations are

$$\begin{aligned} rV_{10} &= (p_{11} + \beta p_{21}) \max(V_{11} - V_{10}, 0) \\ &\quad + (p_{12} + \beta p_{22}) \max(V_{12} - V_{10}, 0) \\ rV_{11} &= (p_{10}\Pi_{11} + \beta p_{20}\Pi_{21})(u + V_{10} - V_{11}) \\ rV_{12} &= (p_{10}\Pi_{12} + \beta p_{20}\Pi_{22})(u + V_{10} - V_{12}). \end{aligned}$$

Something similar holds for type 2. As usual, the best response conditions are:  $\pi_{ij} = 1$  if  $V_{ij} > V_{i0}$ ,  $\pi_{ij} = 0$  if  $V_{ij} < V_{i0}$ , and  $\pi_{ij} = [0, 1]$  if  $V_{ij} = V_{i0}$ , for  $i, j = 1, 2$ . An equilibrium is defined in the obvious way as a fixed point.

We restrict attention here to equilibria where agents use pure strategies, and also where  $\Pi_{11} = \Pi_{22} = 1$  so that each currency is always accepted by citizens of the country that issues it. Such equilibria can be described by pairs  $(\Pi_{12}, \Pi_{21})$  satisfying the relevant versions of the best response conditions. There are four possible cases:  $(\Pi_{12}, \Pi_{21})$  is either  $(1, 1)$ ,  $(0, 0)$ ,  $(1, 0)$ , or  $(0, 1)$ . When  $(\Pi_{12}, \Pi_{21}) = (1, 1)$  we have two international currencies, in the sense that both are accepted by everyone. When  $(\Pi_{12}, \Pi_{21}) = (0, 0)$  we have two national currencies, each accepted only by citizens of the country that issues it. When  $(\Pi_{12}, \Pi_{21}) = (0, 1)$ , or vice-versa, we have one national and one

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<sup>16</sup>Zhou (1994) considers an extension of the model where the countries produce different goods and agents have tastes that vary over time, and shows that this can lead to currency exchanges in equilibrium. See the exercises.

international currency.

To see which of these possibilities actually constitutes an equilibrium, it is simply a matter of inserting the candidate strategies and checking the best response conditions for each case. The first thing we need to do is solve for the steady state. This requires solving

$$\begin{aligned} p_{10}(p_{11} + \beta p_{21}) &= p_{11}(p_{10} + \beta p_{20}\Pi_{21}) \\ p_{10}(p_{12} + \beta p_{12})\Pi_{12} &= p_{12}(p_{10}\Pi_{12} + \beta p_{20}) \end{aligned}$$

using the identities  $p_{i0} = 1 - p_{i1} - p_{i2}$  for  $i = 1, 2$  and  $p_{1j} + p_{2j} = 2M_j$  for  $j = 1, 2$  (the latter saying that all of the money is held by someone). For example, in the case  $(\Pi_{12}, \Pi_{21}) = (0, 0)$ , where agents never accept foreign currencies, we have  $p_{11} = 2M_1$ ,  $p_{12} = p_{21} = 0$ , and  $p_{22} = 2M_2$ . The other cases are left as exercises.

The value functions for type 1 in the case  $(\Pi_{12}, \Pi_{21}) = (0, 0)$  reduce to

$$\begin{aligned} rV_{10} &= 2M_1(V_{11} - V_{10}) \\ rV_{11} &= (1 - 2M_1)(u + V_{10} - V_{11}) \\ rV_{12} &= \beta(1 - 2M_2)(u + V_{10} - V_{12}). \end{aligned}$$

Although in this equilibrium type 1 is not supposed to accept money 2,  $V_{12}$  is defined as the value if he deviates from the candidate strategy. For  $\Pi_{12} = 0$  to be a best response condition, we need to check  $V_{12} \leq V_{10}$ .<sup>17</sup> Simple algebra implies that this condition holds if and only if

$$\beta(1 + r)(1 - 2M_2) \leq 2M_1(1 - 2M_1). \quad (6.17)$$

If (6.17) and the symmetric condition for type 2 are both satisfied,  $(\Pi_{12}, \Pi_{21}) = (0, 0)$  is an equilibrium. Clearly, these conditions hold if  $\beta$  is small but not if  $\beta$  is large. The intuition is straightforward. If the degree of economic integration is low, in the sense that the probability of dealing with a foreigner is

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<sup>17</sup>We are applying the *unimprovability* principle of dynamic programming here, which says that if no one-shot deviation from a candidate strategy improves your payoff then the strategy is optimal; see the Appendix.

small, then when your fellow citizens do not accept foreign currency your best response is to do likewise; but if the probability of dealing with a foreigner is high, then you should accept foreign money even if your fellow citizens do not. The other parameters enter (6.17) in a similarly intuitive way.

The analysis of the other cases is similar, and the results can be summarized as follows. An equilibrium with two international monies exists for *all* parameter values. Intuitively, if agents regard the monies as perfect substitutes then they will be, and in such a situation all that matters is the total stock of money,  $M_1 + M_2$ . This equilibrium is essentially the same as a model with one money. An equilibrium with one international and one national money also exists for some parameter values, determination of which is left to the exercises.

The point of the analysis is that there can exist several qualitatively different equilibria, with one or both or neither money emerging as an international means of payment. Note that these equilibria can differ in terms of welfare (see the exercises). To some extent, which equilibria exist depends on fundamentals, including  $M_1$  and  $M_2$ , and the degree of economic integration. At the same time, multiple equilibria may exist for some parameter values. Hence, which monies get accepted in which countries can depend in part on self-fulfilling expectations.

**Exercises:**

1. Prove that there are no currency exchanges in pure strategy equilibrium.
2. Solve for the steady state inventory distributions for the cases not given in the text.
3. Derive conditions under which equilibrium with one international currency exists. Under what conditions do multiple equilibria exist?
4. Can there be equilibria (and under what conditions) where a local currency is *not* accepted by local agents?

5. What can be said about the welfare properties of the various outcomes. For simplicity, you may wish to concentrate on the symmetric case where  $M_1 = M_2$ .
6. Redo the analysis allowing for barter in addition to monetary exchange.
7. What happens if policy makers in the two countries choose  $M_1$  and  $M_2$ , independently in order to maximize the welfare of their own citizens? Compare this to the outcome if they cooperate.
8. As in Zhou (1994), consider a version of the model where agents differentiate between home and foreign goods. A given agent gets utility  $u$  from his preferred good and  $v < u$  from the other good, where his preferred good fluctuates between domestic and imported, and back again, according to Poisson processes. Do agents consume only preferred goods, or some of both, in equilibrium? Can there now be currency exchanges in pure strategy equilibria?

## 6.8 Commodity Money

In this section we present the model of commodity money in Kiyotaki and Wright (1989). Rather than looking for symmetric equilibria, where no goods are treated as special and therefore none are used as money, the focus here is precisely on equilibria in which one or more of the consumption goods does emerge endogenously as a medium of exchange. Although there can also exist equilibria with fiat currency in addition to commodity money in this model, we will not consider them in this section. Also, to keep things manageable, we consider the version of the model with exactly three types of goods and agents and we focus only on steady states.<sup>18</sup>

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<sup>18</sup>Aiyagari and Wallace (1991) generalize the model to  $N$  types. Kehoe et al. (1993) and Renero (1994) analyze non-steady state equilibria. Other extensions include Marimon, McGrattan and Sargent (1990), who examine learning, and Cuadras-Morató (1993) and Li (1994), who introduce private information as to the quality of commodities, including

Let  $i = 1, 2, 3$  index the agent types and also the (indivisible) consumption goods. Specialization is incorporated by assuming that type  $i$  consumes only good  $i$  and produces only good  $i+1$  (modulo 3); that is, type 1 produces good 2, type 2 produces good 3, and type 3 produces good 1. The utility for type  $i$  from consuming one unit of good  $i$  is  $u$ . After consumption, production is both instantaneous and free. Commodities are storable, one unit at a time, but only at a cost that depends on the good: the cost of storing good  $i$  is  $c_i$ , where  $c_1 < c_2 < c_3$ . It is a venerable notion that which commodities emerge as media of exchange depends at least in part on their intrinsic properties, and *storability* is the property that is the focus here.

Type  $i$  always accepts good  $i$ , immediately consumes it, and produces a new unit of good  $i + 1$  (at least if  $u$  is sufficiently big). The only aspect of trading strategies that we need to determine is whether type  $i$  trades his production good  $i + 1$  for good  $i + 2$  in an attempt to facilitate acquisition of his consumption good, or whether he holds onto good  $i + 1$  until he can trade it directly for consumption. Let  $s_i = 1$  if  $i$  trades good  $i + 1$  for good  $i + 2$ , 0 otherwise. Then a (pure) strategy profile is given by  $\mathbf{s} = (s_1, s_2, s_3)$ . If  $s_i = 1$ , then for type  $i$  good  $i + 2$  is a medium of exchange or commodity money.

Since type  $i$  consumes good  $i$  as soon as he gets it, his inventory always consists of either good  $i + 1$  or good  $i + 2$ . This implies  $\mathbf{p} = (p_1, p_2, p_3)$  completely describes the inventory distribution, where  $p_i$  is the proportion of type  $i$  agents holding good  $i + 1$ . As meetings are random and the types exist in equal numbers, the rate at which any agent meets type  $i$  with good  $i + 1$  is  $p_i/3$ . Then, given  $\mathbf{s}$ , a steady state inventory distribution satisfies

$$p_i p_{i+1} s_i = (1 - p_i) p_{i+2}. \tag{6.18}$$

for  $i = 1, 2, 3$  (where addition is modulo 3).<sup>19</sup>

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those that may be used as money.

<sup>19</sup>Consider, for instance, type 1. First note that two agents of the same type never trade with each other in equilibrium. Now if type 1 has good 2 he can acquire good 3 only by

Let  $V_{ij}$  be the value function of type  $i$  holding good  $j$ . For type 1, for example, standard techniques yield

$$rV_{12} = -c_2 + \frac{1}{3}(1 - p_2 + p_3 s_3)u + \frac{1}{3}p_2 \max(V_{13} - V_{12}, 0) \quad (6.19)$$

$$rV_{13} = -c_3 + \frac{1}{3}p_3(u + V_{12} - V_{13}). \quad (6.20)$$

For example, (6.19) sets the flow payoff for type 1 with good 2 equal to the instantaneous return,  $-c_2$ , plus the expected utility from consumption, which occurs when he meets either a type 2 with good 1 or a type 3 with good 1 who is willing to trade, plus the option value of trading when he meets type 2 with good 3. The best response condition for type 1 is  $s_1 = 1$  if  $V_{13} > V_{12}$ ,  $s_1 = 0$  if  $V_{13} < V_{12}$ , and  $s_1 = [0, 1]$  if  $V_{13} = V_{12}$ .

Manipulating (6.19) and (6.20) we find that  $V_{12} - V_{13}$  is proportional to  $D_1$ , where

$$D_1 = c_3 - c_2 + \frac{1}{3}(1 - p_2 - p_3 + p_3 s_3)u. \quad (6.21)$$

The term  $c_3 - c_2$  is the change in storage cost when type 1 trades good 2 for good 3, which tells us which good is a better a store of value or has a higher instantaneous return. The other term measures the change in the probability of consuming when he trades good 2 for good 3, which tells us which good is better as a medium of exchange, or has more liquidity. Deciding whether to trade good 2 for good 3 amounts to comparing instantaneous return and liquidity. The important point to note is that liquidity is endogenous — it depends on the strategy profile  $\mathbf{s}$ , and also on  $\mathbf{p}$  which itself depends on  $\mathbf{s}$ .

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meeting a type 2 agent with good 3, which occurs with probability  $p_2/3$ , and trading, which occurs if and only if  $s_1 = 1$  (because type 2 always wants good 2). In steady state, type 1 never switches from good 3 to good 2 directly, since if he preferred good 2 he would not have traded it for good 3 in the first place; however, he can switch by trading good 3 for good 1, consuming, and producing a new unit of good 2. This happens when he meets type 3 with good 1, which occurs with probability  $p_3/3$ . Steady state requires the number of type 1 agents who switch from good 2 to good 3 equals the number who switch back, which is (6.18) with  $i = 1$ .

Analogous conditions can be derived for types 2 and 3 simply by modulating the subscripts in (6.21). Checking the best response conditions for each type now amounts to checking the sign of  $D_i$ . A steady state equilibrium is a pair  $(\mathbf{s}, \mathbf{p})$  satisfying the steady state and the best response conditions. Since there are only a small number of possible (symmetric, pure strategy) steady state strategy profiles, it is only a matter of algebra to check each case. In this way we can find all such equilibria.

The results can be summarized as follows: if  $c_3 - c_2 > \frac{1}{6}$  then the unique such equilibrium involves  $\mathbf{s} = (0, 1, 0)$ , which implies that good 1 is the unique commodity money; if  $c_3 - c_2 < \frac{1}{3}(\sqrt{2} - 1)$  then the unique such equilibrium involves  $\mathbf{s} = (0, 1, 1)$ , which implies that goods 1 and 3 are both commodity monies; these are the only (symmetric, pure strategy) steady state equilibria. In both of the equilibria, type 2 agents use good 1 as a medium of exchange while type 3 agents keep their production good until they can trade directly for consumption. In the first equilibrium, type 1 does not use good 3 as a medium of exchange because it is too costly to store relative to good 2; in the second equilibrium, he does use good 3 as a medium of exchange because it is not that much more costly to store.

Notice that there is a range for  $c_3 - c_2$  in which neither equilibrium exists; however, one can construct a natural mixed strategy equilibria in this range. Also note that the two pure strategy equilibria never coexist; however, there are regions of parameter space where one of the pure strategy equilibria coexists with a mixed strategy equilibrium. Moreover, for a different version of the model that reorders storage costs so that  $c_1 > c_2 > c_3$ , there can exist multiple pure strategy equilibria. See the exercises for more on all of these claims.

Multiplicity is of interest because it implies that a commodity money may be determined by more than fundamental considerations, like storage costs or other intrinsic properties. We saw similar results in the international currency model for fiat monies. The commodity money model makes it even more clear that there is a tension between fundamental properties of assets and

endogenous liquidity when it comes to determining the equilibrium medium of exchange.<sup>20</sup>

**Exercises:**

1. Verify the assertion in the text about the set of pure strategy equilibria.
2. Show that there exists a mixed strategy equilibrium  $\mathbf{s} = (\sigma, 1, 0)$ , where  $\sigma \in (0, 1)$ , for the parameters such that pure strategy equilibria do not exist. Show that there can exist a mixed strategy equilibrium  $\mathbf{s} = (1, 0, \sigma)$  in a region of parameter space where  $\mathbf{s} = (1, 1, 0)$  is also an equilibrium. See Kehoe et al. (1993).
3. In discrete time, show that for some parameter values there exists a dynamic equilibrium  $\mathbf{s}_t = (\sigma_t, 1, 0)$ , where  $\sigma_t = 0$  if  $t$  is even and  $\sigma_t = 1$  if  $t$  is odd. See Kehoe et al. (1993).
4. Consider a version of the model with  $c_1 > c_2 > c_3$ . Verify the following results:  $\mathbf{s} = (1, 0, 1)$  is an equilibrium for all parameter values;  $\mathbf{s} = (0, 1, 1)$  is an equilibrium for some parameter values; and these are the only (pure strategy, symmetric) steady state equilibria. Hint: This is simply a relabeling of Model B in Kiyotaki and Wright (1989).
5. Consider a two type version of the model, where type 1 produces good 2 and type 2 produces good 1. Show that fiat money can have value under certain conditions on the storage costs for the two goods and money, and interpret this result. See Aiyagari and Wallace (1992). What if we allow agents in this model to choose their type?

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<sup>20</sup>Wright (1995) considers an extension of the model that allows for an arbitrary distribution (rather than equal numbers) of types. This is relevant because the relative numbers of agents who produce and consume the different goods can also influence which of them serve as money. That paper also considers the case where agents can choose their type. This is relevant because whether goods serve as money also influences the numbers of agents who produce and consume the different goods. See also Cuadras-Morató and Wright (1995).

6. Let the distribution of types be given by  $\theta = (\theta_1, \theta_2, \theta_3)$ , where  $\theta_i$  is the fraction of type  $i$ . Characterize the set of equilibria. See Wright (1995).