Cash-In-Advance

Randall Wright

1 Basic Assumptions

We begin with a very simple model: an endowment economy with homogeneous agents. The representative agent chooses a sequence for consumption c_t to solve

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to recursive budget and CIA constraints

$$\begin{aligned} p_t c_t &= p_t e_t + m_t + T_t - m_{t+1} \\ p_t c_t &\leq m_t + T_t, \end{aligned}$$

where p_t is the nominal price level, e_t is the endowment, m_t is money hodlings at the begining of period t, and T_t is a transfer of money from the government, that could be negative and that the agent regards as lump sum. The CIA constraint requires that consumption be financed out of cash on hand at the start of the period, including the transfer.[1] In particular, one cannot use the $p_t e_t$ dollars one receives from the sale of one's endowment at t to finance c_t; it must be carried forward and used for c_{t+1}.

There are different interpretations of this assumption. One that is similar to a story Lucas proposed is as follows. First, each household consists of a pair, say a "worker" and a "shopper." Second, there are several types of

[1] Alternatively, one can assume T_t is *not* available that period, so that the CIA constraint would be $p_t c_t \leq m_t$; see below.

households, and each type lives in a distinct physical location, say an island. To motivate gains from trade, assume that the consumption good comes in K varieties, say different colors, and that each type produces good k but wants to consume good $k+1 \pmod K$. So each period "shoppers" of each type k simultaneously take cash to the next island in order to purchase color $k+1$ consumption goods, while their "worker" partners – perhaps better called "vendors," really – stay home waiting to sell their endowment of color k goods for money. Goods cannot be bartered directly if $K > 2$.[2] Clearly, cash aquired from today's sales cannot be used until next period, since the shopper leaves before the money rolls in. This motivates the CIA constraint. What is relevant is the timing, not that households come in pairs; e.g., the "vendor" agent can be replaced by "vending machine" that collects cash while an individual is out shopping.

We can rewrite the CIA contraint using the budget constraint as $m_{t+1} \geq p_t e_t$. Hence, the assumption can be reinterpretted as saying that agents are forced to hold at least the nominal value of their endowment as money from each period t into $t+1$. Although not usually stated this way, this makes it pretty obvious what the CIA constraint is doing – simply imposing a demand for money. The supply of money at t is denoted M_t, where the initial stock $M_0 > 0$ is given exogenously. The government budget constraint here is $T = M' - M$. Formulating the problem in dynamic programing terms, after eliminating c_t, Bellman's equation can be written

$$V(m) = \max_{\substack{m' \\ s.t.\ m' \geq pe}} \left\{ u\left(e + \frac{m + T - m'}{p}\right) + \beta V(m') \right\},$$

where we surpress the time subscript and use a "prime" to indicate next period (e.g., $m_t = m$ and $m_{t+1} = m'$). We ignore nonnegativity constraints for now, but they will not bind in equilibrium here anyway. Also, we express

[2] In principle, goods could be traded indirectly, with perhaps some colors emerging as commodity money; this can be ruled out simply by assuming that they cannot be stored (say, for more than one period).

the CIA constraint in real terms as $m'/p \geq e$ and let λ denote the Lagrange multiplier, so that we can rewrite the problem as

$$V(m) = \max_{m'} \left\{ u\left(e + \frac{m + T - m'}{p}\right) + \lambda\left(\frac{m'}{p} - e\right) + \beta V(m') \right\}.$$

The usual dynamic programming arguments can be used to show that V is differentiable and strictly concave; hence the solution to the maximization problem is characterized by the first order condition

$$-\frac{u_1(c)}{p} + \frac{\lambda}{p} + \beta V'(m') = 0.$$

The envelope condition is $V'(m) = u_1(c)/p$. Updating this one period and inserting into the previous equation, we have the Euler equation

$$u_1(c) - \lambda = \frac{\beta}{\pi} u_1(c'),$$

where $\pi = p'/p$, so that $\pi - 1$ is the rate of inflation and $1/\pi$ the gross rate of return on money. An equilibrium here can be defined as a sequence for (c, m, λ, p) satisfying the Euler equation, the CIA constraint, the government budget constraint, and the market clearing conditions, $c = e$ and $m = M$, at every date. Note that the government and individual budget constraints together imply that as long as one of the two market clearing conditions holds, the other holds automatically (Walras Law).

Using the fact that $c = e$ at every date, in equilibrium, we write the Euler equation as

$$\lambda = u_1(e) - \frac{\beta}{\pi} u_1(e').$$

Since the multipler λ must be nonnegative, we see that $\pi \geq \beta u'(e')/u'(e)$ in any equilibrium; as a special case, if e is constant then we must have $\pi \geq \beta$ in any equilibrium. The CIA constraint is binding whenever $\pi > \beta u'(e')/u'(e)$, in which case $m'/p = c = e$ in equilibrium. Equating $m' = M'$, we see that $p = M'/e$ at every date; i.e., the nominal price level is proportional to next period's money stock.[3] Suppose e is constant, and consider the money supply rule $M' = \mu M$, so that $\mu - 1$ is the growth rate of M. Then $\pi = p'/p = M'/M = \mu$. More generally, if e is not constant, then inflation is equal to the rate of monetary expansion minus the growth rate in e. In any event, given any path for the money supply, this economy has a unique monetary equilibrium, where $c = e$ and $p = M'/e$.[4]

In the above formulation there was no asset market, which is not restrictive because there can be no asset trading in equilibrium given a representative agent. However, as always, we could open up a bond market simply to see what the bond prices or interest rates would have to be so that the market cleared with no trades. Rewrite the budget equation as

$$pc = pe + m + T - m' + pbR - pb',$$

where b is the number of bonds held at the start of the period, measured in units of the consumption good, and R is the gross real rate of interest. Agents start with $b_0 = 0$. As is standard, the credit market clears (i.e., no ones buys or sells bonds) as long as

$$R' = \frac{u_1(e)}{\beta u_1(e')}.$$

Recall that $\lambda \geq 0$ is equivalent to $u_1(e) \geq \frac{\beta}{\pi} u_1(e')$, which we can now write $R'\pi \geq 1$. This is sometimes expressed as saying the nominal interest rate must be positive, which follows as long as one defines the nominal rate i to be $R'\pi - 1$.

[3]If we alternatively assume that the transfer T is not available to satisfy the CIA constraint that period, the same methods lead to $p = M/e$ – i.e. the price level is proportional to the current money supply. Of course, this has no real implications in this simple economy, since $c = e$ at every date in either case.

[4]It is not so clear how to define a nonmonetary equilibrium here. Normally, one would say that such an equilibrium is one where the value of money is 0 – i.e., $p = \infty$ – but in this model, if $p = \infty$ the CIA constraint implies $c = 0$, and so markets do not clear. It seems reasonable to say that $p = \infty$ constitutes a nonmonetary equilibrium despite the fact that $c < e$.

Of course, if e is constant then $R' = 1/\beta$, and $\pi/\beta \geq 1$ is a necessary condition for equilibrium to exist, which puts constraints on feasible monetary policies since $\pi = \mu$. Hence, we can contract the money supply but not faster than the rate implied by $\mu \geq \beta$. The intuitive reason is that if we tried to set $\mu < \beta$ then agents would have incentive to set $c < e$ and store the difference in cash, since cash is appreciating faster than the rate of time preference. Indeed, if we allow them to issue bonds (i.e., borrow), given $R = 1/\beta$ there would be arbitrage profits available. This is not crucial to the result however: as we saw above, equilibrium requires $u_1(e) \geq \frac{\beta}{\pi}u_1(e')$, which can always be expressed as $\pi R' \geq 1$ by defining $R' = u_1(e)/\beta u_1(e')$, even if we do not allow bond trading. Notice that while this rules out $\pi < \beta$, nothing rules out $\pi > \beta$; although $\pi > \beta$ gives agents an incentive to set $c > e$, the CIA constraint prevents them from doing so, and markets clear at $c = e$.

2 Cash Goods and Credit Goods

Other than recognizing that there is a lower bound to the rate of monetary contraction in equilibrium, there are of course no welfare implications to monetary policy in the above example, because we always have $c = e$; there is no margin that can be distorted to affect the real allocation, even though prices do depend on the money supply. To make the analysis interesting along this dimension, we need to give agents the opportunity to take some more interesting decisions. A simple way to do this is to assume that there are two goods, so that $u(c) = u(c_1, c_2)$, but only c_1 is subject to a CIA constraint; sometimes c_1 is called a *cash good* while c_2 is called a *credit good.* Moreover, for now assume that there is a linear technology for converting the endowment good into either of the consumption goods, say $e = c_1 + c_2$; this implies that in equilibrium the two consumption goods must have the same price.

The constraints are

$$\begin{aligned} pc_1 &= pe - pc_2 + m + T - m' \\ pc_1 &\leq m + T, \end{aligned}$$

since only c_1 is subject to CIA. Eliminating c_1 using the budget condition, the CIA constraint becomes $m' \geq pe - pc_2$; this says that any money generated from the sale of e, if it is not spent on c_2 must be held into the next period. Again writing the constraing in real terms as $m'/p \geq e - c_2$ and letting λ be the multipler, Bellman's equation is

$$V(m) = \max_{m', c_2} \left\{ u\left(e - c_2 + \frac{m + T - m'}{p}, c_2\right) + \lambda\left(\frac{m'}{p} - e + c_2\right) + \beta V(m') \right\}.$$

As above, one can show that V is differentiable and strictly concave; hence, an interior solution (we will consider corner solutions below) to the maximization problem is characterized by the first order conditions

$$\begin{aligned} -\frac{u_1(c)}{p} + \frac{\lambda}{p} + \beta V'(m') &= 0 \\ -u_1(c) + u_2(c) + \lambda &= 0. \end{aligned}$$

The envelope condition is $V'(m) = u_1(c)/p$, which implies the Euler equation

$$u_1(c) - \lambda = \frac{\beta}{\pi} u_1(c').$$

While we discuss dynamic equilibria later, let us begin with the case where the endowment e and the rate of monetary expansion μ are constant, and look for steady state equilibria in which comsumption c and real balances $z = m/p$ are constant (so that prices evolve at the same rate as M). In steady state, the Euler equation implies

$$\lambda = u_1(c)\left(1 - \frac{\beta}{\pi}\right),$$

so that again we must have $\pi \geq \beta$. Combining this with the other first order condition, we have

$$E(c_2) = u_2(e - c_2, c_2) - \frac{\beta}{\pi} u_1(e - c_2, c_2) = 0,$$

after inserting $e - c_2 = c_1$. A solution to this equation is an interior steady state equilibrium as long as it satisfies the nonnegativity conditions strictly, $c_2 \in (0, e)$.

We could also have a corner solution in equilibrium. If $u_2(e, 0) \leq \frac{\beta}{\pi} u_1(e, 0)$ then we have an equilibrium where only cash goods are consumed; in this case things operate just like the previous model, with no credit goods. Alternatively, if $u_2(0, e) \geq \frac{\beta}{\pi} u_1(0, e)$ then we have an equilibrium where no cash goods are consumed and money is simply not valued. If these two inequalities do not hold – which would be the case, e.g., as long as we impose the standard curvature conditions $u_1(0, c_2) = u_2(c_1, 0) = \infty$ – then there exists an interior monetary steady state, that is, a solution $c_2 \in (0, e)$ to $E(c_2) = 0$. Differentiation implies $E' = \frac{\beta}{\pi} u_{11} - (1 + \frac{\beta}{\pi}) u_{12} + u_{22}$; this cannot be signed for arbitrary values of π, but at $\pi = \beta$ we know $E' < 0$ for any concave u. Hence, we can guarantee a unique equilibriuim for low inflation rates, but not in general.

Summarizing, we have shown the following. First, there exists a monetary steady state as long as $u_2(0, e) < \frac{\beta}{\pi} u_1(0, e)$, since it is only when $u_2(0, e) \geq \frac{\beta}{\pi} u_1(0, e)$ that we have a corner solution where no cash goods are consumed. A monetary steady state could be interior, or it could have only cash goods consumed, which occurs when $u_2(e, 0) \leq \frac{\beta}{\pi} u_1(e, 0)$. Thus, with low inflation it is possible to have only cash goods consumed, with high inflation it is possible to have only credit goods, and with intermediate inflation we have both. Of course, curvature conditions like $u_2(e, 0) = u_1(0, e)$ imply that both goods are consumed for any $\pi \geq \beta$. The monetary steady state is unique if π is close to β, but we cannot guarantee this for a general π. We could guarantee a unique steady state for any π if we assume $u_{12} \geq 0$ (including the case where u is separable), since then clearly $E' < 0$. Actually, all we really need is the weaker assumption that the marginal rate of substitution along the line $c_1 = e - c_2$ is decreasing in c_2:

$$\frac{\partial}{\partial c_2} \frac{u_2(e - c_2, c_2)}{u_1(e - c_2, c_2)} = u_2 u_{11} - (u_1 + u_2) u_{12} + u_1 u_{22} < 0. \tag{1}$$

Since $u_2 = \frac{\beta}{\pi} u_1$ in equilibrium, (1) says $E' < 0$. Moreover, one can show that (1) necessarily holds if both goods are normal, since c_1 normal if and only if $u_1 u_{22} - u_2 u_{12} < 0$ and c_2 normal if and only if $u_2 u_{11} - u_1 u_{12} < 0$.[5]

Consider welfare. It should be obvious that efficiency requires $u_2 = u_1$ (solve the planner's problem without the CIA constraint). Hence, the optimal monetary policy requires $\pi = \beta$ at every date, which means $\mu = \beta$. This policy a simple version of what is referred to as the *Friedman rule*, something that will be encountered frequently in what follows. One way to interpret such a policy is to say that it "undoes" the CIA constraint. It does so by making agents "happy" to hold cash from one period to the next by effectively giving money a rate of return equal to the rate of time preference. In principle one could to do this by paying interest on currency, although presumably it is administratively easier to contract the money supply, at least given that we have recourse to nondistorting taxes (see below). Alternatively, if bonds satisfied the CIA constraint, which is another way of saying that money pays interest, one can support the efficient outcome by endowing the economy with a fixed stock of bonds.[6]

[5] Consider maximizing $u(c_1, c_2)$ subject to $c_1 + Pc_2 = e$. The first order condition is $-Pu_1 + u_2 = 0$, and the second order conditon always holds since $Q = P^2 u_{11} - 2P u_{12} + u_{22} < 0$. We have

$$\frac{\partial c_1}{\partial e} = -Q^{-1} \left(\frac{u_2}{u_1} u_{12} - u_{22} \right) \text{ and } \frac{\partial c_2}{\partial e} = -Q^{-1} \left(u_{12} - \frac{u_2}{u_1} u_{11} \right).$$

Since $Q < 0$, normal goods means $u_{22} - \frac{u_2}{u_1} u_{12} < 0$ and $\frac{u_2}{u_1} u_{11} - u_{12} < 0$.

[6] Although we will not do do so explicitly here, one can introduce bonds (that do not satisfy the CIA constraint) into the model with both cash and credit goods exactly as in the model with only cash goods. As in that model, $R' = u_2(c)/\beta u_2(c')$ means that bonds

The situation is depicted in Figure 1 in terms of the trade off between c_2 at one date and c_1 at the next date, since this is the interesting margin: an agent who reduces consumption of the credit good today can keep the money for one period when he can then consume more cash goods. An equilibrium satisfies the marginal condition $u_2/\beta u_1 = p/p' = 1/\pi$, as well as feasibility, $c_2 = e - c_1$. Differentiation implies $\partial c_2/\partial\pi$ is proportional to $-E'$, where $E(c_2)$ was derived above. Again, E' is ambiguous in general, but definitely negative at $\pi = \beta$. Thus, near the optimal policy more inflation necessarily increases the consumption of credit goods and reduces the consumption of cash goods. One way to see it is to note that the function E shifts up with an increase in π. If $E' < 0$, which must be the case when $\pi = \beta$, there is a unique equilibrium and $\partial c_2/\partial\pi > 0$. If there are multiple solutions to $E = 0$, then of course E' and therefore $\partial c_2/\partial\pi$ alternates in sign across solutions. Since E increases with π, the equilibria with $\pi = \beta$ yields the lowest value of c_2 across any equilibria.[7]

Now consider dynamic equilibria. In general, a dynamic equilibrium satisfies

$$u_2(c_1, c_2) = \frac{\beta}{\pi} u_1(c_1', c_2').$$

Suppose that $\pi > \beta$, so that the CIA constraint binds, at every date. Then $c_1 = M'/p = \mu M/p = \mu z$, where z denotes real balances, and by feasibility

do not trade, which is necessary for equilibrium, and so in steady state we have $R = 1/\beta$. Given $R' = u_2(c)/\beta u_2(c')$, we can state the policy results as follows: $\mu R' \geq 1$ is necessary for equilibrium to exist, and the (efficient) policy is $\mu R' = 1$.

[7] These effects can be illustrated using Figure 1. First, starting at the optimum, if π increases we need to find a point where an indifference curve is flatter on the line $c_2 = e - c_1$, which means moving southeast; hence, $\partial c_2/\partial\pi > 0$ at $\pi = \beta$. However, starting at an arbitrary equilibrium, the indifference can in principle become flatter as we move in either direction along this line. Another way to say it is that starting with an arbitrary policy, $\partial c_2/\partial\pi$ has a negative substitution effect and an ambiguous wealth effect, but at the optimal policy the wealth effect vanishes because $\partial V/\partial\pi = 0$. The fact that c_2 is lowest at the optium $\pi = \beta$ can also be seen by the fact that any intersection of the line $c_2 = e - c_1$ with an indifference curve that is flatter than this line must be to the right of the optimum. In particular, this implies is that in order for an increase in π to increase c_1 and decrease c_2, the increase in π must improve welfare.

Figure 1: Equilibrium with Cash and Credit Goods

$c_2 = e - \mu z$. Moreover, $\pi = p'/p = \frac{p'/M'}{p/\mu M} = \mu z/z'$. Hence, the previous condition can be written

$$F(z, z') = \mu z u_2(\mu z, e - \mu z) - \beta z' u_1(\mu z', e - \mu z') = 0.$$

This implicitly defines a difference equation $z' = f(z)$. An equilibrium is a bounded solution to $z' = f(z)$ (it must be bounded for the usual reasons – if not, and real balances become arbitrarily large, at some point the representative agent could get more utility from spending all of his money than he could get in any equilibrium).

Notice that $f(0) = 0$, and that

$$\frac{dz'}{dz} = f'(z) = \frac{\beta u_1 + \beta \mu z'(u_{11} - u_{12})}{\mu u_2 + \mu^2 z(u_{21} - u_{22})}.$$

In particular, $f'(0) = \beta u_1(0,e)/\mu u_2(0,e)$, which means that $f'(0) < 1$ if and only if $u_2(0,e) < \frac{\beta}{\mu} u_1(0,e)$, which is the condition derived above that guarantees there exists a monetary steady state. Figure 2 shows the situation in the case where there exists a (unique) monetary steady state, z^*, which means $f'(0) < 1$, which means that $f(z)$ cuts the 45^o line from below at z^*. This means that the monetary equilibrium is unstable, and the nonmonetary equilibrium is stable. Hence, there always exist dynamic equilibria where we start at some $z_0 \in (0, z^*)$ and $z \to 0$. Indeed, if $f'(z) > 0$, as shown, then for any $z_0 \in (0, z^*)$ we have $z \to 0$, and therefore there is an equilibrium where real balances converge monotonically to 0 over time. we do not generate an equilibrium if we start at $z_0 > z^*$ in this case (where $f' > 0$), since then z becomes unbounded.

We need to check one detail concerning the dynamic paths for z described above – and that is we need to verify that the CIA constraint binds at all points along the path, as we have assumed in the argument. Note that CIA binds if and only if $\lambda > 0$, where

$$\lambda = u_1(\mu z, e - \mu z) - u_2(\mu z, e - \mu z)$$

at each point in time, from the first order conditions. Since this expression is unambiguously decreasing in z, if $\lambda > 0$ at some $\hat{z}$ then $\lambda > 0$ for all $z < \hat{z}$. At the steady state z^*, we have $u_2 = \frac{\beta}{\mu} u_1$, and so as we established earlier $\lambda > 0$ in steady state as long as $\mu > \beta$. Hence, when $z < z^*$ along the entire path, as when $f' > 0$ and $z_0 \in (0, z_0)$, we know that $\lambda > 0$ along the entire path.

If $f' > 0$ does not hold, even more interesting outcomes are possible. For example, if $f'(z^*) < 0$ then z oscillates around z^*, and if f^{-1} has a slope less than -1 at z^* then $z' = f(z)$ will display limit cycles around z^*, by the standard arguments (see, e.g., Azariadis). This is displayed in Figure 3; clearly, if f^{-1} has slope less than -1 at z^* then f and f^{-1} must intersect not only at z^*, but also at (z_L, z_H), say; hence, $z_L = f(z_H)$ and $z_H = f(z_L)$.

Figure 2: Dynamic Equilibria with $f' > 0$

Of course, given that z is not monotone decreasing, the above argument to establish that $\lambda > 0$ along the entire path does not work. However, since $\mu > \beta$ implies $\lambda > 0$ at steady state z^*, continuity implies that $\lambda > 0$ for all $z < \hat{z}$ for some $\hat{z} > z^*$. Hence, we can construct equilibria where z cycles and the CIA constraint is binding at every date at least as long as the cycle is not too big.

Consider the following example:

$$u(c) = \frac{(b + c_1)^{1-\alpha}}{1-\alpha} + c_2,$$

where $\alpha > 0$. For this case, $F(z, z') = 0$ can be solved explicity for

$$z = f^{-1}(z') = \frac{\beta z'}{\mu\,(b + \mu z')^{\alpha}},$$

and for the steady state

$$z^* = \frac{\left(\frac{\beta}{\mu}\right)^{1/\alpha} - b}{\mu}.$$

Moreover, $\lambda = u_1 - u_2 = (b + \mu z)^{-\alpha} - 1 > 0$ as long as $z < \hat{z} = (1 - b)/\mu > z^*$ (assuming $\mu > \beta$). For example, if $\mu = 3$, $\beta = 0.95$, $b = 0.47$, and $\alpha = 5$, a two period cycle emerges around the steady state $z^* = 0.108$, where z fluctuates between approximately $z_L = 0.072$ and $z_H = 0.151$. Since $\hat{z} = 0.177$, CIA binds at z_L and z_H. Indeed, in the figure, where we start at z_0 just below $\hat{z}$ and watch z converge to the cycle, CIA binds along the entire path.[8]

3 Endogenous Labor

One intuitive breakdown of goods into cash and credit goods is to say that consumption of one's own leisure does not require cash, while consumption of market goods does. Although such a model is basically a reinterpretation of what we have already seen in the model with cash and credit goods, it is worth considering since it is one popular specification in the literature. Suppose $u = u(c, 1 - h)$, where c is consumption and subject to CIA, while h is hours worked so that (by normalizing total time to 1) we can interpret $1 - h$ as lesiure. Let the nominal wage be w; the real wage is of course w/p, and will be determined below. The idea here will be that workers get paid in nominal wages that cannot be spent in the current period, but everything goes through exactly the same, if instead we say that they are paid in real

[8]The construction of the cycle used the fact that, in this example, the slope of f^{-1} at $z^*$is easily calculated to be $\left[1 - \alpha + \alpha b(\mu/\beta)^{1/\alpha}\right]^{-1}$, which equals -1 at $b = (\beta/\mu)^{1/\alpha}(\alpha - 2)/\alpha$. In the neigborhood of this value of b, a two period cycle emerges.

Figure 3: Limit Cycle Equilibrium

goods, but then have to sell these for cash (in the same period) to finance consumption next period. We also pay out profits to the representative agent as a nominal dividend, d.

An individual has constraints

$$\begin{aligned} pc &= wh + d + m + T - m' \\ pc &\leq m + T, \end{aligned}$$

since c must be financed with cash brought into the period, not current receipts. Eliminating c using the budget condition as we did above, the CIA constraint becomes $m' \geq wh + d$, which says that nominal income earned in one period must be held as money into the next period. As above, we can write the constraint in real terms as $\frac{m'}{p} \geq \frac{w}{p}h + \frac{d}{p}$, and Bellman's equation becomes

$$\begin{aligned} V(m) &= \max_{m',h} \left\{ u\left(\frac{wh+d+m+T-m'}{p}, 1-h\right) \right. \\ &\quad \left. +\lambda\left(\frac{m'-wh-d}{p}\right) + \beta V(m') \right\}. \end{aligned}$$

The first order conditions are

$$\begin{aligned} -\frac{u_1(c,1-h)}{p} + \frac{\lambda}{p} + \beta V'(m') &= 0 \\ \frac{w}{p}u_1(c,1-h) - u_2(c,1-h) - \frac{w}{p}\lambda &= 0. \end{aligned}$$

Using $V'(m) = u_1(c,1-h)/p$, we can write the first of these as

$$u_1(c,1-h) - \lambda = \frac{\beta}{\pi}u_1(c',1-h').$$

The obvious equilibrium conditions for competitive firms imply $w = pf'(h)$ and $d = pf(h) - wh$, and then the budget equation implies $c = f(h)$; hence we can write the other first order condtion as

$$f'(h)\left[u_1(c,1-h) - \lambda\right] = u_2(c,1-h),$$

where $c = f(h)$. Combing these two equations, we have

$$u_2(c,1-h) = \frac{\beta}{\pi}u_1(c',1-h')f'(h). \tag{2}$$

Things are very much analogous to the previous model, except that now the relevant trade off can be said to be between leisure today and consumption tomorrow, since working today yields money income that must be carried over one period before it can be turned into consumption. In particular, in a steady state we must have $\pi \geq \beta$. Also, efficiency implies $\pi = \beta$.

One difference from the previous model is that the relative price of leisure, the real wage, is endogenous here since $w = pf'(h)$, while the relative price of credit goods in terms of cash was fixed at 1. Of course, one could augment the previous model, say by assuming $c_1 = g(c_2)$ instead of the extreme case $c_1 = e - c_2$. In any case, having the price endogenous complicates things only a little. For example, $\partial h / \partial \pi$ now takes the same sign as

$$\beta u_1 f'' + \beta f'^2 u_{11} - (\beta + \pi) f' u_{12} + \pi u_{22},$$

where the first term accounts for the equilibrium change in wages while the rest of the terms are analogous to the expression for $\partial c_2 / \partial \pi$. Hence, at least at the optimum $\pi = \beta$, we know that hours are decreasing in π. Moreover, h is maximized at $\pi = \beta$ (not only locally, but globally, by the same argument that told us c_2 is minimized globally at $\pi = \beta$).

One reason to interpret the credit good as leisure is that we can bring firms into the picture in a more interesting way. In particular, suppose we change the model by saying that consumers do not have a CIA constraint at all, but firms do. Thus, consumers maximize subject only to the budget equation $pc = hw + d$ at every date. In particular, notice that since consumers are not constrained to hold money, they never do, so m does not appear in their problem.[9] Hence, workers solve a sequence of static problems, the solution to which is characterized by the first order condition $\frac{w}{p} u_1(c, 1-h) = u_2(c, 1-h)$ at each date. Since $c = f(h)$ in equilibrium, this condition can be solved for h at each point in time. Of course, given an arbitrary interest rate consumers might also like to save or borrow so as to smooth consumption over time, but this will not be possible in equilibrium, because we have homogeneous agents. As usual, interest rates will adjust so that $u_1(c, 1-h) = \beta R' u_1(c', 1-h')$, and no lending or borrowing takes place, which means that we can solve the

[9] Also, here we give all transfers of new money to firms, so T does not appear in the consumer problem either. This is merely for notational convenience, however; it would be equivalent to give these transfers to consumers, who would simply spend them in each period.

consumer problem as described above.

The firm problem, however, now explicitly contains dynamic elements due to their CIA constraint. As is standard, we assume that they maximize the (real) present discounted value of their dividend stream, $\sum \Delta_t \frac{d_t}{p_t}$, where $\Delta_t = 1/(R_1 \cdots R_t)$ is the relative price of a unit of consumption at t in terms of date 0 goods, under the interpretation of R_t as the gross interest rate on a one-period bond maturing at t. Firms are subject to two constraints. First, $d = pf(h) - wh + m + T - m'$, since dividends must come from either profits or adjustments to their money inventories. Second, they face the CIA constraint $wh \leq m + T$, which says that they must pay nominal wages out of cash on hand at the start of the period (and not out of cash receipts from sales within the period). Writing the CIA constraint in real terms, Bellman's equation becomes

$$V(m) = \max_{m',h} \left\{ \frac{pf(h) - wh + m + T - m'}{p} + \lambda \left(\frac{m + T - wh}{p} \right) + \frac{1}{R'} V(m') \right\}$$

where the effective discount factor is $\Delta'/\Delta = 1/R'$.

The first order conditions imply

$$\begin{aligned} -\frac{1}{p} + \frac{V'(m')}{R'} &= 0 \\ f'(h) - \frac{w}{p} - \lambda \frac{w}{p} &= 0. \end{aligned}$$

As $V'(m) = (1 + \lambda)/p$, the first equation can be written $\pi R' = 1 + \lambda'$, which again implies $\pi R' \geq 1$ as a necessary condition for equilibrium. The other first order condition can now be rearranged as

$$f'(h') = \frac{w'}{p'}(1 + \lambda') = \frac{w'}{p'} \pi R'.$$

From the consumers problem, $\frac{w'}{p'} = \frac{u_2(c', 1-h')}{u_1(c', 1-h')}$ and $R' = \frac{u_1(c, 1-h)}{\beta u_1(c', 1-h')}$. Hence, equilibrium satisfies

$$u_2(c', 1 - h') = \frac{\beta}{\pi} \frac{u_1(c', 1 - h')^2}{u_1(c, 1 - h)} f'(h'). \tag{3}$$

Notice that this is different from the equilibrium condition when CIA is imposed on consumers, given by (2).

The intuition is as follows. In the case of CIA on consumers the relevant trade off for an individual is between leisure today and consumption tomorrow, by analogy to the trade off between credit goods today and cash goods tomorrow depicted in Figure 1. The MRS is $\frac{u_2(c,1-h)}{\beta u_1(c',1-h')}$, while the MRT is $\frac{f'(h)}{\pi}$ since by giving up a unit of leisure one acquires $f'(h)$ units of output today that can be used to buy $1/\pi$ units of consumption tomorrow. Equating MRS and MRT yields (2). With CIA on firms, however, the relevant trade off is between consumption today and consumption tomorrow. The MRS is $\frac{u_1(c,1-h)}{\beta u_1(c',1-h')}$, while the MRT is $\frac{u_1(c',1-h')}{u_2(c',1-h')}\frac{f'(h')}{\pi}$, since by giving up one unit of consumption today through lower real dividends, a shareholder allows the firm to hire $\frac{u_1(c',1-h')}{u_2(c',1-h')}\frac{1}{\pi}$ units of labor next period, which then yields $f'(h')$ units of output. Equating MRS and MRT in this case yields (3). Despite all this, it is quite interesting that the steady state implications are identical, since in steady state (2) and (3) reduce to the same thing.

4 Investment and Growth

We want to consider economies with capital. A simple set up is provided by the model with a linear technology, $y = f(k) = Bk$ (we ignore labor for the moment), which is interesting because it can generate long run growth. Also, let $A = B + 1 - \delta$, where δ is the depreciation rate on capital, so that feasibility can be written $c = f(k) + (1-\delta)k - k' = Ak - k'$. As a benchmark, let us review the case where there is no money. Then the equilibrium will be efficient and solve the planner's problem described by

$$V(k) = \max_{k'} \left\{ u\left(Ak - k'\right) + \beta V(k') \right\}.$$

Assuming an iterior solution (we will check this below), the first order condition is $u'(Ak-k') = \beta V'(k')$, the envelope condition is $\beta V'(k) = Au'(Ak-k')$, and so the Euler equation is

$$u'(c) = \beta A u'(c').$$

Since $c = Ak - k'$, this is a second order difference equation in k, which has many solutions given the single initial condition k_0. The unique optimal path for k is the solution that also satisfies the usual transversality condition (TVC), which here we write as $\lim_{T\to\infty} \beta^T u'(Ak_T - k_{T+1})k_{T+1} = 0$.

We are interested in a balanced growth path – that is, a solution to the above problem where k grows at a constant rate: $k' = \gamma k$, which implies $c' = \gamma c$. It is well known that this does not obtain for arbitrary preferences, and only in the case of constant relative risk aversion (CRRA), $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. Given this utility function, and given balanced growth, the Euler equation becomes

$$c^{-\sigma} = \beta A(\gamma c)^{-\sigma},$$

which simplifies to $\gamma = (\beta A)^{1/\sigma}$. Hence, the Euler equation is satisfied by the balanced growth path $k' = \gamma k$ with $\gamma = (\beta A)^{1/\sigma}$. We have $k_t = \gamma^t k_0$ for all t, and the net growth rate is positive if $\gamma > 1$, which means $\beta A > 1$. The TVC holds if $\beta\gamma < 1$, which means $\beta^{1+\sigma} A > 1$ (notice that this implies $c = Ak - k' > 0$ automatically). So, we simply assume that $\beta < A^{-1/(1+\sigma)}$, so that TVC holds (as is nonnegativity), and then observe that there are still two cases: $\beta < A^{-1}$ implies $\gamma < 1$, so $k \to 0$, and $\beta > A^{-1}$ implies $\gamma > 1$, so k grows without bound.[10]

Now consider the same economy with a CIA constraint on consumption, but for now not investment. Using the fact that the equilibrium interest rate (from the profit maximization condition) must be $r = B$, so that $[r + (1 - \delta)]k = Ak$, the budget and cash constraints are

$$\begin{aligned} pc &= pAk - pk' + m + T - m' \\ pc &\leq m + T. \end{aligned}$$

[10]Notice that in this model we converge to the balanced growth path immediately – that is, $k_{t+1} = \gamma k_t$ from the very first period.

As usual, we combine these to write CIA as $m' \geq p(Ak - k')$. Hence, we have

$$V(k,m) = \max_{k',m'} \left\{ u\left(Ak - k' + \frac{m + T - m'}{p}\right) + \lambda\left(\frac{m'}{p} + k' - Ak\right) + \beta V(k', m') \right\}.$$

The first order conditions are $u'(c) - \lambda = \beta V_1(k', m')$ and $u'(c) - \lambda = p\beta V_2(k', m')$, the envelope conditions are $V_1(k,m) = Au'(c) - A\lambda$ and $V_2(k,m) = u'(c)/p$, and so the Euler equations are

$$\begin{aligned} u'(c) - \lambda &= \beta A\left[u'(c') - \lambda'\right] \\ u'(c) - \lambda &= \frac{\beta}{\pi} u'(c'). \end{aligned}$$

Togther these imply $(\pi A - 1)u'(c') = \pi A \lambda'$, and so we see that $\pi A \geq 1$ is necessary for equilibrium. Recalling that $A = r$, this is the same as the condition we saw in the other models, $\pi r \geq 1$.

Solving the second Euler equation for λ and inserting into the first yields $\frac{\beta}{\pi} u'(c') = \beta A \frac{\beta}{\pi'} u'(c'')$, or

$$u'(c') = \beta A \frac{\pi}{\pi'} u'(c'').$$

Hence, at least if π is constant, we have $u'(c) = \beta A u'(c')$ at every date, which is identical to the equilibrium condition from the model with no CIA constraint. That is, given CRRA utility, equilibrium in the model with CIA is given by the same balanced growth path we had without CIA, with $\gamma = (\beta A)^{1/\sigma}$. Moreover, given $M' = \mu M$, the CIA condition at equality implies $c = \mu M/p = \mu z$. So on the balance growth path $z' = \gamma z$, and therefore $\pi = \mu/\gamma$ (the inflation rate is the rate of monetary expansion minus the economic growth rate). This means that the condition $\pi A \geq 1$ becomes $\mu \geq \beta^{1/\sigma} A^{(1-\sigma)/\sigma}$. This puts a bound on how fast we can contract the money supply, but otherwise there are no implications for policy: as long as μ satisfies this condition, the equilibrium in independent of μ and identical to the model without CIA.

Stockman pointed out that the above very strong implication follows because we have CIA on consumption only (although he did so in a different model, with a steady state rather than a balanced growth path, which we analyze below). Consider imposing CIA on consumption and investment: $pc + p(k' - k) \leq m + T$. As usual, we use the budget equation to write the CIA constraint as $\frac{m'}{p} \geq Ak - k$, which contrasts with the previous model with CIA only on consumption, where $\frac{m'}{p} \geq Ak - k'$. We assume $A > 1$.[11] The Bellman equation is

$$V(k, m) = \max_{k', m'} \left\{ u\left(Ak - k' + \frac{m + T - m'}{p}\right) + \lambda\left(\frac{m'}{p} + k - Ak\right) + \beta V(k', m')\right\}.$$

The first order conditions are $u'(c) = \beta V_1(k', m')$ and $u'(c) - \lambda = p\beta V_2(k', m')$, the envelope conditions are $V_1(k, m) = Au'(c) - (A-1)\lambda$ and $V_2(k, m) = u'(c)/p$, and so the Euler equations are

$$\begin{aligned} u'(c) - \beta\lambda' &= \beta A\left[u'(c') - \lambda'\right] \\ u'(c) - \lambda &= \frac{\beta}{\pi} u'(c'). \end{aligned}$$

The second Euler equation is the same as in the model with CIA on c only, but the first Euler equation is different. Eliminating λ now leads to

$$u'(c') = \beta u'(c') + \frac{\beta^2(A-1)}{\pi} u'(c'').$$

Assuming CRRA utility and balanced growth, this can be written

$$c^{-\sigma} = \beta(\gamma c)^{-\sigma} + \frac{\beta^2(A-1)}{\pi}\left(\gamma^2 c\right)^{-\sigma},$$

which is a quadratic in γ^σ and has solution

$$\gamma^\sigma = \frac{\beta}{2}\left[1 + \sqrt{1 + 4(A-1)/\pi}\right].$$

Hence, the growth rate is decreasing in π. As above, $\pi = \mu/\gamma$, which in principle we could insert into this equation to solve for γ as a function of μ;

[11] Recalling that $A = B + 1 - \delta$, CIA implies here that $(B - \delta)k$, and so the assumption is $B > \delta$.

alternatively, we could think of the policy as the direct choice of π, which implies γ and then μ endogensouly. In any case, recalling that the equilibrium without CIA is $\gamma^\sigma = \beta A$, the efficient policy can be seen to be $\pi A = 1$ or $\mu = \beta^{1/\sigma} A^{(1-\sigma)/\sigma}$. Moreover, CRRA implies $\lambda = c^{-\sigma}(1 - \gamma^{-\sigma}\beta/\pi)$, or $\pi \geq \beta\gamma^{-\sigma}$, which combines with the above solution for γ to yield $\pi A \geq 1$ or $\mu \geq \beta^{1/\sigma} A^{(1-\sigma)/\sigma}$ in any equilibrium, analogous to what we found earlier.

The bottom line is when we impose CIA on consumption only then the rate of monetary expansion μ is neutral, but when we impose CIA on consumption plus investment μ affects the allocation. We leave is as an exrcise to show that if we have CIA on investment only then the outcome is the same as CIA on consumption plus investment (not too surprsingly, given that CIA on consumption only did not matter). Intuitively, CIA on consumption only cannot be avoided, since the only alternative to consumption is investment, and investment merely creates future output that still cannot be consumed without holding cash for one period. Hence, CIA on consumption does not distort any decisions. By contrast, CIA on investment does distort decisions, since one can aviod it by consuming more and saving less. As usual, in this case the optimal policy is the Friedman rule, which is to deflate so that $\pi A = 1$, which effectively undoes the CIA constraint.

Of course, if we impose CIA only on consumption, but assume credit goods or leisure that are not subject to CIA as well as cash goods that are, policy will matter in this model, just as it did in the models without capital. For instance, suppose $u = u(c, 1-h)$ and $y = A(h)k$. Notice that since this technology displays increasing returns, we do not assume agents rent labor and capital in competitive factor markets here, because profit maximization will not be well defined. Rather, we assume agents operate their own individual technologies.[12] Their resource constraint is $c =$

[12] To motivate this, consider a static (nonmonetary) economy. If competitive firms maximize profit $A(h)k - wh - rk$, the second order conditions fail. However, if individuals maximize $V = u[A(h)k - k', 1-h] + \beta v(k')$, where here v is an (exogenous) increasing and concave function, the problem is well behaved. To see this, note that the first order

$A(h)k - k' + (m+T-m')/p$, and for now we impose CIA only on consumption, $pc \le m+T$, which becomes $m'/p \ge A(h)k - k'$ using the resource constraint. Hence, we have

$$\begin{aligned} V(k,m) &= \max_{k',m',h} \left\{ u\left[A(h)k - k' + \frac{m+T-m'}{p}, 1-h \right] \right. \\ &\quad \left. + \lambda \left[\frac{m'}{p} + k' - A(h)k \right] + \beta V(k',m') \right\}. \end{aligned}$$

The usual methods yield a first order condition for h and two Euler equations for k and m, which we write as

$$\begin{aligned} u_1(c,1-h) - \lambda &= u_2(c,1-h)/kA'(h) \\ u_1(c,1-h) - \lambda &= \beta A(h') \left[u_1(c',1-h') - \lambda' \right] \\ u_1(c,1-h) - \lambda &= \frac{\beta}{\pi} u_1(c',1-h'). \end{aligned}$$

Proceeding as always, the two Euler equations imply $\lambda' = u_1(c', 1 - h')[\frac{1}{\pi A(h')} - 1]$, and so $\pi A(h') \ge 1$ in any equilibrium. By eliminating λ the Euler equations also imply

$$u_1(c',1-h') = \beta A(h') \frac{\pi}{\pi'} u_1(c'',1-h''),$$

which is of the same form as the equilibrium condition in the basic model with CIA on c only; in particular, if π is constant we have $u_1(c,1-h) = \beta A(h) u_1(c',1-h')$. Therefore, investment is efficient if and only if h is at its efficient level, h^*. However, the above also imply

$$u_2(c,1-h) = kA'(h) \frac{\beta}{\pi} u_1(c',1-h'),$$

which generally distorts the labor decision here for exactly the same reason it does in the model without capital – recall (2). We leave it as an exercise

conditions are $V_1 = -u_1 + \beta v' = 0$ and $V_2 = u_1 kA' - u_2 = 0$, and the second order conditions are $V_{11} = u_{11} + \beta v'' < 0$, $V_{22} = (kA')^2 u_{11} - 2(kA') u_{12} + u_{22} < 0$, and (after simplification) $V_{11}V_{22} - V_{12}^2 = u_{11}u_{22} - u_{12}^2 + \beta v'' V_{22} > 0$.

to work out equilibrium for the case where preferences satisfiy the balanced growth conditions for an economy with endogenous labor – that is, $c' = \gamma c$ and h constant – which means that either $u = \frac{c^{1-\sigma}}{1-\sigma}v(1-h)$, or $u = \log(c) + v(1-h)$, for some well behaved function v – and to show that the Friedman rule is once again efficient.

One can consider other combinations of the above models. For instance, consider endogenous labor with CIA on investment only, $\frac{m+T}{p} \geq k' - k$. The usual methods now lead to the equilibrium conditions

$$\begin{aligned}
u_1(c, 1-h) &= u_2(c, 1-h)/kA'(h) \\
u_1(c, 1-h) + \lambda &= \beta \left[A(h')u_1(c', 1-h') + \lambda'\right] \\
u_1(c, 1-h) &= \frac{\beta}{\pi} u_1(c', 1-h').
\end{aligned}$$

Or, consider cash and credit goods in the growth model. The equilibrium conditions are now

$$\begin{aligned}
u_1(c_1, c_2) - \lambda &= u_2(c_1', c_2') \\
u_1(c_1, c_2) - \lambda &= \beta A(h') \left[u_1(c_1', c_2') - \lambda'\right] \\
u_1(c_1, c_2) - \lambda &= \frac{\beta}{\pi} u_1(c_1', c_2').
\end{aligned}$$

We leave it as an exercise to find γ under the assumption that preferences generate balanced growth, to explore the effect of an increase in π on γ and h, and to show that the Friedman rule continues to be efficient in these models.[13]

One could also work with a more standard production function, say $y = f(k, h)$, as much of the literature does. For example, suppose we impose CIA on consumpton only, $pc \leq m + T$. The budget constraint is $pc = wh + p(r + 1 - \delta)k + d - pk' + m + T - m'$, where d represents (nominal)

[13] Note that the balanced growth conditions are different in these two cases: with endogenous labor, we want $c' = \gamma c$ and $h' = h$; with cash and credit goods, we would presumably want $c_1' = \gamma c_1$ and $c_2' = \gamma c_2$.

dividends that the firm pays out each period (they will be 0 in equilibrium if we assume constant returns). This leads to

$$\begin{aligned} V(k,m) &= \max_{k',m',h} \left\{ u\left[\frac{wh+d+m+T-m'}{p} + (r+1-\delta)k - k', 1-h\right] \right. \\ &\quad \left. +\lambda\left[\frac{m'-wh-d}{p} - (r+1-\delta)k + k'\right] + \beta V(k',m') \right\}. \end{aligned}$$

Deriving the Euler equations and inserting the factor prices from profit maximization, $w/p = f_2(k,h)$ and $r = f_1(k',h')$, we have the equilibrium conditions

$$\begin{aligned} u_1(c,1-h) - \lambda &= u_2(c,1-h)/f_2(k,h) \\ u_1(c,1-h) - \lambda &= \beta\left[f_1(k',h') + 1 - \delta\right]\left[u_1(c',1-h') - \lambda'\right] \\ u_1(c,1-h) - \lambda &= \frac{\beta}{\pi} u_1(c',1-h'). \end{aligned}$$

Letting $R' = f_1(k',h') + 1 - \delta$, these imply $\lambda = u_1(R'\pi - 1)/R'\pi$, which means $R'\pi \geq 1$ in any equilibrium. As we will soon see, in a steady state $R = 1/\beta$, and so this says we must have $\pi \geq \beta$ in steady state.

We can eliminate λ to reduce the model to the following two equations in two unknowns:

$$u_2(c,1-h) = \frac{\beta}{\pi} f_2(k,h) u_1(c,1-h) \quad (4)$$

$$u_1(c,1-h) = \beta R' u_1(c',1-h'). \quad (5)$$

These conditions must hold at every date along any equilibrium path, along with $c = f(k,h) + (1-\delta)k - k'$.[14] In a steady state, $k = k'$ and so $c = f(k,h) - \delta k$. Moreover, in steady state, the second equation becomes $\beta R' = 1$, so we have $R' = 1/\beta$, or $f_1(k,h) = \rho + \delta$ where ρ is given by $\beta = \frac{1}{1+\rho}$. For any

[14] Using the conditions, $m = M$ and $M + T = M'$, we have $c = \frac{w}{p}h + (r+1-\delta)k + \frac{d}{p} - k' = f(k,h) + (1-\delta)k - k'$ since $\frac{d}{p} = f(k,h) - \frac{w}{p}h - rk$.

$h > 0$ we can solve $f_1(k,h) = \rho + \delta$ for a unique $k = k(h)$, at least as long as we assume the usual curvature assumptions , $f_1(0,h) = \infty$ and $f_1(\infty,h) = 0$. While we cannot solve for $k(h)$ explicity, we do know $k'(h) = -f_{12}/f_{11}$. Writing $c(h) = f[k(h),h] - \delta k(h)$, this allows us to reduce the equilibrium conditions to one equation, $T(h) = 0$, where

$$T(h) = f_2[k(h),h]u_1\left[c(h), 1-h\right] - \pi(1+\rho)u_2\left[c(h), 1-h\right].$$

The usual curvature assumptions guarantee $T(0) > 0$ and $T(1) < 0$. Hence, there always exists a steady state level of h and $k = k(h)$. One can compute

$$\begin{aligned} T'(h) &= \frac{u_1}{f_{11}}\left(f_{11}f_{22} - f_{12}^2\right) - \frac{\rho f_2 f_{12}}{u_2 f_{11}}\left(u_2 u_{11} - u_1 u_{12}\right) \\ &\quad + \frac{f_2^2}{u_2}\left(u_2 u_{11} - u_1 u_{12}\right) + \frac{f_2}{u_2}\left(u_1 u_{22} - u_2 u_{21}\right) \end{aligned}$$

at any solution to $T(h) = 0$ (we have inserted $f_2 = \frac{\pi}{\beta}\frac{u_2}{u_1}$ and also $f_1 - \delta = \rho$). In general, the sign of this expression is ambiguous, and so we cannot guarantee a unique steady state. Of course functional form assumptions, such as $f_{12} \geq 0$ and $u_{12} \geq 0$, can be imposed to guarantee $T'(h) < 0$ and therefore uniqueness. In fact, if consumption and leisure are normal goods then $u_1 u_{22} - u_2 u_{21} \leq 0$ and $u_2 u_{11} - u_1 u_{12} \leq 0$, respectively, and so normality is the only assumption on preferences that we really need to guarantee $T' < 0$, at least as long as $f_{12} \geq 0$. In fact, a weaker condition than $f_{12} \geq 0$ will do: given that c and $1-h$ are normal goods, a sufficient condition for $T' < 0$ is

$$f_2 f_{11} \leq \rho f_{12}, \tag{6}$$

where $\rho = f_1 - \delta > 0$ from the steady state condition.[15]

[15] Group terms as

$$\begin{aligned} T'(h) &= \frac{u_1}{f_{11}}\left(f_{11}f_{22} - f_{12}^2\right) + \frac{f_2}{u_2}\left(u_1 u_{22} - u_2 u_{21}\right) \\ &\quad + \left(f_2 f_{11} - \rho f_{12}\right)\frac{f_2}{u_2 f_{11}}\left(u_2 u_{11} - u_1 u_{12}\right). \end{aligned}$$

Since T obviously shifts down with π, whenever there is a unique equilibrium we can be sure that $\partial h/\partial\pi < 0$. Hence, the above restrictions (e.g., normal goods and $f_{12} \geq 0$) that guarantee uniqueness also guarantee hours are decreasing with inflation – naturally, since this is simply substitution of of the cash good c into the credit good $1-h$. In general, of course, there can be multiple equilibria and in this case the effect of π on h is different in every alternate solution to $T(h) = 0$. But in any case, even if h is nonmonotonic in π, we know that h is globally maximized at $\pi = \beta = 1/R$ (if there are multiple steady states at $\pi = \beta$, pick the one with the highest h). Once one knows $\partial h/\partial\pi$, we have $\partial k/\partial\pi = k'(h)\partial h/\partial\pi = -\frac{f_{12}}{f_{11}}\partial h/\partial\pi$. If $f_{12} > 0$, for example, then with normal goods we know $\partial h/\partial\pi < 0$ and therefore $\partial k/\partial\pi < 0$; with $f_{12} < 0$, however, h and k will move in opposite directions when π increases. Similarly

$$\frac{\partial c}{\partial\pi} = \frac{f_2 f_{11} - \rho f_{12}}{f_{11}} \frac{\partial h}{\partial\pi}.$$

With normal goods and the condition in (6), we know $\partial c/\partial\pi < 0$.[16]

We also point out that if h is exogenously set at $h = 1$ – i.e., leisure

Then normal goods implies $T' < 0$ if $f_2 f_{11} - \rho f_{12} \leq 0$, as claimed. We also note that this discussion of uniqueness applies to an arbitrary value of π. If we assume $\pi = \beta$ then we can rearrange as

$$\begin{aligned} T'(h) &= \frac{u_1}{f_{11}}\left(f_{11}f_{22} - f_{12}^2\right) + \left(\frac{u_2}{u_1}\right)^2 u_{11} - 2\frac{u_2}{u_1}u_{12} + u_{22} \\ &\quad - \frac{\rho f_{12}}{u_1 f_{11}}\left(u_2 u_{11} - u_1 u_{12}\right). \end{aligned}$$

Now only the final terms is ambiguous, and it is negative if $f_{12} \geq 0$ and leisure is not inferior. So, setting $\pi = \beta$ does not help us here as much as it did in previous models, but it helps a little.

[16] One can pursue things further. Thus,

$$\frac{\partial}{\partial\pi}\frac{w}{p} = -f_{11}(f_{11}f_{22} - f_{12}^2)\frac{\partial h}{\partial\pi},$$

and so the real wage move in the same direction as h if f exhibits decreasing returns and is invariant to changes in π with constant returns, since that implies $f_{11}f_{22} - f_{12}^2 = 0$. Also, one can easy show that $\partial r/\partial\pi = 0$, so the rental rate is always invariant to π.

does not enter utility – then (4) is irrelevant and the remaining equilibrium condition is (5), which we rewrite here as

$$u_1(c) = \beta \left[f_1(k') + 1 - \delta \right] u_1(c')$$

where we abuse notation slightly by writing $f(k, 1) = f(k)$, and $c = f(k) + (1-\delta)k - k'$. This is exactly the same condition found in the standard model with no CIA constraint, and so the equilibrium is invariant to changes in π. This is not surprising, given what we found in the linear model with CIA on consumption only. As we pointed out in another model, things change if we alternatively impose CIA on consumption plus investment; that is $p(c + k' - k) \leq m + T$.[17] The usual methods in this case (still assuming $h = 1$) now lead to

$$u_1(c) = \beta u_1(c') + \frac{\beta^2}{\pi} \left[f_1(k') - \delta \right] u_1(c'').$$

At steady state, $1 - \beta = \frac{\beta^2}{\pi} \left[f_1(k) - \delta \right]$. Clearly, $\partial k / \partial \pi < 0$.

[17] Things are the same is we impose CIA on just investment.