Chapter 9

Bargaining Theory

In this section we present the basics of bilateral bargaining theory. We begin by discussing the axiomatic model of Nash (1950). We then describe a simple version of the strategic model developed by Rubinstein (1982). We then generalize the simple strategic model, and analyze the relationship in this general context between the Nash and Rubinstein solutions. Finally, we examine bargaining in nonstationary environments, including cases where preferences are changing over time.

9.1 An Axiomatic Bargaining Model

Consider two agents, labelled $j = 1, 2$, trying to come to an agreement over alternatives in some arbitrary set $\mathcal{A}$. Each agent j has a von Neumann - Morgenstern utility function u_j defined over $\mathcal{A} \cup \{D\}$, where D represents the outcome if they fail to reach an agreement. From these, we can construct the set of all utility pairs that result from some agreement,

$$\mathcal{S} = \{[u_1(a), u_2(a)] \subset \mathcal{R}^2 : a \in \mathcal{A}\},$$

as well as the pair $d = (d_1, d_2)$, where $d_j = u_j(D)$ is referred to as the *disagreement point* or *threat point*.

Following Nash (1950), we take the pair $(\mathcal{S}, d)$ to define a bargaining problem, and assume that $\mathcal{S}$ is compact and convex, that $d \in \mathcal{S}$, and that for some $s \in \mathcal{S}$ we have $s_j > d_j$. We are interested in a *bargaining solution*, by which we mean a function f that specifies a unique outcome $f(\mathcal{S}, d) \in \mathcal{S}$ for every bargaining problem $(\mathcal{S}, d)$. Rather than specifying an explicit model of the bargaining procedure, the idea behind the axiomatic approach is to impose properties that one wants a bargaining solution to satisfy, and then look for solutions with these properties.

Nash specifies four axioms, which we simply state without comment (see Osborne and Rubinstein [1990], e.g., for a discussion).

A1. Invariance to equivalent utility representations: If we transform a bargaining problem $(\mathcal{S}, d)$ into $(\mathcal{S}', d')$ by taking $s'_j = \alpha_j s_j + \beta_j$ and $d'_j = \alpha_j d_j + \beta_j$, where $\alpha_j > 0$, then $f_j(\mathcal{S}', d') = \alpha_j f_j(\mathcal{S}, d) + \beta_j$.

A2. Symmetry: If the bargaining problem is symmetric, in the sense that $d_1 = d_2$ and $(s_1, s_2) \in \mathcal{S} \Longleftrightarrow (s_2, s_1) \in \mathcal{S}$, then $f_1(\mathcal{S}, d) = f_2(\mathcal{S}, d)$.

A3. Independence of irrelevant alternatives: If $(\mathcal{S}, d)$ and $(\mathcal{S}', d)$ are bargaining problems with $\mathcal{S} \subset \mathcal{S}'$, and $f(\mathcal{S}', d) \in \mathcal{S}$, then $f(\mathcal{S}, d) = f(\mathcal{S}', d)$.

A4. Pareto efficiency: If $(\mathcal{S}, d)$ is a bargaining problem with $s, s' \in \mathcal{S}$ and $s'_j > s_j$, $j = 1, 2$, then $f(\mathcal{S}, d) \neq s$.

Nash shows that there exists a unique bargaining solution satisfying these axioms, and it takes the simple form

$$f(\mathcal{S}, d) = \arg\max \, (s_1 - d_1)(s_2 - d_2), \tag{9.1}$$

where the maximization is over $s \in \mathcal{S}$, and is subject to the constraints $s_j \geq d_j$, $j = 1, 2$. The maximand on the right hand side of (9.1) is called the *Nash product*, and the solution f is called the *Nash bargaining solution*.

To sketch Nash's proof, first scale utility by additive constants so that $d_1 = d_2 = 0$. Now the graph of the set $\mathcal{S}$ in (u_1, u_2) space contains the origin in its interior. See Figure 1. Since $\mathcal{S}$ is compact and convex, there exists a unique point (u_1^*, u_2^*), with $u_j^* > 0$, that maximizes $u_1 u_2$ over $\mathcal{S}$. Indeed, we can always scale utility by multiplicative constants so that $(u_1^*, u_2^*) = (u^*, u^*)$

is on the 45^o line, as shown in the figure. We wish to show that $f(\mathcal{S}, 0) = (u^*, u^*)$, since that is exactly the assertion in (9.1), given our normalizations.

We take an indirect route to this end. First, draw a line through (u^*, u^*) with slope -1. The box $\mathcal{B}$ shown in the figure has this line as one side, contains $\mathcal{S}$, and is symmetric around the 45^o line. Consider the bargaining problem $(\mathcal{B}, 0)$. Since this problem is symmetric, the solution $f(\mathcal{B}, 0)$ lies on the the 45^o line by Assumption A2. Furthermore, $f(\mathcal{B}, 0)$ lies on the frontier of $\mathcal{B}$ by Assumption A4. Therefore $f(\mathcal{B}, 0) = (u^*, u^*)$. This shows that the solution to bargaining problem $(\mathcal{B}, 0)$ solves $\max u_1 u_2$. But, by Assumption A3, $f(\mathcal{S}, 0) = f(\mathcal{B}, 0)$, and the proof is complete.

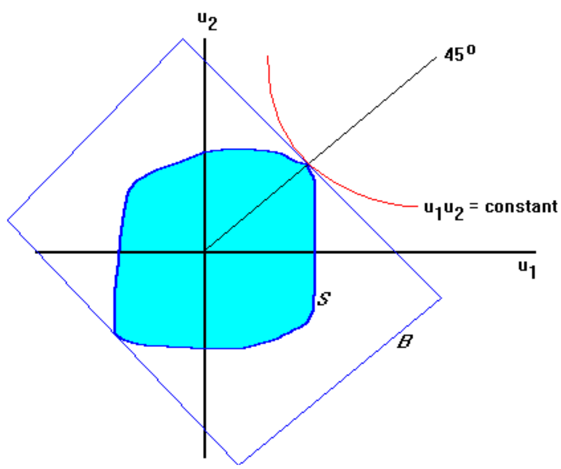

Figure 9.1: Nash's Axiomatic Bargaining Solution

To make things more concrete, suppose that the agents are negotiating over how to divide a "pie" of size 1. Let q be the share going to agent 1, and (using Pareto efficiency) $1 - q$ the share going to agent 2. Let the utility of agent j be given by $u_j(q)$, where u_1 is increasing and u_2 is decreasing (since q is the share going to agent 1). Assume u_j is concave. Let d_j denote the utility of j if they never split the pie. Then the Nash solution is

$$q = \arg\max\ [u_1(q) - d_1][u_2(q) - d_2], \tag{9.2}$$

subject to $q \in [0,1]$. For example, if $u_1 = q$, $u_2 = 1-q$, and $d_1 = d_2$, then $q = \frac{1}{2}$ and they split the pie right down the middle.

If we drop axiom A.2 (symmetry), one can show that for any $\theta \in (0,1)$ there is a bargaining solution f_θ satisfying the other three axioms, given by

$$f_\theta(\mathcal{S}, d) = \arg\max\, (s_1 - d_1)^\theta (s_2 - d_2)^{1-\theta}, \tag{9.3}$$

where again the maximization is subject to $s_j \geq d_j$, $j = 1, 2$ (see Osborne and Rubinstein [1990] and the references contained therein). We call f_θ the *generalized* Nash solution, and θ the *bargaining power* of agent 1. Of course, if $\theta = \frac{1}{2}$ then (9.3) reduces to (9.1). In the limit, as $\theta \to 1$ the solution maximizes s_1 subject to $s_2 \geq d_2$, and as $\theta \to 0$ the solution maximizes s_2 subject to $s_1 \geq d_1$.

In the pie-splitting example, (9.3) becomes

$$q = \arg\max\, [u_1(q) - d_1]^\theta [u_2(q) - d_2]^{1-\theta}, \tag{9.4}$$

which is characterized by the first order condition

$$\theta[u_2(q) - d_2]u_1'(q) + (1-\theta)[u_1(q) - d_1]u_2'(q) = 0, \tag{9.5}$$

subject to $u(q_j) \geq d_j$. For example, let $u_1 = q$ and $u_2 = 1 - q$, and define the *surplus* as the total available utility over and above the total utility generated by disagreement, $1 - d_1 - d_2$. Then (9.5) implies

$$\begin{aligned} q &= d_1 + \theta\,(1 - d_1 - d_2)\,, \\ 1 - q &= d_2 + (1-\theta)(1 - d_1 - d_2). \end{aligned}$$

Hence, each agent gets his threat point plus a share of the surplus corresponding to his bargaining power. Based on this, the Nash solution is sometimes described as "splitting the surplus."

It is perhaps worth emphasizing that nothing requires the utility functions to be defined over one dimensional arguments (as in the pie-splitting example). Suppose, for instance, that agent 1 is a worker and agent 2 a firm, and they are bargaining over both compensation and hours, and write the payoffs as $u_i = u_i(c, h)$. Assuming concavity, the generalized Nash problem (9.3) has a solution characterized by the first order conditions

$$\theta[u_2(c,h) - d_2]\frac{\partial u_1}{\partial c} + (1-\theta)[u_1(c,h) - d_1]\frac{\partial u_2}{\partial c} = 0,$$

$$\theta[u_2(c,h) - d_2]\frac{\partial u_1}{\partial h} + (1-\theta)[u_1(c,h) - d_1]\frac{\partial u_2}{\partial h} = 0.$$

In particular, the ratio of these two equations implies that the solution is on the "contract curve" in (c, h) space. For example, if $u_2(c, h) = F(h) - c$, where F is the production function, F' is equated to the worker's marginal rate of substitution.

9.2 A Strategic Bargaining Model

The Nash bargaining model is inherently static, or timeless, in the sense that only the outcome, and not the bargaining procedure, is analyzed. This conveys some advantages, such as tractability. There are circumstances, however, in which it is useful to look into the bargaining process. For instance, we may be interested in knowing how outcomes are affected by changes in procedure.

Moreover, in applications (especially when the bargaining problem is embed into an equilibrium model), it is not always obvious what form the Nash solution should take. In many interesting situations there can be ambiguity with respect to the appropriate threat points or bargaining power. As emphasized by Binmore et al. (1986), one can get guidance on these issues by studying explicit models of strategic bargaining.

To this end, we now present the model developed by Rubinstein (1982), in which the procedure is modeled explicitly as a game in real time. That is, there is a well-defined sequence of moves and agents have preferences over the time of agreement as well as the terms of agreement. Time proceeds without end as $t = 0, 1, 2, \ldots$, there is a fixed pie of size 1 over which the agents are negotiating, and an agreement gives q to agent 1 and $1-q$ to agent 2, with $q \in [0,1]$.

The procedure is as follows. At $t=0$, one player, say agent 1, offers a value of q that player 2 can either accept or reject. If he accepts, the offer is implemented and the game ends; if he rejects, they wait until $t=1$, at which point player 2 makes an offer, and so on, with the agent making the offer alternating at each date. If q is accepted at date t, the payoff to agent 1 is $\delta^t q$ and the payoff to agent 2 is $\delta^t(1-q)$, where $\delta \in (0,1)$ is the discount factor. If no offer is ever accepted the payoffs are 0. A strategy for j is generally a function that specifies a value of q if it is his turn to make an offer, and an element of $\{Accept,\ Reject\}$ if it is his turn to respond, as a function of the history of the game to that point.

Rubinstein (1982) shows that there is a unique subgame-perfect equilibrium in this game, and it has the following properties. There is a pair of numbers (q_1, q_2), where q_j will be termed the *reservation value* of player j, such that in any subgame, player 1 accepts any offer $q \geq q_1$ and player 2 accepts any offer $q \leq q_2$, and each player always proposes the reservation value of the other agent. It turns out that $q_1 = \delta/(1+\delta)$ and $q_2 = 1/(1+\delta)$. Therefore, the outcome is that at date $t=0$ player 1 offers $q_2 = 1/(1+\delta)$, player 2 accepts, and the game ends.

Consider the following modification (due to Shaked and Sutton [1984]) of Rubinstein's proof of uniqueness of the subgame-perfect equilibrium payoffs. Let M be the greatest payoff that player 1 gets in any subgame-perfect equilibrium starting at $t=2$. Then at $t=1$, player 1 must accept an offer by player 2 of δM in any subgame-perfect equilibrium. Hence, player 2 must get a payoff of at least $1-\delta M$ at $t=1$ in any subgame-perfect equilibrium. But then at $t=0$, player 2 must get a payoff of at least $\delta(1-\delta M)$, and so player can get at most a payoff of $1-\delta(1-\delta M)$ in any subgame-perfect equilibrium. But since the subgames at $t=0$ and $t=2$ are the same, $M = 1-\delta(1-\delta M)$, which means $M = 1/(1+\delta)$.

Now let m be the lowest payoff player 1 gets in any subgame-perfect equilibrium starting at $t = 2$. Then at $t = 1$, player 1 will get a payoff of no less than δm, and so player 2 can get a payoff of no more than $1 - \delta m$, in any subgame-perfect equilibrium. But then at $t = 0$, player 2 will get a payoff of no more than $\delta(1 - \delta m)$ in any subgame-perfect equilibrium. Hence, player 1 gets a payoff of at least $1 - \delta(1 - \delta m)$ in any subgame-perfect equilibrium starting at $t = 0$. Since the subgames at $t = 0$ and $t = 2$ are the same, $m = 1 - \delta(1 - \delta m)$, which means $m = 1/(1 + \delta)$.

We have just shown that the greatest payoff player 1 can get and the lowest payoff player 1 can get in any subgame-perfect equilibrium starting at $t = 0$ are the same: $1/(1 + \delta)$. Hence, this is the unique payoff he gets in any subgame-perfect equilibrium, and therefore the unique payoff player 2 gets is $\delta/(1 + \delta)$. This outcome is supported by the strategies described above, using the reservation values q_1 and q_2.

Notice that a smaller δ implies a bigger q_2, and therefore a greater the share of the pie for agent 1. This is because it is agent 1 who makes the first offer, and this offer is disciplined by the threat of agent 2 rejecting anything less than q_2. Player 1's offer makes player 2 indifferent between accepting and rejecting in order to propose a counteroffer next period; the smaller is δ the more agent 2 is willing to give up to avoid waiting. In particular, as $\delta \to 0$ we have $U_1 \to 1$ and $U_2 \to 0$, and as $\delta \to 1$ we have $U_1 \to \frac{1}{2}$ and $U_2 \to \frac{1}{2}$.

9.3 Generalized Strategic Models

Here we present several extensions of the Rubinstein's model, and pursue the relationship between strategic models and the Nash solution.

To begin, suppose the player who gets to make the offer is determined randomly each period, and let it be agent j with probability π_j, where $\pi_j > 0$ and $\pi_1 + \pi_2 = 1$. Let the payoffs from settlement at date t be given by $\delta_1^t u_1(q)$ and $\delta_2^t u_2(q)$, where u_1 is increasing and u_2 is decreasing in q. Assume u_j is concave. Moreover, suppose that the length of a period is a variable, given by Δ, so that $t = 0, \Delta, 2\Delta, ...$, and write $\delta_j = 1/(1 + r_j\Delta)$, where r_j is the discount rate of agent j.

The main result of the previous section − that there is a unique subgame-perfect equilibrium that can be characterized in terms of reservation values q_1 and q_2 − continues to hold in this model. The reservation values satisfy the following recursive relations:

$$u_1(q_1) = \frac{1}{1 + r_1\Delta}[\pi_1 u_1(q_2) + \pi_2 u_1(q_1)] \tag{9.6}$$

$$u_2(q_2) = \frac{1}{1 + r_2\Delta}[\pi_1 u_2(q_2) + \pi_2 u_2(q_1)]. \tag{9.7}$$

For example, (9.6) says that agent 1 is indifferent between accepting q_1 and rejecting for a chance at a counteroffer after waiting Δ.

One can show that $q_2 > q_1$ for any $\Delta > 0$, but that q_2 and q_1 approach the same limit as $\Delta \to 0$. Thus, as the period of delay shrinks, so does the advantage from being the proposer. For example, consider the case where $u_1(q) = q$ and $u_2(q) = 1 - q$. Then (9.6) and (9.7) are linear and can be solved for

$$q_1 = \frac{\pi_1 r_2}{\pi_1 r_2 + \pi_2 r_1 + r_1 r_2 \Delta} \tag{9.8}$$

$$q_2 = \frac{\pi_1 r_2 + r_1 r_2 \Delta}{\pi_1 r_2 + \pi_2 r_1 + r_1 r_2 \Delta} \tag{9.9}$$

Clearly, $q_1 > q_2$ for all $\Delta > 0$, and both converge to the same limit as $\Delta \to 0$.

Returning to the general case, let $q = \pi_1 q_2 + \pi_2 q_1$ be the average offer, which is arbitrarily close to both q_1 and q_2 for small Δ. To describe q in more detail, consider a first-order Taylor's approximation of (9.6) and (9.7) around q:

$$u_1(q) + (q_1 - q)u_1'(q) = \frac{u_1(q)}{1 + r_1\Delta} + o(\Delta) \tag{9.10}$$

$$u_2(q) + (q_2 - q)u_2'(q) = \frac{u_2(q)}{1 + r_2\Delta} + o(\Delta) \tag{9.11}$$

where $o(\Delta)$ is any function with the property that $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$. If we multiply (9.10) by $(1 + r_1\Delta)(1 + r_2\Delta)\pi_2 u_2'$ and (9.11) by $(1 + r_1\Delta)(1 +$

$r_2\Delta)\pi_1 u_1'$, then add the equations and simplify, we arrive at

$$(1 + r_1\Delta)\pi_1 r_2 u_2(q)u_1'(q) + (1 + r_2\Delta)\pi_2 r_1 u_1(q)u_2'(q) = \frac{o(\Delta)}{\Delta}.$$

As $\Delta \to 0$, this tends to

$$\pi_1 r_2 u_2(q)u_1'(q) + \pi_2 r_1 u_1(q)u_2'(q) = 0. \tag{9.12}$$

Comparing (9.12) and (9.5), we see that q solves a generalized Nash bargaining problem of the form (9.4), where, in this case, $d_1 = d_2 = 0$ and

$$\theta = \frac{\pi_1 r_2}{\pi_1 r_2 + \pi_2 r_1}. \tag{9.13}$$

Hence, the Nash solution can be regarded as an approximation to the equilibrium outcome of the strategic bargaining game when Δ is small – a point first noted by Binmore (1987). The advantage of using the strategic approach, even if we are mainly interested in the case where Δ is small, is that it explicitly delivers the threat points and bargaining power and indicates how they depend on features of the underlying game; see Binmore et al. (1986) for further discussion.

An important extension to the above model is to allow for the possibility of exogenous breakdowns in the negotiations. Let λ_j be the Poisson arrival rate with which j *believes* an exogenous breakdown will occur; i.e., $\lambda_j\Delta + o(\Delta)$ is the probability a breakdown will occur in an interval of time of length Δ. (We do not necessarily impose $\lambda_1 = \lambda_2$, for reasons that will become more clear later). For now, assume that when a breakdown occurs the game ends, and let b_j be the (exogenous) utility of agent j in this event.

The generalized versions of (9.6) and (9.7) can be written

$$u_1(q_1) = \frac{1}{1 + r_1\Delta}\{\lambda_1\Delta b_1 + (1 - \lambda_1\Delta)[\pi_1 u_1(q_2) + \pi_2 u_1(q_1)]\} + o(\Delta) \tag{9.14}$$

$$u_2(q_2) = \frac{1}{1 + r_2\Delta}\{\lambda_2\Delta b_2 + (1 - \lambda_2\Delta)[\pi_1 u_2(q_2) + \pi_2 u_2(q_1)]\} + o(\Delta). \tag{9.15}$$

Let $q = \pi_1 q_1 + \pi_2 q_2$. Exactly the same method that led to (9.12) can be used here. That is, approximate these equations around q, simplify, and let $\Delta \to 0$ to get

$$\frac{\pi_2}{r_2 + \lambda_2}[u_1 - \frac{\lambda_1 b_1}{r_1 + \lambda_1}]u_2' + \frac{\pi_1}{r_1 + \lambda_1}[u_2 - \frac{\lambda_2 b_2}{r_2 + \lambda_2}]u_1' = 0. \tag{9.16}$$

Comparing (9.16) and (9.5), we see that in this model the limit q also solves a generalized Nash problem, where now the threat points are given by

$$d_j = \frac{\lambda_j b_j}{r_j + \lambda_j}, \tag{9.17}$$

and the bargaining power of agent 1 is given by

$$\theta = \frac{\pi_1(r_2 + \lambda_2)}{\pi_1(r_2 + \lambda_2) + \pi_2(r_1 + \lambda_1)}. \tag{9.18}$$

One interpretation of the threat point in (9.17) is that d_j is the utility j can expect to get by delaying settlement forever, appropriately discounted. As $r_j \to 0$ (the case discussed in Binmore et al. [1986]) we have $d_j \to b_j$, and the threat points are the exogenous payoffs in the event of a breakdown.

In many applications, such as bargaining in the context of a "market" model like the one in like Rubenstein and Wolinsky (1985) or Binmore and Herarro (1987), b_j is made endogenous by assuming that after a breakdown agents get to look for new partners. Hence, b_j is the value of search. If α_j is the Poisson arrival rate of new partners for agent j, then the standard Bellman equation implies the value of search satisfies

$$r_j b_j = \alpha_j(u_j - b_j) \tag{9.19}$$

where u_j should be interpreted as the expected payoff of the bargaining subgame.

One way to proceed is to rearrange (9.16) as

$$\pi_2 \left[(r_1 + \lambda_1)(u_1 - b_1) + r_1 b_1\right] u_2' + \pi_1 \left[(r_2 + \lambda_2)(u_2 - b_2) + r_2 b_2\right] u_1' = 0.$$

Then, inserting (9.19), we have

$$\pi_2(r_1+\lambda_1+\alpha_1)(u_1-b_1)u_2' + \pi_1(r_2+\lambda_2+\alpha_1)(u_2-b_2)u_1' = 0.$$

Hence, one could say that q solves a generalized Nash problem with $d_i = b_i$ and

$$\theta = \frac{\pi_1(r_2+\alpha_2+\lambda_2)}{\pi_1(r_2+\alpha_2+\lambda_2)+\pi_2(r_1+\alpha_1+\lambda_1)}. \tag{9.20}$$

However, since b_j is endogenous, we can alternatively use (9.19) to eliminate b_i from (9.16) and write

$$\frac{\pi_2(r_1+\lambda_1+\alpha_1)r_1}{r_1+\alpha_1}u_1u_2' + \frac{\pi_1(r_2+\lambda_2+\alpha_2)r_2}{r_2+\alpha_2}u_2u_1' = 0.$$

Hence, one can also say that q solves a generalized Nash problem with $d_i = 0$ and

$$\theta = \frac{\pi_1 r_2(r_1+\alpha_1)(r_2+\alpha_2+\lambda_2)}{\pi_1 r_2(r_1+\alpha_1)(r_2+\alpha_2+\lambda_2)+\pi_2 r_1(r_2+\alpha_2)(r_1+\alpha_1+\lambda_1)}. \tag{9.21}$$

This shows that, in some cases, there may be more than one Nash representation of the same strategic bargaining outcome. To pursue this further, consider a Rubinstein-Wolinsky style "market" model where the only source of breakdowns is that new agents sometimes arrive between rounds in the bargaining game, and when a new type agent arrives he replaces the incumbent. If a new type 1 agent arrives, for example, the old type 1 agent must leave and the type 2 agent begins bargaining with the interloper (he cannot negotiate with both simultaneously, nor can he pick one strategically). Hence, the breakdown rate for type 1 is the arrival rate for type 2 and vice-versa: $\lambda_1 = \alpha_2$ and $\lambda_2 = \alpha_1$.

To reduce notation, assume $r_1 = r_2 = r$ and $\pi_1 = \pi_2 = \frac{1}{2}$. Then (9.20) implies

$$\theta = \frac{1}{2},$$

regardless of α_1 and α_2. Hence, different arrival rates show up in terms of different threat points, since b_j depends on α_j, but not in terms of different bargaining power. Alternatively, (9.21) implies

$$\theta = \frac{r + \alpha_1}{2r + \alpha_1 + \alpha_2}.$$

Now different arrival rates show up in terms of different bargaining power, but the same threat point since $b_1 = b_2 = 0$ in this representation.

Everything we have been saying up to now is predicated on the assumption that the two agents want to trade. Suppose that agent j gets some utility Φ_j if he opts out of the bargaining, sometimes referred to as an *outside option*. For example, if he can derive the same payoff by leaving his bargaining partner voluntarily as he can derive in the event of an exogenous breakdown, then $\Phi_j = b_j$. In any case, we assume that the only time that a player can opt out is after rejecting an offer (he cannot, e.g., opt out after the other player rejects an offer); as discussed in Osborne and Rubinstein (1990), this is important.

The recursive equations describing the reservation values, like (9.6) and (9.7), can be amended to take outside options into account by writing

$$u_1(q_1) = \max\left\{\tfrac{1}{1+r_1\Delta}[\pi_1 u_1(q_2) + \pi_2 u_1(q_1)], \Phi_1\right\}$$

$$u_2(q_2) = \max\left\{\tfrac{1}{1+r_2\Delta}[\pi_1 u_2(q_2) + \pi_2 u_2(q_1)], \Phi_2\right\}.$$

For example, let $u_1 = q$ and $u_2 = 1 - q$, let $\pi_1 = \pi_2$, and consider the case where $\Delta \to 0$. Without outside options, the solution is $q = \frac{1}{2}$. With outside options, the solution can be described as follows: if $\Phi_1 \leq \frac{1}{2}$ and $\Phi_2 \leq \frac{1}{2}$ then $q = \frac{1}{2}$; if $\Phi_1 > \frac{1}{2}$ and $\Phi_2 \leq 1 - \Phi_1$ then $q = \Phi_1$; if $\Phi_2 > \frac{1}{2}$ and $\Phi_1 \leq 1 - \Phi_2$ then $q = 1 - \Phi_2$; and if $\Phi_1 + \Phi_2 > 1$ then there are no gains from trade and there will be no agreement.

9.4 Nonstationary Bargaining Models

Although the Rubinstien model is dynamic in the sense that it specifies the sequence of moves and how payoffs depend on the time of settlement as well as the terms of settlement, the analysis so far has assumed the environment is stationary in the sense that preferences or opportunities do not vary with time. In some applications, however, bargaining problems come up in non-stationary models.

The first thing we do in this section is to consider is the case where the bargaining game has a finite end. For instance, suppose two agents are dividing a pie of unit size, and offers can be made at only the following dates: $t = 0, 1, 2, ...T$. If no offer is accepted by date T, the game ends with zero payoffs. If the game ends with an accepted offer at $t \leq T$, the payoff of agent j is $\delta_j^t u_j(q)$, with $u_1'(q) > 0 > u_2'(q)$. Assume agent j makes the offer at each date with probability π_j.

The reservation values at T are given by $q_1(T) = 0$ and $q_2(T) = 1$, since agent 1 will accept any amount greater than 0 and agent 2 will pay any amount less than 1 at the last possible chance for settlement. The reservation values at any date $t < T$ are given by the following recursive relations:

$$u_1[q_1(t)] = \delta_1\{\pi_1 u_1[q_2(t+1)] + \pi_2 u_1[q_1(t+1)]\} \tag{9.22}$$

$$u_2[q_2(t)] = \delta_2\{\pi_1 u_2[q_2(t+1)] + \pi_2 u_2[q_1(t+1)]\} \tag{9.23}$$

From these, one can solve for the entire sequence $[q_1(t), q_2(t)]$ starting with $t = T$ and using backward induction.

For example, let $u_1(q) = q$ and $u_2(q) = 1 - q$, and assume $\delta_1 = \delta_2 = \delta$. Given $q_1(T) = 0$ and $q_2(T) = 1$, (9.22) and (9.23) imply $q_1(T-1) = \delta\pi_1$ and $q_2(T-1) = 1 - \delta\pi_2$. Applying (9.22) again, we have

$$q_1(T-2) = \delta[\pi_1(1 - \delta\pi_2) + \pi_2\delta\pi_1] = \delta\pi_1.$$

Similarly, applying (9.23), we have $q_2(T-2) = 1 - \delta\pi_2$. Hence, $q_1(t) = \delta\pi_1$ and $q_2(t) = 1 - \delta\pi_2$ for all $t < T$, which are the reservation values in the infinite horizon game. If we do not assume $\delta_1 = \delta_2$, then the solution does not converge to the equilibrium of the infinite horizon game after one step, but it does converge as $T \to \infty$.[1]

[1] Assume agent 1 proposes at T, agent 2 proposes at $T-1$, and so on, and let $\hat{q}(t)$ be

A more general specification than a finite horizon lets $u_j(q,t)$ depend on time in some arbitrary way, subject only to the restriction that u_j and $\partial u_j/\partial t$ are bounded in t. Here we will only consider Markov equilibria (i.e., strategies will not depend on histories), and equilibria which involve immediate trade as soon as the agents meet; nevertheless, the terms of trade will generally depend on when they meet. Of course, for immediate trade to be consistent with equilibrium we require some assumptions – for example, if their utility functions are increasing very fast over time the agents may prefer to delay settlement. We proceed by analyzing the outcome assuming that trade is immediate, and then checking that $e^{-r_j t}u_j(q,t)$ is decreasing in t for both j, which suffices to guarantee that this is an equilibrium.

Given a Markov equilibrium with immediate trade, there exist reservation values $[q_1(t), q_2(t)]$ that satisfy the following recursive equations

$$u_1[q_1(t),t] = \delta_1\{\pi_1 u_1[q_2(t+\Delta), t+\Delta] + \pi_2 u_1[q_1(t+\Delta), t+\Delta]\} \tag{9.24}$$

$$u_2[q_2(t),t] = \delta_2\{\pi_1 u_2[q_2(t+\Delta), t+\Delta] + \pi_2 u_2[q_1(t+\Delta), t+\Delta]\} \tag{9.25}$$

for all t, where π_j and δ_j are assumed for simplicity to be independent of t. It may be shown that for all t, q_1 and q_2 both converge to $q = \pi_1 q_2 + \pi_2 q_1$ as $\Delta \to 0$ (see Coles and Wright [1994]).

We can study the behavior of q (as a function of t) by following the same procedure used in the stationary model. First, take a Taylor's approximation of (9.24) and (9.25) around q:

$$u_1[q(t),t] + [q_1(t) - q(t)]\frac{\partial u_1[q(t),t]}{\partial q} = \delta_1 u_1[q(t+\Delta), t+\Delta] + o(\Delta)$$

---

the offer at t. Clearly, $\hat{q}(T) = 1$. Then $\hat{q}(T-1) = \delta_1$, since agent 2 will have to offer enough to make 1 accept. Then $\hat{q}(T-2) = 1 - \delta_2 + \delta_1\delta_2$, since agent 1 will have to offer enough to make 2 accept, and so on. The offers of agent 1 converge to

$$(1-\delta_2)(1+\delta_1\delta_2+\delta_1^2\delta_2^2+...) = \frac{1-\delta_2}{1-\delta_1\delta_2},$$

which is the reservation value of agent 2 in the infinite horizon alternating offers game, or in the infinite horizon random offers game when $\pi_1 = \pi_2$. Similarly, the offers of agent 2 converge to the reservation value of agent 1.

$$u_2[q(t),t] + [q_2(t) - q(t)]\frac{\partial u_2[q(t),t]}{\partial q} = \delta_2 u_2[q(t+\Delta), t+\Delta] + o(\Delta).$$

Then, multiply the first of these by $\pi_2 \partial u_2[q(t),t]/\partial q$ and the second by $\pi_1 \partial u_1[q(t),t]/\partial q$, add the equations and simplify to get

$$\begin{aligned} & \pi_1 \left\{ u_2[q(t),t] - \delta_2 u_2[q(t+\Delta), t+\Delta] \right\} \frac{\partial u_1[q(t),t]}{\partial q} \\ + \quad & \pi_2 \left\{ u_1[q(t),t] - \delta_1 u_1[q(t+\Delta), t+\Delta] \right\} \frac{\partial u_2[q(t),t]}{\partial q} \quad = o(\Delta). \end{aligned} \tag{9.26}$$

Comparing (9.26) with (9.5), for small Δ, $q(t)$ looks like the generalized Nash solution with bargaining power $\theta = \pi_1$ and threat points

$$d_j = \delta_j u_j[q(t+\Delta), t+\Delta].$$

The threat point d_j has a very natural interpretation as what agent j can get by rejecting an offer and settling next period. Unfortunately, however, d_j depends on the (endogenous) $q(t+\Delta)$. Therefore, we pursue another way of characterizing q.

Setting $\delta_j = 1/(1 + r_j \Delta)$ in (9.26), rearranging, and taking the limit as $\Delta \to 0$, we get

$$\pi_1(r_2 u_2 - \frac{\partial u_2}{\partial t} - \frac{\partial u_2}{\partial q}\dot{q})\frac{\partial u_1}{\partial q} + \pi_2(r_1 u_1 - \frac{\partial u_1}{\partial t} - \frac{\partial u_1}{\partial q}\dot{q})\frac{\partial u_2}{\partial q} = 0.$$

This also has an interpretation in terms of a generalized Nash solution with time varying threat points. Alternatively, it can be rewritten

$$\dot{q} = \frac{\pi_1(r_2 u_2 - \partial u_2/\partial t)}{\partial u_2/\partial q} + \frac{\pi_2(r_1 u_1 - \partial u_1/\partial t)}{\partial u_1/\partial q}. \tag{9.27}$$

Hence, in any Markov subgame-perfect equilibrium with immediate trade, the limiting value of q (as $\Delta \to 0$) satisfies a relatively simple differential equation. A version of this differential equation can be used in nonstationary models with bargaining in much the same way as the Nash solution can be used in stationary applications.

Suppose the utility functions settle down over time: $u_j(q,t) \to \bar{u}_j(q)$, where $\bar{u}_j$ satisfies our usual assumptions. Then $q \to \bar{q}$ as $t \to \infty$, where $\bar{q}$ satisfies

$$\pi_1 r_2 \bar{u}_2(\bar{q})\bar{u}_1'(\bar{q}) + \pi_2 r_1 \bar{u}_1(\bar{q})\bar{u}_2'(\bar{q}) = 0. \tag{9.28}$$

Comparing this to (9.5), $\bar{q}$ is the generalized Nash solution with $d_j = 0$ and $\theta = \pi_1 r_2/(\pi_1 r_2 + \pi_2 r_1)$. Hence, if preferences settle down, q converges to a particular Nash solution as $t \to \infty$.

This result suggests that there may be a Nash representation for q of the form

$$q(t) = \arg\max\, [u_1(q,t) - d_1]^{\theta}[u_2(q,t) - d_2]^{1-\theta}$$

that holds for all t, and not just as $t \to \infty$. One candidate is what we call the *myopic* Nash solution, defined by setting $d_j = 0$ and $\theta = \pi_1 r_2/(\pi_1 r_2 + \pi_2 r_1)$, which we have just seen are the values for d_j and θ that apply in the limit. It turns out that the myopic Nash solution does *not* generally coincide with the solution to differential equation (9.27).

To see this, consider the following example: $u_1(q,t) = q^{\rho}$, with $0 < \rho < 1$, and $u_2(q,t) = e^{-\gamma t} - q$, with $\gamma > 0$. Then (9.27) becomes

$$\dot{q} = \frac{r(1+\rho)q - \rho(r+\gamma)e^{-gt}}{2\rho},$$

and (9.28) implies $q \to 0$. The solution to this differential equation subject to this boundary condition is[2]

$$q^* = \frac{\rho\,(r+\gamma)\,e^{-\gamma t}}{r(1+\rho) + 2\gamma\rho}.$$

By way of comparison, the myopic Nash solution is

$$q^n = \frac{\rho e^{-\gamma t}}{1+\rho}.$$

Notice that $q^n > q^*$ for all $\gamma > 0$. Of course, if $\gamma = 0$ then $q^n = q^*$ (since the model is then stationary). In fact, one can show the following: if

[2]Notice that $e^{-rt}u_j(q,t)$ is decreasing in t, so that immediate trade is consistent with equilibrium.

$u_j(q,t) = \eta_j q + v_j(t)$, where η_j is constant, and if $r_1 = r_2$, then the q that solves (9.27) coincides with the myopic Nash solution for all t, and not just in the limit as $t \to \infty$ (see Coles and Wright [1994] for details). But, in general, if utility is nonlinear in q, or nonseparable in q and t, or if $r_1 \neq r_2$, then the myopic Nash solution does not give the same path as (9.27).

Finally, we provide without derivation versions of the above results that apply in models with exogenous breakdowns. First, (9.27) generalizes to

$$\begin{aligned} \dot{q} &= \frac{\pi_1[(r_2+\lambda_2)u_2 - \lambda_2 b_2 - \partial u_2/\partial t]}{\partial u_2/\partial q} \\ &\quad + \frac{\pi_2[(r_1+\lambda_1)u_1 - \lambda_1 b_1 - \partial u_1/\partial t]}{\partial u_1/\partial q} \end{aligned}$$

where λ_j is the probability with which agent j expects a breakdown and b_j is his utility in this event (assumed for simplicity to be independent of t). Second, if the utility functions settle down over time, then q converges to a steady state which coincides with the Nash solution for the stationary model with threat points given by (9.17) and bargaining power given by (9.18). Moreover, if $u_j(q,t) = \eta_j q + v_j(t)$ and $r_1 + \lambda_1 = r_2 + \lambda_2$, then q coincides with the myopic Nash solution for all t and not just in the limit.