Technical Appendix to 'Efficiency of Simultaneous Search'

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This note provides the technical details on the omitted setup and proof in "Efficiency of Simultaneous Search" by the author. For notation, please see that paper.

Extended Setup for Section 5:

We generalize the setting to the case where workers can send any number $i \in \mathbb{N}_0$ of applications, at a cost $c(i)$. We have $c(0) = 0$, increasing marginal costs $c_i = c(i) - c(i-1)$, and largest integer N such that $c(N) \leq 1$. Since many arguments are straightforward generalizations of that special case, we focus mainly on the changes that are necessary to adapt the prior setup.

The extension requires mainly adaptations of the workers' setup, while it remains essentially unchanged for firms. The workers' strategy is now a tuple $(\boldsymbol{\gamma}, \boldsymbol{G})$, where $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, ..., \gamma_N) \in \triangle_N$ and $\boldsymbol{G} = (G^1, G^2, ..., G^N) \in \times_{i=1}^N \Phi^i$, where $\triangle_N$ is the N-dimensional unit simplex and Φ^i the set of cumulative distribution functions of $[0,1]^i$. γ_i denotes the probability of sending i applications, and G^i denotes the cumulative distribution function over $[0,1]^i$ that describes the application behavior. Let $(w_1, ..., w_i)$ satisfy $w_1 \leq w_2 \leq ... \leq w_i$ and let G^i_j denote the marginal distribution of G^i over w_j. A worker who applies to $(w_1, ..., w_i)$ attains in analogy to (3) the utility

$$U_i(w_1, ..., w_i) = \sum_{j=1}^{i} \left[\prod_{k=j+1}^{i} (1 - p(w_k)) \right] p(w_j) w_j - c(i). \tag{3'}$$

A worker who applies nowhere attains $U_0 = 0$. Instead of (7) the relevant condition is now

$$\sum_{i=1}^{N} \left[\gamma_i \sum_{j=1}^{i} G^i_j(w) \right] = v \int_0^w \lambda(\tilde{w}) \, dF(\tilde{w}). \tag{7'}$$

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To specify $\psi(w)$ in the extended setup, consider a firm at wage w that receives an application and let $\hat{G}(\tilde{\mathbf{w}}|w)$ denote probability that the sender applied with his other $N-1$ applications to wages weakly below $\tilde{\mathbf{w}}$. If the sender only sent $i < N-1$ other applications, then we code (only for this definition) the additional $N-1-i$ applications as going to wage -1. So $\tilde{\mathbf{w}} = (\tilde{w}_1, ..., \tilde{w}_{N-1}) \in ([0,1] \cup \{-1\})^{N-1}$. Let $h(\tilde{\mathbf{w}}|w)$ count the number of applications sent to wage w when the worker applies to $\tilde{\mathbf{w}}$ and w. Replacing (8) we now specify

$$\psi(w) = \int \left[1 - \frac{1-(1-p(w))^{h(\tilde{\mathbf{w}}|w)}}{p(w)h(\tilde{\mathbf{w}}|w)} \prod_{\tilde{w}_j > w} [1-p(\tilde{w}_j)]\right] d\hat{G}(\tilde{\mathbf{w}}|w). \tag{8$'$}$$

The product $\prod_{\tilde{w}_j > w}[1-p(\tilde{w}_j)]$ describes the probability that the applicant does not take a job at a strictly better wage. Its multiplier gives the probability that a worker does not turn down a job offer because of a job at another firm with the same wage, conditional on failing at higher wages (see e.g. Burdett, Shi and Wright (2001), equation (6)). Then the integrand gives the probability that the worker takes the job at a different firm.

The definitions for all other variables, i.e. μ, p and η and π remain unchanged. With these adjustments the equilibrium definition extends to this section.

Proof of Proposition 10:

Consider some (candidate) equilibrium with some $\boldsymbol{\gamma}$ that we fix for the moment. Denote by $\hat{i}$ the highest integer for which $\gamma_i > 0$. By straightforward generalization of Proposition 1 we have

$$\begin{aligned} p(w) &= 1 \qquad \forall w \in [0, \bar{w}_0], \text{ and} && \text{(A.1)} \\ p(w)w + (1-p(w))u_{i-1} &= u_i \quad \forall w \in [\bar{w}_{i-1}, \bar{w}_i] \quad \forall i \in \{1, ..., N\}, && \text{(A.2)} \end{aligned}$$

where $u_i \equiv \max_{w \in [0,1]} p(w)w + (1-p(w))u_{i-1}$ for all $i \in \{1, 2, ...\hat{i}\}$, $u_0 \equiv 0$, $\bar{w}_0 = u_1$. The Market Utility Assumption implies that workers cannot receive more than the Market Utility, which implies that $u_i - u_{i-1} = c_i$ for $i > \hat{i}$. Indifference then yields $\bar{w}_i = u_{i-1} + [u_i - u_{i-1}]^2/(u_i - u_{i-1} - c_{i+1})$. If this is in $[0,1]$ then this gives the appropriate boundary, otherwise $[\bar{w}_i, \bar{w}_{i+1}]$ is empty.

Using (A.2), we can rewrite the profit function for a firm who offers a wage $w \in [\bar{w}_{i-1}, \bar{w}_i]$ with $\bar{w}_{i-1} < 1$ as

$$\pi(\mu) = (1-e^{-\mu})(1-u_{i-1}) - \mu(u_i - u_{i-1}), \tag{A.3}$$

where $\mu = \mu(w)$. The logic is similar to (17). If $\bar{w}_{i-1} = 1$ the profit is trivially zero. Proposition 2, stating that there exists no equilibrium in which only one wage is offered, can now easily be shown with similar techniques whenever $\gamma_i > 0$ for some $i > 1$. By a similar argument it is straightforward that at least i wages have to be offered in equilibrium whenever $\gamma_i > 0$. Given that (A.3) is strictly concave, it is also immediate that all firms within the same interval offer the same wage, yielding exactly $\hat{i}$ wages when some workers send $\hat{i}$ applications.

We call the group of firms that ends up offering the i'th highest wage as group i and index all their variables accordingly. It is convenient to denote by $\Gamma_i = \sum_{k=i}^{\hat{i}} \gamma_k$ the fraction of workers who apply to at least i firms. Then at wage i the probability of retaining an applicant is $(1 - \psi_i) = \sum_{j=i}^{\hat{i}} \frac{\gamma_j}{\Gamma_i} [\prod_{k=i+1}^{j} (1 - p_k)]$, since a fraction γ_j / Γ_i of applicants sends j applications and does not get a better job with probability $\prod_{k=i+1}^{j} (1 - p_k)$. The effective queue length at wage i is given by $\mu_i = (1 - \psi_i)\lambda_i$, where $\lambda_i = \Gamma_i / v_i$ is the gross queue length. For $i < \hat{i}$ the unique offered wage in $[\bar{w}_{i-1}, \bar{w}_i]$ is obtained by the first-order-conditions of (A.3), which are given by

$$u_i - u_{i-1} = e^{-\mu_i}(1 - u_{i-1}). \tag{A.4}$$

Therefore (A.3) can be rewritten as

$$\pi_i = (1 - e^{-\mu_i} - \mu_i e^{-\mu_i})(1 - u_{i-1}). \tag{A.5}$$

Free entry implies that $\pi_i = K$, which together with (A.4) implies that $\mu_i = \mu_i^*$ and $u_i = u_i^*$ as defined in the main body. By a similar argument as for (25) and (26) the condition $\pi_i = K$ defines for a given vector $\boldsymbol{\gamma}$ of applications the unique measure v_i of firms in each group. Existence and – except for the case where $c_{i^*} = u_i^* - u_{i-1}^*$ – uniqueness follow by similar arguments as in the case of at most two applications.

To show constrained efficiency, we first consider search efficiency for given $\boldsymbol{\gamma}$ and v. Let $\hat{i}$ still denote the maximum number of applications that workers send. We first consider a planner that only uses $\hat{i}$ wages and, thus, $\hat{i}$ groups of firms, with $\rho_i v$ firms in each groups. That is, a worker who applies to i firms applies once to each of the lowest i wages and accepts an offer from a higher wage firm over an offer from a lower wage firm. We call an allocation of firms across groups that leads to the maximum number of matches $\hat{i}$-group-efficient. Compare two adjacent groups of firms i and $i-1$ with total measure $\nu = v_i + v_{i-1}$. We show that the only efficient way of dividing this measure up between the two groups is the equilibrium division.

The maximal total number of matches within these groups is given by

$$\max_{\rho\in[0,1]} M(\rho) = \nu\rho(1-e^{-\mu_i}) + \nu(1-\rho)(1-e^{-\mu_{i-1}}). \tag{A.6}$$

It can be shown that a boundary solution cannot be optimal, as it means that one application is waisted. Noting that $(1-\psi_{i-1}) = \gamma_{i-1}/\Gamma_{i-1}+(1-\psi_i)(1-p_i)\Gamma_i/\Gamma_{i-1}$ we can write $\mu_i = (1-\psi_i)\lambda_i$ and $\mu_{i-1} = [\gamma_{i-1}/\Gamma_{i-1} + (1-\psi_i)(1-p_i)\Gamma_i/\Gamma_{i-1}]\lambda_{i-1}$. The first derivative is then

$$\begin{aligned}\frac{dM(\rho)}{d\rho}\frac{1}{\nu} &= 1-e^{-\mu_i} - (1-e^{-\mu_{i-1}}) + e^{-\mu_i}\rho(1-\psi_i)\frac{d\lambda_i}{d\rho} \\ &+ e^{-\mu_{i-1}}(1-\rho)\left[(1-\psi_{i-1})\frac{d\lambda_{i-1}}{d\rho} - \frac{\Gamma_i}{\Gamma_{i-1}}(1-\psi_i)\frac{dp_i}{d\rho}\lambda_{i-1}\right]\end{aligned}$$

We can use similar substitutions as for (29), with the adjustment that now $d\mu_i/d\rho = -\mu_i/\rho = -\nu\mu_i^2/[(1-\psi_i)\Gamma_i]$, to show that the last term in the first line equals $-\mu_i e^{-\mu_i}$, and the second line reduces to $e^{-\mu_{i-1}}[\mu_{i-1} - (1-e^{-\mu_i} - \mu_i e^{-\mu_i}]$. Therefore we obtain the following first order condition

$$\frac{dM(\rho)}{\nu d\rho} = (1-e^{-\mu_i} - \mu_i e^{-\mu_i})(1-e^{-\mu_{i-1}}) - (1-e^{-\mu_{i-1}} - \mu_{i-1}e^{-\mu_{i-1}}) = 0. \tag{A.7}$$

For given ν this uniquely characterizes the optimal interior ρ, since similar substitutions as above yield $d^2M/d\rho^2 = -\nu[\mu_2^2 e^{-\mu_2}(1-e^{-\mu_1})/\rho + e^{-\mu_1}(1-e^{-\mu_2} - \mu_2 e^{-\mu_2} - \mu_1)^2/(1-\rho)] < 0$. A similar construction as in the proof of Proposition 6 shows that $\hat{i}$ groups are sufficient to achieve the constrained optimal outcome.

Next, we establish that the overall entry of firms and the measure of firms in each group under equilibrium conditions 1a), 1b), 2a) and 3) yields optimal entry and optimal application decisions simultaneously, taking $\boldsymbol{\gamma}$ as given. Let $\boldsymbol{\rho}(v) = (\rho_1(v), \rho_2(v), .., \rho_{\hat{i}}(v))$ be the fraction of firms in each of the $\hat{i}$ groups under constrained optimal search given v and $\boldsymbol{\gamma}$. Again let $M^*(\boldsymbol{\gamma}, v, \boldsymbol{\rho}(v))$ denote the constrained efficient number of matches given v and $\boldsymbol{\gamma}$. Similar to (30) the objective function is given by $\max_{v\geq 0} M^*(\boldsymbol{\gamma}, v, \boldsymbol{\rho}(v)) - vK$. When $\hat{i} > 0$, then $K < 1$ ensures that the optimal solution is in the interior of $[0, V]$. We show that the first order condition uniquely determines the solution and corresponds to the free entry condition.

By the envelope theorem the impact of a change of the fraction $\rho_i(v)$ of firms in each group on the measure of matches can be neglected, i.e. $\frac{dM^*}{d\rho_i}\frac{d\rho_i}{dv} = 0$ at the $\hat{i}$-group-efficient ρ_i. We get as first order condition

$$dM^*(\boldsymbol{\gamma}, v, \boldsymbol{\rho})/dv = K, \tag{A.8}$$

where $\boldsymbol{\rho} = \boldsymbol{\rho}(v)$. Writing $M^*(\boldsymbol{\gamma}, v, \boldsymbol{\rho}) = [1 - \sum_{i=1}^{\hat{i}}[\gamma_i \prod_{j=1}^{i}(1-p_j)]$ we have

$$\frac{dM^*(\boldsymbol{\gamma}, v)}{dv} = \sum_{i=1}^{\hat{i}}\Big[\gamma_i \sum_{j=1}^{i}[\frac{dp_j}{dv}\prod_{\substack{k\leq i\\ k\neq j}}(1-p_k)]\Big] = \sum_{i=1}^{\hat{i}}\Big[\frac{dp_i}{dv}\Gamma_i(1-\psi_i)\prod_{k<i}(1-p_k)\Big], \tag{A.9}$$

where the equality line is obtained by rearranging the terms for each dp_i/dv. To simplify notation, define the partial sum

$$\xi_{i'} = \sum_{i\geq i'}^{N}[\frac{dp_i}{dv}\Gamma_i(1-\psi_i)\prod_{k<i}(1-p_k)]. \tag{A.10}$$

Since $p_{\hat{i}} = (1-e^{-\mu_{\hat{i}}})/\mu_{\hat{i}}$ we have $dp_{\hat{i}}/dv = -(1/\mu_{\hat{i}}^2)(1-e^{-\mu_{\hat{i}}} - \mu_{\hat{i}}e^{-\mu_{\hat{i}}})(d\mu_{\hat{i}}/dv)$. Since $\mu_{\hat{i}} = \gamma_{\hat{i}}/(\rho_{\hat{i}}v)$, we have $d\mu_{\hat{i}}/dv = -\gamma_{\hat{i}}/(\rho_{\hat{i}}v^2) = -\rho_{\hat{i}}\mu_{\hat{i}}/\gamma_{\hat{i}}$. So we get $dp_{\hat{i}}/dv = -\rho_{\hat{i}}(1-e^{-\mu_{\hat{i}}}-\mu_{\hat{i}}e^{-\mu_{\hat{i}}})/\gamma_{\hat{i}}$. Noting that $\Gamma_{\hat{i}}(1-\psi_{\hat{i}}) = \gamma_{\hat{i}}$, we have established that

$$\xi_{\hat{i}} = \rho_{\hat{i}}(1-e^{-\mu_{\hat{i}}} - \mu_{\hat{i}}e^{-\mu_{\hat{i}}})\prod_{k<\hat{i}}(1-p_k). \tag{A.11}$$

By induction we can establish the following lemma, which we prove subsequently because it would distract from the argument at this point.

Lemma A1 *For all i it holds that*

$$\xi_i = (\sum_{k=i}^{N}\rho_k)(1-e^{-\mu_i} - \mu_i e^{-\mu_i})\prod_{j<i}(1-p_k). \tag{A.12}$$

This implies that $\xi_1 = 1 - e^{\mu_1} - \mu_1 e^{-\mu_1}$. The first order condition $\xi_1 = K$ uniquely defines μ_1, and corresponds to the free entry condition of the lowest wage firms. By (A.7) it also determines μ_i uniquely for all $i \in 2, ..., \hat{i}$, which in turn determines v_i uniquely for all $i \in 1, ..., \hat{i}$. Thus equilibrium entry and search is constrained optimal given $\boldsymbol{\gamma}$.

Finally, when we endogenize $\boldsymbol{\gamma}$, again note that the number of applications of other workers in equilibrium is not important for the marginal benefits of each individual worker, which are always $u_i^* - u_{i-1}^*$. Therefore again the decision on the number of applications is constrained efficient, establishing constrained efficiency overall. *Q.E.D.*

Proof of Lemma A1:

We are left to show that the following holds for all $i \in \{1, ..., \hat{i} - 1\}$:

$$\xi_{i+1} = (\sum_{k=i+1}^{\hat{i}} \rho_k)(1 - e^{-\mu_{i+1}} - \mu_{i+1} e^{-\mu_{i+1}}) \prod_{k<i+1} (1 - p_k). \tag{A.13}$$

It clearly holds for $i = \hat{i} - 1$ by $(A.11)$. Now assume it holds for some i. We consider ξ_i. We know that

$$\xi_i = \xi_{i+1} + \Gamma_i (1 - \psi_i) \frac{dp_i}{dv} \prod_{k<i} (1 - p_k). \tag{A.14}$$

The second summand can be written as

$$\frac{dp_i}{dv} \prod_{k<i} (1 - p_k) = -\frac{1 - e^{-\mu_i} - \mu_i e^{-\mu_i}}{\mu_i^2} [\frac{d\mu_i}{dv} \prod_{k<i} (1 - p_k)] \tag{A.15}$$

Since $\mu_i = \lambda_i (1 - \psi_i) = \lambda_i (\sum_{j=i}^{\hat{i}} \frac{\gamma_j}{\Gamma_i} (\prod_{k=i+1}^{j} (1 - p_k)))$ we can write the term in square brackets in (A.15) as

$$\frac{d\mu_i}{dv} \prod_{k<i} (1 - p_k) = \frac{\lambda_i \ \xi_{i+1}}{\Gamma_i (1 - p_i)} - \frac{\Gamma_i (1 - \psi_i)}{\rho_i v^2} \prod_{k<i} (1 - p_k) = \frac{\lambda_i \ \xi_{i+1}}{\Gamma_i (1 - p_i)} - \frac{\rho_i \mu_i^2}{\Gamma_i (1 - \psi_i)} \prod_{k<i} (1 - p_k).$$

Observing that $\frac{1}{\mu_i} (1 - e^{-\mu_i} - \mu_i e^{-\mu_i}) = p_i - e^{-\mu_i}$, we can substitute the prior equation into (A.15) and multiply by $\Gamma_i (1 - \psi_i)$ to get

$$\Gamma_i (1 - \psi_i) \frac{dp_i}{dv} \prod_{k<i} (1 - p_k) = \frac{p_i - e^{-\mu_i}}{1 - p_i} \xi_{i+1} + \rho_i (1 - e^{-\mu_i} - \mu_i e^{-\mu_i}) \prod_{k<i} (1 - p_k).$$

We can substitute this into (A.14), and use (A.13) and the property of $\hat{i}$-group-efficient search in (A.7) to obtain

$$\xi_i = (\sum_{k=i}^{N} \rho_k)(1 - e^{-\mu_i} - \mu_i e^{-\mu_i}) \prod_{k<i} (1 - p_k). \tag{A.16}$$

Q.E.D.