4. Homework in Econ 682 - Game Theory and Applications

The homework is due on April 23nd at the beginning of class. Answers have to be typed, with the exception of graphs, which may be drawn by hand.

Question1:

1. Answer problem 4.2 in the book.

   Answer: We will first rule out that there exists a pure strategy equilibrium. In any pure strategy equilibrium player 2 plays either L' or R'. If he plays L' the best reply by player 1 is to play L, but the best reply of player 2 against L is R' (which violates that 2 plays L'). Similarly, if player 2 plays R' the best reply by player 1 is to play R, but the best reply of player 2 against R is L' (which violates that 2 plays R'). Therefore, no pure strategy equilibrium exists.

   Consider the following mixed strategy equilibrium. Player 2 plays L' with probability q (and R' with complementary probability $1-q$). Consider e.g. $q = 1/2$. In order to randomize, player 1 has to find both actions equally attractive, which is only the case if he holds belief $p = 1/2$. Then player 1 playing R is a best response because it gives a payoff of 2 for player 1 while playing M or L only gives payoff 1.5. Playing R also validates the belief that $p = 1/2$ because the information set is off-the-equilibrium-path. Let r be the probability of playing R and m the probability of playing M, then $\{r = 1, m = 0, p = 1/2, q = 1/2\}$ constitutes a Perfect Bayesian Nash Equilibrium.[1]

2. Answer problem 4.3 part a) in the book.

   Answer: Let p be receiver's belief that the sender is type t_1 if he played L, and let q be the belief that the sender is type t_1 if he played R. We will show that $((R, R), (u, d), p, q = .5)$ is a PBNE if $p \geq 2/3$.

   If both receiver types play R, then the receiver does not learn from the equilibrium action and $q = 1/2$. Given this belief, his optimal action after observing R is to play d since $.5 * 2 > .5 * 1$. This implies that the receiver of type t_1 obtains 3, which is larger than any other payoff he could have obtained in the game and therefore he does not have any incentive to deviate. The receiver of type t_2 on the other hand is only willing to play R if he gets less than 2 by playing L, which only happens if the receiver plays u. Playing u is only optimal for the receiver after observing L if his belief is $p \geq 2/3$ because then $q * 2 \geq (1 - q) * 1$.

[1] Note that any $q \in [1/3, 2/3]$ and the structure $\{r = 1, m = 0, q, p = 1/2\}$ is an equilibrium, since the payoff of playing M or L does not exceed 2.

3. Consider the game in question 4.4 b) in the book. Is there a separating equilibrium? Is there more than one? Justify each of your answers (i.e. derive the equilibrium, or show that such an equilibrium does not exist).

   Answer: Yes, there exist (exactly) two separating equilibria.

   A separating equilibrium means that both players play a different action. Assume type t_1 plays R and type t_2 plays L. Let p be receiver's belief that the sender is type t_1 if he played L, and let q be the belief that the sender is type t_1 if he played R. Then the equilibrium beliefs have to be $p = 0$ and $q = 1$. Under the beliefs it is optimal for the sender to play (u, d), i.e. to play u after L and to play d after R. This means that sender type t_1 obtains 4 in equilibrium, which is more than he could have gotten at any other end-node and therefore he has no incentives to deviate. Sender type t_2 obtains 3 in equilibrium and would only obtain 2 after deviating. Therefore $((R, L), (u, d), p = 0, q = 1)$ is a PBNE (and it is the only PBNE where t_1 plays R and t_2 plays L).

   When the sender plays (R, L) we do not have flexibility either in terms of the beliefs or in terms of the equilibrium strategy of the receiver. Therefore any other separating PBNE has to involve the sender playing (L, R). In this case equilibrium beliefs have to be $p = 1$ and $q = 0$, which means that the optimal equilibrium actions by the receiver are (d, u). Then sender t_1 obtains 1 in equilibrium and would obtain only 0 by playing R. Similarly, sender t_2 obtains 1 in equilibrium and would obtain only 0 by playing L. Therefore, none of them has an incentive to deviate, and $((L, R), (d, u), p = 1, q = 0)$ is a PBNE (and it is the only PBNE where t_1 plays L and t_2 plays R).

Question 2: Consider a setting with one seller and two potential buyers of an indivisible good. Assume buyers' valuations are i.i.d. draws from a uniform distribution on $[0, 1]$. The valuation of each buyer is his private information. Assume the seller runs the following auction: Both potential buyers submit a bid. The buyer with the higher bid wins the object. Each potential buyer pays the bid (whether he wins the object or not - this is an all-pay auction!). Solve for the symmetric equilibrium in which both buyers use a strictly increasing and differentiable bidding strategy.

Answer: An all-pay auction means that a bidder who bids b_i and has valuation v_i obtains the following utility dependent on the bid b_j of the other player

$$u_i(v_i, b_i, b_j) = \begin{cases} v_i - b_i & if \quad b_i > b_j \\ \frac{1}{2}v_i - b_i & if \quad b_i = b_j \\ -b_i & if \quad b_i < b_j \end{cases} .$$

Assume that player 2 uses the strictly increasing and differential bidding function $B(v)$. When player 1 draws valuation v he maximizes the following program:

$$\begin{aligned} &\max_b && \Pr\{b > B(v)\}v - b \\ \Leftrightarrow &\max_b && \Pr\{B^{-1}(b) > v\}v - b \\ \Leftrightarrow &\max_b && B^{-1}(b)v - b, \end{aligned}$$

where we arrive from the first to the second line when b is in the domain of the bids of the other player, i.e. $b \in [B(0), B(1)]$. (We will be true later when we impose symmetry). If the choice of b is interior (we will verify this later), then the optimal b is given by the following first order condition:

$$\frac{1}{B'(B^{-1}(b))}v = 1.$$

Imposing symmetry such that $b = B(v)$ we get

$$\begin{aligned} \frac{1}{B'(v)}v &= 1. \\ \Leftrightarrow v &= B'(v). \end{aligned}$$

Integration both sides yields

$$\frac{1}{2}v^2 + C = B(v),$$

where C is a constant of integration. Since $B(0)$ cannot be larger than zero (otherwise the bidder with valuation zero would not participate in the auction) and $B(0)$ cannot be smaller than zero (when we impose that the seller does not give the good away at negative prices) we have $C = 0$. Therefore, the bidding function is

$$\frac{1}{2}v^2 = B(v),$$

which implies that the assumptions we made in the derivation are correct. Also, one can easily check that the second order condition for a maximum is fullfiled, and so we found the symmetric equilibrium bidding behavior.

Question 3: Consider a setting with one seller and two potential buyers as in the question before. But now assume that the buyer also has a private value v_s for the good which is drawn from a uniform distribution on $[0, 1]$. Assume a mechanism designer that wants to maximize the surplus in this setting. How does the mechanism design program look like? Explain what you write up. (If you cannot solve this problem, consider the case where there is only one buyer as done in class and explain what you write up).

Answer: Consider any extensive form game. Let the players be denoted by index $i \in \{1, 2, s\}$. If buyer 1 has type v_1 and buyer 2 has type v_2 and the seller has valuation v_s, the (possibly mixed equilibrium strategy) will lead to some probability $X_i(v_1, v_2, v_s)$ that player i holds the good at the end of the game. Similarly, $\Pi_i(v_1, v_2, v_s)$ is the expected payment of player i (which tends to be negative for the seller, i.e. the seller tends to collect money). The utility of each agent is then

$$U_i(v_1, v_2, v_s) = X_i(v_1, v_2, v_s)v_i - \Pi_i(v_1, v_2, v_s)$$

We intend to maximize the value that is created in the exchange process

$$\max_{X_1(.),X_2(.),X_s(.),\Pi_1(.),\Pi_2(.),\Pi_s(.)} \int\int\int [X_1(v_1, v_2, v_s)v_1 + X_2(v_1, v_2, v_s)v_2 + X_3(v_1, v_2, v_s)v_3]\, dv_1 dv_2 dv_s,$$

where $X_i(.)$ and $\Pi_i(.)$ are functions of v_1, v_2 and v_3. We know that any equilibrium outcome can also implemented by a game in which we ask the agents about their type and promise payoffs that reveal the truth. So the maximization problem has the following truth-telling (incentive compatibility) constraint for each agent $i \in \{1, 2, s\}$, where we integrate over the types of the other players j and k :

$$\int\int X_i(v_i, v_j, v_k)v_i - \Pi_i(v_i, v_j, v_k)dv_j dv_k \geq \int\int X_i(\hat{v}_i, v_j, v_k)v_i - \Pi_i(\hat{v}_i, v_j, v_k)dv_j dv_k$$

for all $v_i, \hat{v}_i, v_j, v_k$. Moreover, if the agent already knows his type, we have to give him at least as much utility as he could have gotten by not participating in the mechanism. Therefore we get the following interim individual rationality (IR) constraint for each buyer $i \in \{1, 2\}$

$$\int\int X_i(v_i, v_j, v_k)v_i - \Pi_i(v_i, v_j, v_k)dv_j dv_k \geq 0$$

for all v_i. For the seller we get[2]

$$\int\int X_s(v_s, v_j, v_k)v_s - \Pi_s(v_s, v_j, v_k)dv_j dv_k \geq v_s$$

for all v_s. Finally, the probabilities of keeping the good have to be positive and add to one, and the transfers have to add to zero (what somebody pays somebody else has to get). That is, the maximization problem has to obay

$$\begin{aligned} X_i(v_1, v_2, v_s) &\geq 0, \\ \sum_i X_i(v_1, v_2, v_s) &= 1, \\ \sum_i \Pi_i(v_1, v_2, v_s) &= 0. \end{aligned}$$

[2]By subtracting v_s from both sides of the inequality we get

$$\int\int [1 - X_s(v_s, v_j, v_k)][-v_s] - \Pi_s(v_s, v_j, v_k)dv_j dv_k \geq 0.$$

which has the same interpretation we used in class: The seller looses v_s when he trades the good (where $1 - X_s(v_s, v_j, v_k)$ is the probability that he trades).

This program characterizes the optimal outcome in any exchange process in which agents have private information (and know this information before they decide whether they want to participate or not).

Question 4: Solve for a pooling equilibrium and a separating equilibrium in Spence (1974) signalling model as it is presented in the book, under the special assumption that $c(H,e) = e^2$, $c(L,e) = 2e^2$, $y(H,e) = 4$, $y(L,e) = 1$.

Answer: In this setting education is not productive. If the workers ability were observable, none of the workers would aquire education, the high type would earn a wage of 4 and the low type a wage of 1.

Since ability is not observable, we have to analyze a signalling game. The qustion did not provide the probability q that the worker is a high type. Therefore we will consider any $q \in (0,1)$. Let's first consider a pooling equilibrium. An easy starting point is pooling on the full-information education level $e_p = 0$. If we can find an equilibrium where $e(H) = e(L) = 0$, the equilibrium belief has to be $\mu(H|e=0) = q$. Off-the-equilibrium-path beliefs are not determined, and the belief that makes it easiest to sustain a pooling equilibrium is the belief that $\mu(H|e) = 0$ if $e \neq 0$. The wages by the firms given their beliefs are then

$$w(e) = \begin{cases} 4q + (1-q) & if \quad e = 0 \\ 1 & if \quad e \neq 0 \end{cases} . \tag{1}$$

Clearly, given this wage schedule every worker finds it optimal to obtain zero units of education. Therefore, we have verified that $e(H) = 0, e(L) = 0$, $w(e)$ according to (1), $\mu(H,0) = q$ and $\mu(H|e) = 0$ if $e \neq 0$ constitutes a pooling PBNE.[3]

Let's now consider a separating equilibrium. First, oberve that in a separating equilibrium the low type chooses $e(L) = 0$ levels of education. To see that, assume that $e(L) = e > 0$ could be sustained in a separating equilibrium. Since the equilibrium is separating the firms infer after observing e that the worker is of a low type, and will offer him a wage of 1. This gives an equilibrium utility level $1 - 2e^2$. If the low type deviates and gets education level zero, he gets a wage of at least 1 because firms expect at worst a low type for sure, and he does not have to pay education costs, and so education level zero is strictly better than the equilibrium level of eduction.

Define e_s as the indifference for the high type between getting the low output an acquiring zero education and getting the high output but having to acquire

[3]There are other pooling perfect Bayesian Nash Equilibria. For example, we can have $e_p = 0$ but at some high levels of education some (small) belief that the worker is high. This would still sustain e_p as a pooling equilibrium. (It is easy to verify that $\mu(H|2) = .1$ and associated $w(2) = .1 * 4 + .9 * 1$ together with the rest of the earlier equilibrium still constitutes an equilibrium, since it is not worthwhile for either agent to aquire 2 units of education even with this slightly improved wage). Similar, it is easy to see that other pooling education levels can be sustained, such as $e_P = .1$, because the threat to be taken as a low type at any other level of education is just too high.

e_s units of eduction:

$$\begin{aligned} 1 &= 4 - 2e_s^2 \\ &\Leftrightarrow \quad e_s = \sqrt{1.5}. \end{aligned}$$

Let's sustain an equilibrium where $e(H) = e_s = \sqrt{1.5}$. Let's assume the beliefs are

$$\mu(H|e) = \begin{cases} 1 & if \quad e \geq \sqrt{1.5} \\ 0 & if \quad e < \sqrt{1.5} \end{cases} \tag{2}$$

and the associated wages are

$$w(e) = \begin{cases} 4 & if \quad e \geq \sqrt{1.5} \\ 1 & if \quad e < \sqrt{1.5} \end{cases}. \tag{3}$$

This is optimal for the firms given their belief, and the belief is consistent with the education choices of the workers. For the workers, only zero or $\sqrt{1.5}$ units of education can be optimal (any other level increases the education costs without increasing the wage). By construction, the low type worker is indifferent between zero and $\sqrt{1.5}$ units of education and has no incentive to deviate. For the high type worker the equilibrium utility is $4 - 1.5 = 2.5$. Deviating to education level zero would yield a utility of $1 < 2.5$. Thus, no worker type has an incentive to deviate. We have shown that $e(H) = \sqrt{1.5}$, $e(L) = 0$, μ as in (2) and wages as in (3) constitute a perfect Bayesian Nash Equilibrium.[4]

Question 5: (Adverse Selection) Consider a setting with two insurance companies called 1 and 2 and one individual who wants to buy insurance. The individual has probability p of becoming sick. He has income of 400, but if he is sick he has to give up 300 in terms of medical bills. One dollar buys one unit of consumption, and the individual is risk averse over consumption c: his utility function is $u(c) = \sqrt{c}$. The insurance companies are risk neutral (have utility functions that is linear in money with slope 1).

Assume that the individual is a high risk type with probability 1/2, in which case $p = .6$. With complementary probability he is a low risk type, in which case $p = .4$. The individual knows his type, the insurers do not.

To make the question simpler, assume that fees for insurance are fixed by the government: Insurance companies have to charge a fee $F = 150$ which the individual has to pay if he signs up for insurance and stays healthy. Each insurance company $j \in \{1, 2\}$ is free to choose the amount of insurance premium I_j that it will pay to the individual in case the individual becomes sick.

The parties play the following game. First, both insurance companies simultaneously announce the insurance amount I_j that they will pay in case the individual becomes sick. Then the individual decides whether he wants to sign up for insurance or not. He can sign up with at most one company, but can choose not to sign up with any company. Finally, the health outcome is realized (and the insured person pays F if he stays healthy and collects I but has to pay

[4] Again, other pooling equilibria can be sustained. See the book for more details.

the medical bills if he becomes sick; while an uninsured person pays neither F nor can he collect I).

NOTE: The fee $F = 150$ makes it possible for the insurance company to break even AND to offer full insurance to both types IF both types are forced to participate in the insurance program. Full incurance means that consumption is equal in both states, i.e. that $400 - F = 100 + I$, which yields full insurance amount $I = 150$. In this case the average probability of becoming sick is .5, and so under full insurance the expected loss is $.5 * I = .5 * 150 = 75$. The insurance company obtains $.5 * F = .5 * 150 = 75$. Therefore, full insurance is possible if both types have to participate in the mechanism.[5]

Please answer (for some of the values you might want to use a calculator):

1. What is the strategy set for each player? (Remember that a strategy is an action for each information AND type).

   Answer: Denote the type space for the individual as $T = \{L, H\}$, where H stands for "high risk" and L stands for "low risk". A strategy for the individual is a function f that maps the type space $\{H, L\}$ and the insurance offers $\mathbb{R}_+^2$ into $\{A_1, A_2, R\}$, where R stands for "reject", A_1 stands for "accept the insurance from firm 1" and A_2 stands for "accept the insurance from firm 2". That is: $f(H, 100, 50) = A_1$ means that the worker who is a high-risk type and is offered insurance $I_1 = 100$ and $I_2 = 50$ chooses to accept the insurance offer from firm 1. The strategy space of the worker is the set of all such functions f.

   The strategy space for firm $j \in \{1, 2\}$ is a number in the positive real line $\mathbb{R}_+$ that represent its insurance amount I_j. Since it can offer any amount and does not observe anything before making its offer, its strategy space is just the entire positive real line $\mathbb{R}_+$.

2. What is the belief of the insurance companies? (The individual knows his type and therefore his belief is trivial. The belief of the insurance companies about the type of the individual is also simple to find, because the insurance companies do not observe any action by the individual before they set the contract and therefore they cannot learn anything.)

   Answer: Since the insurance company has no ability to learn anything, its belief about the type of the agent is simply that the individual is a high risk type with probability 1/2.

3. If the individual knows his type before signing the contract (but the insurance company does not know his type when deciding what I to offer), full insurance $I_j = 150$ is not an equilibrium outcome.

[5] Competition among insurers will deliver such outcome. Moreover, it is easy to see that full insurance is possible if the individual only learns his type after he has signed the contract (i.e. at the beginning of his life). Moreover, in this case the two insurers would drive the price F down to 150 even if this were not government mandated (the competition between two firms is used in many models to give the same outcome as in a competitive market where many firms compete - because two firms suffice to drive profits to zero). Private information destroys this full insurance outcome, as we will see in the following.

Answer: Assume the firms offer $I_j = 150$. Consider first the low risk type. If the low type accepts, his utility is $\sqrt{250} \approx 15.81$. If he rejects, his utility is $.4 * 10 + .6 * 20 = 16$. So the low risk type rejects! Consider now the high risk type. If he accepts, his utility is also $\sqrt{250} \approx 15.81$. If he rejects, his utility is $.6 * 10 + .4 * 20 = 14$. So the high risk type accepts!

But if only the high risk type accepts, the firm will make a loss. Conditional that its offer is accepted (which will mean that the individual is a high risk type) it will collect the fee of 150 with probability .4 and will have to pay the insurance of 150 with probability .6, and therefore it will make a loss. This cannot be an equilibrium since the firms can always insure themselves weakly positive profits by offering $I_j = 0$.

This is called *adverse selection*, since the offer is only selected by the bad types.

4. If the individual knows his type but is sufficiently risk-averse (e.g. for $u(c) = \sqrt[4]{c}$) show that $I_j = 150$ is an equilibrium outcome.

   Answer: When the individual is more risk averse, insurance is more valuable. In particular, a high risk type has even more incentives to accept this contract. Now consider the low risk type. If that type accepts he gets in expectation $\sqrt[4]{250} \approx 3.976$. If this type rejects he gets $.4 * \sqrt[4]{100} + .6 * \sqrt[4]{400} \approx 3.948$, so he accepts. If both types accept, the insurance company whose offer is accepted breaks exactly even. No insurance company wants to deviate because lowering the insurance means that the individual will accept the offer of the other firm, and raising the insurance premium means that the firm will make a loss.