

Final Exam 892 - Game Theory

Please list your NAME: _____

Note: Please answer all questions on the paper provided (feel free to use both sides). There are 6 questions in total. All answers have to be *justified*. All questions refer to *pure strategies* unless otherwise mentioned.

Question 1: Consider the game depicted in Figure 1. Consider the following equilibrium (partially mixed) strategies by players 1 and 2: player 1 randomizes equally between his three choices, and player 2 plays "out" after A, "R" after B and randomizes equally across "R" and "L" after C. What are the equilibrium beliefs of player 3 across the six nodes [4] to [9] in his information set. It might be easiest to write p_i as the probability he assigns to node [i].

Answer: $p_1 = p_2 = p_3 = 0$ because these path are not taken by player 1.

$p_4 = \frac{1/3}{1/3+1/3} = 1/2$ since the overall probability of getting into the information set is $1/3$ and $1/3$, and this particular path has probability $1/3$ in the overall game.

$p_5 = p_6 = \frac{1/2*1/3}{1/3+1/3} = 1/4$ since the overall probability of reaching the information set is $1/3 + 1/3$, and the combination CL has probability $1/2 * 1/3$ in the overall game.

Question 2: Consider the game depicted in Figure 2.

1. Find ALL pure strategy Perfect Bayesian Nash Equilibria! Do they satisfy the forward induction refinements $R5$ (or the "Intuitive Criterion" $R6$) that we introduced in class (and why?).

Answer: Let q be player 2's belief of M and $1 - q$ his belief of R .

One pure strategy equilibrium is $(M, b, q = 1)$. For player 1 this gives the highest payoff in the game, and for player 2 it is better to play b than a if player one plays M . The belief is consistent with the action, and there are no off-equilibrium path beliefs to which refinements can be applied.

Another pure strategy equilibria is (L, a, q) with $q \leq 1/2$. Player 1 plays L only if player 2 plays a , which he is only willing if he is more than 50% sure to be in the branch following R . This does not satisfy $R5$ (nor $R6$) since R is dominated by L (and M is not).

Question 3: Consider the game depicted in Figure 3.

Find a pooling equilibrium! Does it satisfy the forward induction refinements $R5$ (or the "Intuitive Criterion" $R6$) that we introduced in class (and why?).

Answer: There is a pooling equilibrium $((R, R), (u, d), p = 1, q = .5)$. When both players play R , then $q = .5$ and it is optimal for player 2 to play d . Player t_1 gets 4. By deviating he gets only 3 (since after L player 2 plays u). Player t_2 gets 5 and gets only 4 after L . Finally, after L it is optimal for player 2 to play u if his belief is $p = 1$. This belief survives $R5$ and $R6$ because after L there is an action that makes player t_1 better off than his equilibrium payoff.

Question 4: Consider the signalling model of Spence (1973): There is a worker whose type is either H or L . Both types are equally likely. The worker can acquire education at cost $c(e|H) = e^2$ or $c(e|L) = e^3$, depending on his type. The output of an L worker is $y(L, e) = 3e$. (So he would choose $e^*(L) = 1$ if his type were known.) The output for an H type is $y(H, e) = 5e$. There are two firms that bid for the workers services after observing the education level. Find a separating equilibrium.

Answer: The easiest separating equilibrium can be found by looking at the education level e_s that makes the low type indifferent between the high and the low wage:

$$\begin{aligned} 3e^*(L) - e^*(L)^3 &= 5e_s - e_s^3 \\ \Leftrightarrow 2 &= 5e_s - e_s^3 \\ \Leftrightarrow e_s &= 2. \end{aligned}$$

Then $e(L) = 1$, $e(H) = 2$, $\mu(H|e) = 0$ if $e < 2$, $\mu(H|e) = 1$ if $e \geq 2$ and $w(e) = 3e$ if $e < 2$ and $w(e) = 5e$ if $e \geq 2$ is a candidate for a Perfect Bayesian Nash Equilibrium. Up to here already gave nearly all points.

Still it is important to check that this is an equilibrium. The beliefs are consistent with equilibrium behavior at $e = 1$ and $e = 2$ (and we can choose them freely elsewhere, yet note that this satisfies R5 and R6). In the region $e > 2$ the high type has utility

$$5e - e^2$$

which has a derivative

$$5 - 2e.$$

Therefore the optimal choice is 2.5! So he would deviate. Therefore the equilibrium here is e.g. $e(L) = 1$, $e(H) = 2.5$, $\mu(H|e) = 0$ if $e < 2$, $\mu(H|e) = 1$ if $e \geq 2$ and $w(e) = 3e$ if $e < 2$ and $w(e) = 5e$ if $e \geq 2$ is a candidate for a Perfect Bayesian Nash Equilibrium. It is easy to check that this is indeed an equilibrium.

Question 5: Consider the following auction with two buyers who bid for one indivisible good. The valuation for each buyer is drawn i.i.d. from a distribution that is uniform on $[1, 2]$. That is, valuations are between 1 and 2. The bidder with the highest bids wins the object (in case of equal bids the winner is determined by a fair coinflip). The winner has to pay his bid, the looser does not pay anything (first price auction).

1. Solve for the symmetric equilibrium bidding behavior.
2. Write up the program that solves for the optimal revenue that can be achieved in any Perfect Bayesian Nash Equilibrium of any mechanism (you do not need to solve it).

Answer: Part 1: Consider an player 1 with valuation v_1 who considers which bid b_1 to place. Assume some increasing and differentiable bidding function $B(v_2)$ of player two. The first player's optimization problem is then

$$\begin{aligned} \max \Pr\{b_1 > B(v_2)\}(v_1 - b_1) \\ \Leftrightarrow \max B^{-1}(b_1)(v_1 - b_1) \end{aligned}$$

The first order condition for this problem is

$$\frac{1}{B'(B^{-1}(b))}(v_1 - b_1) - B^{-1}(b_1) = 0$$

Given a symmetric equilibrium we have $b_1 = B(v_1)$ so that the FOC simplifies to

$$\begin{aligned} \frac{1}{B'(v_1)}(v_1 - B(v_1)) - v_1 &= 0 \\ \Leftrightarrow B(v_1) + B'(v_1)v_1 &= v_1 \end{aligned}$$

Integrating both sides yields

$$B(v_1)v_1 = \frac{1}{2}v_1^2 + C$$

where the constant of integration C is determined by the fact that the lowest bidder type will not bid less than 0 or more than one. So any $C \in [0, 1]$ will do. Let's go with

$$B(v_1) = \frac{1}{2}v_1.$$

So $B(v) = \frac{1}{2}v$ is a symmetric equilibrium bidding behavior.

Part 2: Let $X_i(v_1, v_2)$ be the probability that buyer i gets the good, and let $\pi_i(v_1, v_2)$ be the expected transfer to the seller. The seller maximizes the transfers:

$$\max_{X_1(\cdot), X_2(\cdot), \pi_1(\cdot), \pi_2(\cdot)} \int_1^2 \int_1^2 [\pi_1(v_1, v_2) + \pi_2(v_1, v_2)] dv_1 dv_2$$

such that each seller type v_i of type i tells the truth:

$$\int_1^2 [X_i(v_i, v_{-i})v_i - \pi_i(v_i, v_{-i})] dv_{-i} \geq \int_1^2 [X_i(\hat{v}_i, v_{-i})v_i - \pi_i(\hat{v}_i, v_{-i})] dv_{-i}$$

for all $v_i \in [1, 2]$, $\hat{v}_i \in [1, 2]$ and all $i \in \{1, 2\}$. Also, each seller has to get at least as much as if he does not trade:

$$\int_1^2 [X_i(v_i, v_{-i})v_i - \pi_i(v_i, v_{-i})] dv_{-i} \geq 0$$

for all $v_i \in [1, 2]$ and all $i \in \{1, 2\}$.

Figure 1:

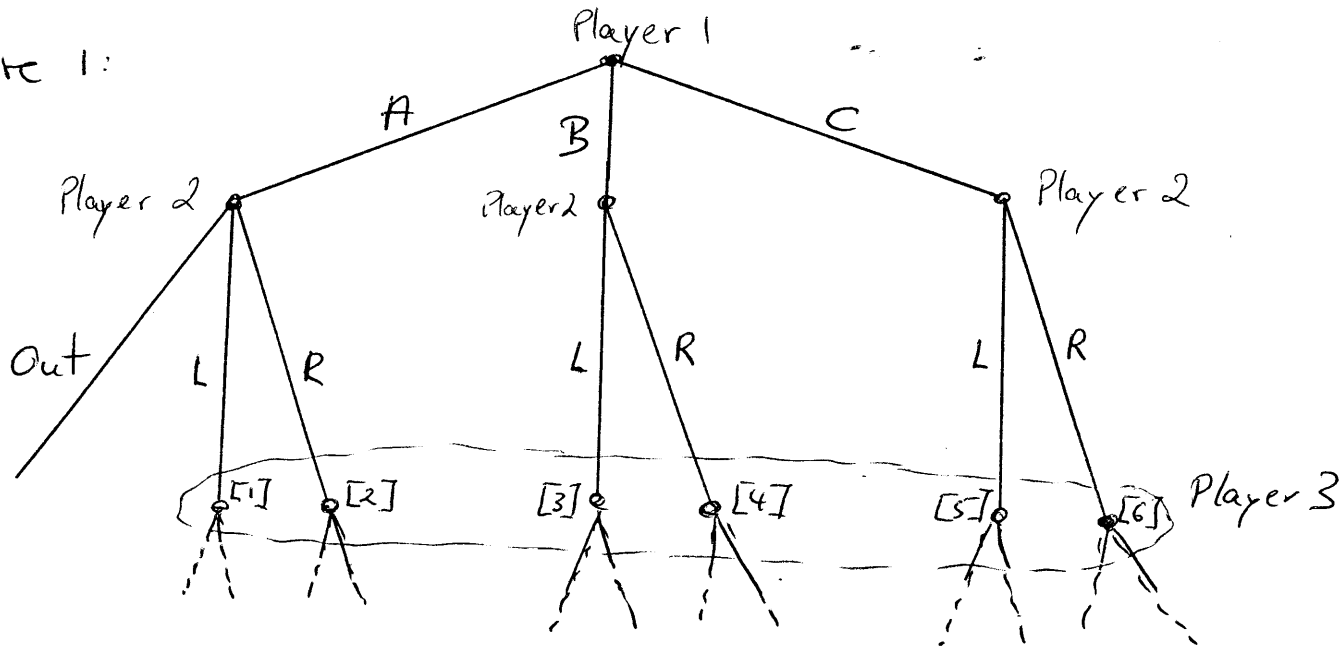


Figure 2:

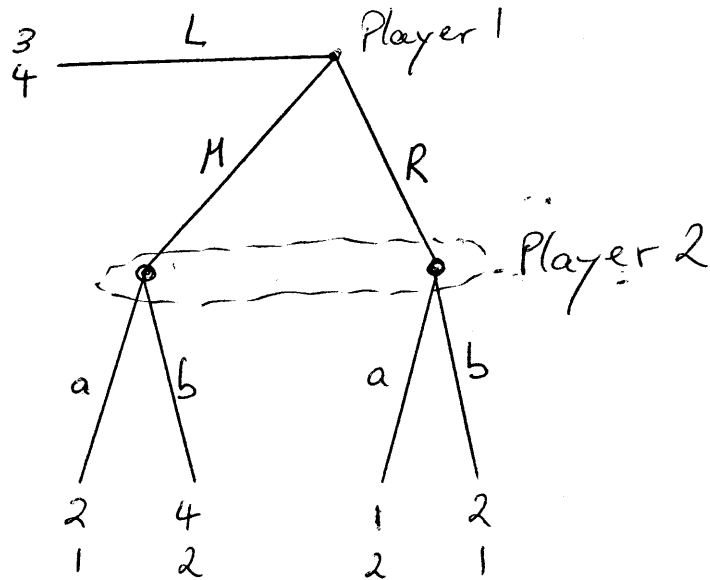


Figure 3

