Optimal Fiscal Policy in a Business Cycle Model without Commitment*

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Abstract

This paper studies optimal taxation in the stochastic growh model when the goverment cannot commit. We use recursive game theory to characterize the set of Sustainable Equilibria and to build strategies that support equilibrium payoffs. We calibrate our model to match U.S. data and compute both the set of sustainable equilibria payoffs, strategies that implement them and triggers. We also look at the Best Equilibrium under no commitment and compare it with the Markov Perfect Equilibrium and with the Ramsey Equilibrium.

Key words: Optimal Fiscal Policy, Business Cycle, Recursive Game Theory, Computational Methods.

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1. Introduction

This paper addresses the following question: how should taxes be set over the business cycle when there is no commitment for the government?

The answer to our question in the case in which the government has access to a commitment technology was developed in an important contribution by Chari, Christiano and Kehoe (1994) who explored the quantitative implications of the theory. That contribution is complemented by Zhu (1992) that showed some interesting theoretical properties and by Stockman (2001) when the government must follow a balanced-budget rule.

However it is also well understood that, in general, the optimal policy prescribed by the theory is time inconsistent. Once capital is accumulated, taxing it is undistortionary and a benevolent government will deviate to a high capital tax to ease the (distortionary) taxation of labor.

Time inconsistency is not a theoretical curiosity. Casual observation of constitutional arrangements in most countries suggest that, in fact, governments are not usually bound in their choice of taxes for more than a relatively short period of time and that changes in the tax arrangements are frequent. This empirical observation raises naturally the problem of how we can characterize the optimal fiscal policy over the cycle when no commitment device is available.

This paper show how to find the set of sustainable equilibria for a calibrated version of the stochastic growth model using the tools developed by Abreu, Pearce and Stacchetti (1990) and by Phelan and Stacchetti (2001). With this set we can build the strategies associated with any particular point in that set and explore the quantitative implications for aggregate quantities and welfare. Sustainable Equilibrium, as defined by Chari and Kehoe (1990) is the natural solution concept is this context since we want to assure sequential rationality for the government and that we always remain in a competitive equilibrium where consumers are small enough to affect, each one of them by itself, aggregate outcomes.

Our results show a relatively small set of sustainable equilibria. The strategies computed for the best equilibrium also imply positive taxation of capital and labor, departing from Ramsey, but with a taxation on capital much lower than in a Markov equilibrium and labor tax rate assuming most of the effects of the stochastic shocks

We can also use for framework for answer several related questions. We measure the improvements from introducing a commitment device and we look at the optimal response function of the government to stochastic shocks.

Our paper relates to a growing literature on the study of optimal policy in the absence of commitment. We can highlight several contributions. Klein and Ríos-Rull (2002) and Klein, Krussel and Ríos-Rull (2002) explore these questions when Markov Perfect equilibria are used as the solution concept. Markov equilibria are natural and intuitively appealing. Also the strategies that support them are very simple to describe. However, since by construction, Markov equilibria abstract from reputational mechanisms it is difficult to asses how much is lost looking at this subset of equilibria. Also some times it is difficult to build this type of equilibria. Benhabib and Rusticcini (1997) argue, in the context of a deterministic model, for the use of optimal control when the problem of the government is appropriately augmented by an additional incentive compatibility constraint. Optimal control simplifies the analysis but we do not have a constructive procedure to write down the value of the deviation needed for the incentive constraint. This limits its utility in practical applications. Closer to us we find Chari and Kehoe (1990), Sleet (1997) and Chang (1998). Chari and Kehoe (1990) proposes a method to check if a given tax policy is time-consistent. Sleet (1997) develops similar tools than Phelan and Stacchetti (2001) and explores the set of equilibria in an overlapping generation economy. Chang (1998) uses recursive game theory to study optimal monetary policy in a very similar spirit to our approach.

We feel that our approach complements the existing results in the literature. With respect to the papers on Markov equilibria we add the possibility of exploring systematically all the set of equilibria and assess whether reputational mechanisms are of quantitative importance and, if they are, whether they are plausible or not. With respect to the literature in optimal control we offer a constructive procedure to find the value associated with a deviation. Finally with respect to the theoretical results, we offer quantitative assessments in the framework of the stochastic neoclassical growth model.

It is important to note that we abstract from several important features of the data. First, for computational reasons[1], we exclude the study of public debt. It is unknown to us how restrictive this abstraction is. Lucas and Stokey (1983) showed in a model without capital the role that public debt may have as a substitute for commitment. Whether any similar intuition can be extended to the stochastic growth model is not intuitive. Also in terms of welfare, since debt works as a shock absorber in the results of Chari, Christiano and

[1]Including debt expands our state space by one additional variable. As we will show below that extension causes a substantial computational burden we want to overcome in the short future.

Kehoe (1994), we may be getting an inaccurate measure of the value of commitment. Second we abstract from any heterogeneity of agents. Different types of agents may raise interesting political-economics issues that deserve careful study. We feel however that we need to explore first the implications of the theory in the simplest case of homogeneous agents.

The rest of the paper is organized as follows. Section 2 presents the stochastic neoclassical growth model with government and taxation. Section 3 discusses how we can generate a recursive formulation of the problem that it is easier to work with than the original formulation. Section 4 outlines some details of the computation used to find the set of sustainable equilibria and the strategies that implement them and the associated triggers. The economy is calibrated to match certain characteristics of the U.S. data in section 5. Section 6 presents our finding and section 7 concludes. An appendix provides further computational details and some proofs omitted in the main text.

2. The Economy

2.1. The Environment

We will consider a production economy populated by a measure one of identical, infinitely lived consumers. In each period $t = 0, 1, ...,$ the economy experiences one possible realization of an stochastic process $s \in S$ where the initial realization s_0 is given. To simplify the analysis we will work with finite sets S. Note that so far we do not require any particular structure in the transition equation for s_t. The realization in period t of the process is denoted by s_t and the history of t realizations from 0 through t by $s^t = (s_0, s_1, ..., s_t)$. Given the structure of uncertainty we can use a commodity space in which goods are indexed by histories. The probability of each of these histories is given by $\pi(s^t)$. With this probability we can define conditional probabilities of future events given a particular history s^t that we will call $\pi(s_{t+j}| s^t)$. For future convenience we also define the binary relation $\succ$ between two realizations of the shock s_t and s_{t+j}, $s_{t+j} \succ s_t$, if the event s_{t+j} is compatible with history s^t, i.e. if $\pi(s_{t+1}| s^t) > 0$. Also we define $s_{t+j} \succeq s_t$ if either $s_{t+j} \succ s_t$ or $s_{t+j} = s_t$.

In each period a competitive-behaving firm has access to a technology to produce the final good, $y(s^t)$, using capital, $k(s^{t-1})$ and labor, $l(s^t)$, given by the neoclassical production function:

$$F\left(k\left(s^{t-1}\right), l\left(s^t\right), s^t\right) \tag{1}$$

Note that we index capital at period t by the history s^{t-1} since it is a predetermined variable at the beginning of the period. Also the history s^t is an argument of the production function. This allows us to think about productivity shocks as possible (but not necessarily the only) events in S. In general we will concentrate in cases where only s_t is relevant to determine the current period technology.

Competitive pricing ensures that input prices equate marginal productivities:

$$r\left(s^t\right) = F_k\left(k\left(s^{t-1}\right), l\left(s^t\right), s^t\right) \tag{2}$$

$$w\left(s^t\right) = F_l\left(k\left(s^{t-1}\right), l\left(s^t\right), s^t\right) \tag{3}$$

The final output can be used for private consumption, $c\left(s^t\right)$, government consumption, $g\left(s^t\right)$ and investment good $i\left(s^t\right)$. The law of motion for capital $k\left(s^t\right)$ is given by:

$$k\left(s^t\right) = i\left(s^t\right) + (1-\delta)\, k\left(s^{t-1}\right) \tag{4}$$

where δ is the depreciation rate on capital and where we impose $i\left(s^t\right) \geq 0$. We will discuss below the role of this nonnegativity constraint.

The preferences of each consumer over sequences of consumption and labor are representable by a time-separable utility function:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) u\left(c\left(s^t\right), l\left(s^t\right)\right) \tag{5}$$

where $0 < \beta < 1$ is the discount factor, $0 \leq l\left(s_t\right) \leq 1$ is the labor supply and the period utility function u satisfies usual assumptions (strict concavity, differentiability and Inada conditions). To save on notation we pick normalized utility functions (i.e. already multiplied by $(1-\beta)$).

The consumer budget constraint associated to this objective function is:

$$c\left(s^t\right) + i\left(s^t\right) = \left(1-\tau_l\left(s^t\right)\right) w\left(s^t\right) l\left(s^t\right) + \left(1-\tau_k\left(s^t\right)\right) r\left(s^t\right) k\left(s^{t-1}\right) \tag{6}$$

$$k\left(s^{-1}\right) > 0 \text{ given} \tag{7}$$

where $\tau_l\left(s^t\right)$ is a proportional tax on labor income, $\tau_k\left(s^t\right)$ a proportional tax on the return of capital and $k\left(s^{-1}\right)$ is the initial endowment of capital of (almost all) consumers. To save in notation we omit in (6) the trading of Arrow-Debreu securities by private agents. Since all our consumers are identical these securities will not be traded in equilibrium. It is important, however, to remember that the securities exist and could be used out of equilibrium. We will discuss the absence of public debt in more detail below.

There is a benevolent government that maximizes:

$$v = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) \left[u\left(c\left(s^t\right), l\left(s^t\right)\right) + G(g(s^t))\right]$$

where G is a (also normalized by $1-\beta$) function of the government consumption $g\left(s^t\right)$. We will assume that $G(\cdot)$ is increasing and concave. We could, equivalently, added that last term in the objective function of the consumer. Separability and the smallness of each consumer that take $g(s_t)$ as given make both choices equivalent. We feel our choice is marginally clearer for the analysis below[2].

In each period the government sets g equal to the revenue raised by the two taxes

$$g\left(s^t\right) = \tau_l\left(s^t\right) w\left(s^t\right) l\left(s^t\right) + \tau_k\left(s^t\right) r\left(s^t\right) k(s^{t-1})$$

This equation implies that we impose a balanced-budget rule event by event. As explained in the introduction, eliminating public debt allow us to reduce the state space of the recursive problem we will develop later. Computational reasons strongly suggest this choice. The balanced budget assumption is a deviation from the basic model of Chari, Christiano and Kehoe (1994) that allow public debt with one-period maturity and a state-contingent return. Stockman (2001) has studied the effects of introducing this balanced-budget rule for the case with commitment.

We will constrain the freedom of the government to set taxes to those rates that belong to the interval $T = \{[\underline{\tau}, \overline{\tau}] | \, 0 < \underline{\tau} < \overline{\tau} < 1\}$. We again depart from the basic framework of Chari, Christiano and Kehoe (1994), specially since we do not allow taxes to be negative. This constraint maybe binding (Zhu (1992) shows that the probability that the optimal tax rate on capital is less than zero in the Ramsey problem is strictly positive). However since the government must balance its budget event by event and, as we will describe below, taxes are announced before the consumer make their choices, a subsidy to an input (a negative tax)

[2] Also this problem is slighlty different from the basuc Chari, Christiano and Kehoe (1994) economy in which the government needs to finance an exogenously given stream of public consumption. In our formulation the government can substitute the public consumption intertemporally.

might not be financed for certain choices of the consumers. Even if this would not happen in equilibrium, we would need to spend some time fixing this possibility without gaining further insight in the problem.

Let us finish the description of the economy mentioning the existence of an exogenous, uniform $[0, 1]$, serially uncorrelated random variable X_t with a publicly observed realization x_t at the beginning of each period t and given initial value x_0. This random variable will be helpful later to convexify the set of equilibria.

2.2. Competitive Equilibria

Even if we are concerned with the case where there is no commitment, it turns out to be convenient to think about competitive equilibria where the government can commit to a certain arbitrary state contingent policy. First the government announces a contingent policy. Call that policy $\tau = \{\tau_l(s^t), \tau_k(s^t)\}_{t=0}^{\infty}$. Given that policy we can define an allocation rule as a sequence of functions $a(\tau)$ that map policies into allocations $\{y(s^t), c(s^t), l(s^t), k(s^t), g(s^t)\}$ for each history s^t. Analogously we can define price rules $w(\tau)$, $r(\tau)$ that map policies into price systems for all histories.

A competitive equilibrium is a state contingent policy τ, an allocation rule $a(\cdot)$ and price rules $w(\cdot)$ and $r(\cdot)$ such that (1) given prices, consumers solve their problem, (2) input prices equate the marginal productivities, (3) the government satisfies its budget constraint period by period and (4) markets clear.

2.3. Sustainable Equilibria

Now we are ready to describe the game associated with our environment, $\Gamma(k(s^{-1}), s^0)$. We will have two types of players, the government and the continuum of anonymous consumers[3]. Anonymity implies that the government cannot observe individual choices of consumers but only their distributions. Similarly consumers observe their own actions, the distribution of other consumers' actions and the government choices.

The timing protocol is as follows. First, in each period t, after x_t and s_t are realized, the government picks the tax rates for the period, $\tau_l(s^t), \tau_k(s^t)$ such that the rates belong to T. The fact that the government only decides taxes for the current period embodies our concept

[3] We also have the representative firm but this player only makes trivial decisions equating marginal productivities to prices, to simplify the exposition we omit further discussion of it.

of lack of commitment[4]. Also, because of anonymity, the tax rates are common for all agents. Then the consumers pick $c(s^t)$, $l(s^t)$ and $k(s^t)$ such that their budget constraint is satisfied. These choices imply a value for $g(s^t)$ that is consumed by the government after tax revenue has been raised.

It is natural to define a public history for the game as $h^t = (h_0, h_1, ..., h_t)$ (not to be confused with the history of the stochastic process s^t) where $h_t = (s_t, x_t, \tau_l(s^t), \tau_k(s^t))$. This public history keeps track of the realizations of the stochastic variables and of the choices of the government[5]. A strategy for the government is a measurable function $\sigma_G(\cdot)$ that maps h^{t-1}, s_t and x_t into tax rates for each period t, $(\tau_l(s^t), \tau_k(s^t)) = \sigma_G(t)(h^{t-1}, s_t, x_t)$. A symmetric public strategy for the consumer is a measurable function $\sigma_C(\cdot)$ that maps h^t into $c(s^t)$, $l(s^t)$ and $k(s^t)$, $(c(s^t), l(s^t), k(s^t)) = \sigma_C(t)(h^t)$. The pair of strategies $\sigma_G(\cdot)$ and $\sigma_C(\cdot)$ is called a symmetric public strategic profile σ. If Σ_G is the set of strategies for the government and Σ_C the set of symmetric public strategies for the consumer, we can define the set of symmetric public strategic profiles as $\Sigma = \Sigma_G \times \Sigma_C$.

Here we want to make two comments. First notice that we are restricting our attention to public strategies in which the consumers ignore their private information in choosing their actions. This constraint avoids the need to account for private histories that will be irrelevant along the equilibrium path because of the convexity of the problem (see Phelan and Stacchetti (2001) for details). It still allows equilibria where a player contemplates deviating to a non-public strategy although the consumers prefer not to do so. Fudenberg, Levine and Maskin (1994) provide further explanation. Second, we allow all the strategies to depend on the random variable x_t. The role of this public randomization is to assure, later on, that the set of equilibria that we characterize present desirable convexity properties.

As a solution concept for the game $\Gamma(k(s^{-1}), s^0)$ we adapt the concept of sustainable equilibrium introduced by Chari and Kehoe (1990) to deal with symmetric public strategies and propose the following definition.

A symmetric public strategy profile σ for the game $\Gamma(k(s^{-1}), s^0)$ is a *sustainable equilibrium* if for any history s^t (with corresponding capital $k(s^{t-1})$):

[4] Of course we could allow the government to announce futures rates but, since it can change that decision in each future event, that announcement would be irrelevant.

[5] We could also account for the distributions of variables as capital (remember that individual consumer's actions are not observed and then they can not be included). However the convexity of the consumer's problem imply that all of them will chose the same actions and that no nontrivial distributions will be ever observed (and consequently that those possible deviations do not have any effects on the equilibrium path). Then from h^t we can always recover all the values of the other variables.

1. The continuation equilibrium payoff for the government is higher than the payoff from any deviation to a different strategy.

2. Given $k(s^{t-1})$ and the tax policy, the sequences $\{y(s^t), c(s^t), l(s^t), k(s^t), g(s^t)\}_{s_j \succeq s_t}$ and $\{w(s^t), r(s^t)\}_{s_j \succeq s_t}$ constitute a Competitive Equilibrium.

The rational behind this solution concept is simple. The government follows a strategy from which there is no profitable deviation after any history. This requirement derives directly from sequential rationality. The consumers always respond to the government strategy with decisions that imply a competitive equilibrium since this is the only situation compatible with feasibility and individual optimization. Note that we are explicitly ruling out the possibility of consumers colluding to punish the government after a deviation with a behavior that implies a lower value of v than the worst competitive equilibrium. Since all the consumers are anonymous, each of them will have an incentive to deviate from such a collusion agreement and maximize its own utility. Consequently any attempt in cooperation cannot be an equilibrium. This point is important for our quantitative analysis because it substantially reduces the worst available punishment after a deviation and thus the associated set of sustainable equilibria.

It is also interesting to compare this definition with that of Ramsey allocation. In both cases the government picks a policy to which the consumers respond with a competitive equilibrium, limiting the implementable allocations. The difference is that in Ramsey the government searches over the space of state-contingent policies to find the optimal one and it sticks with it. In a sustainable equilibrium, the requirement is strengthen to assure that the government, in case of being able to restate its plans after some particular history, cannot find any new deviation that increases its value.

3. Recursive Formulation of the Problem

In this section we show how to rewrite the problem of the consumer in a recursive way. Working with the original problem is a challenging task in particular because we need to keep track of the histories for the game h^t instead of just the states, s_t and $k(s_{t-1})$, as in the standard model with commitment. A recursive formulation allows a better understanding of the problem and more importantly, to apply the tools of recursive game theory as we will do in section 4. To achieve our goal, first we will present the sequence problem of the consumer that comes directly from our description of the environment. Then we will propose a recursive version of the same problem and we will finish showing the equivalence between the two formulations.

3.1. The Sequence Problem

Remember that the consumer's problem, for a given government policy $\tau_l(\cdot)$ and $\tau_k(\cdot)$, was

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) u\left(c\left(s^t\right), l\left(s^t\right)\right) \tag{8}$$

subject to

$$c\left(s^t\right) + i\left(s^t\right) = \left(1 - \tau_l\left(s^t\right)\right) w\left(s^t\right) l\left(s^t\right) + \left(1 - \tau_k\left(s^t\right)\right) r\left(s^t\right) k\left(s^{t-1}\right) \tag{9}$$

$$k\left(s^{-1}\right) > 0 \text{ given} \tag{10}$$

We will call this problem the sequence problem.

Given our assumptions about the utility and production functions, a set of first order conditions for the consumer's problem is given by[6]:

$$u_c\left(s^t\right) = \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) u_c\left(s^{t+1}\right)\left(1 + \left(1 - \tau_k\left(s^{t+1}\right)\right) r\left(s^{t+1}\right) - \delta\right) \tag{11}$$

$$u_l\left(s^t\right) = \left(1 - \tau_l\left(s^t\right)\right) w\left(s^t\right) u_c\left(s^t\right) \tag{12}$$

where we use the notation $u_x\left(s^t\right)$ to denote the marginal utility with respect to the variable x evaluated at the allocation in history s^t. The first equation, 11, is the usual intertemporal Euler condition where we sum over all possible realization of the shocks that can follow the current state. The second equation, 12, is the static optimality relation between labor marginal disutility and consumption marginal utility.

3.2. The Recursive Problem

In order to build our recursive problem, we define first a new variable:

$$m\left(s^t\right) \equiv u_c\left(s^t\right)\left(1 + \left(1 - \tau_k\left(s^t\right)\right) r\left(s^t\right) - \delta\right)$$

[6] For simplicity in the exposition we also ignore for the moment the restriction that $i\left(s^t\right) \geq 0$. In our computations below we would include, though, this constraint.

What is m? The marginal value of capital, the increase in the consumer's utility if it had started with an additional unit of capital and spent all the additional income on consumption. This variable will be key in our recursive formulation below. The observation that the marginal value of capital allows the writing of policy problems recursively comes from Kydland and Prescott (1980) and it has been applied by Marcet and Marimón (1994) and Phelan and Stacchetti (2000). Another way to think about m is as the payoff to the consumer in parallel to v as the payoff to the government.

Then we can consider the following recursive problem also for the same given government policy. For any s^t and $k(s^{t-1})$ the consumer solves:

$$\max_{c(s^t),k(s^t)} u\left(c\left(s^t\right), l\left(s^t\right)\right) + \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) m\left(s^{t+1}\right) k\left(s^t\right)$$

such that:

$$\begin{aligned} c\left(s^t\right) + i\left(s^t\right) &= \left(1-\tau_l\left(s^t\right)\right) w\left(s^t\right) l\left(s^t\right) + \left(1-\tau_k\left(s^t\right)\right) r\left(s^t\right) k\left(s^{t-1}\right) \\ k\left(s^t\right) &= i\left(s^t\right) + (1-\delta) k\left(s^{t-1}\right) \end{aligned}$$

Given the assumptions we made on the utility and production functions this problem has an interior solution and the first order conditions of the consumer's problem are given by:

$$u_c\left(s^t\right) = \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) m\left(s^{t+1}\right) \tag{13}$$

$$u_l\left(s^t\right) = \left(1-\tau_l\left(s^t\right)\right) w\left(s^t\right) u_c\left(s^t\right) \tag{14}$$

3.3. The Equivalence of the Sequence and Recursive Problem

To show the equivalence between the two formulations notice that, if we plug the definition of $m(s^{t+1})$ in (13) we get:

$$u_c\left(s^t\right) = \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) u_c\left(s^{t+1}\right)\left(1+\left(1-\tau_k\left(s^{t+1}\right)\right) r\left(s^{t+1}\right)-\delta\right) \tag{15}$$

$$u_l\left(s^t\right) = \left(1-\tau_l\left(s^t\right)\right) w\left(s^t\right) u_c\left(s^t\right) \tag{16}$$

and then we get the same set of first order conditions that solves the recursive problem than the ones that solve the sequence problem.

Then, if we can show that an appropriate transversality condition is satisfied, the solution to both problems must be the same. In particular if

$$\lim_{t\to\infty}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi\left(s^t\right)m\left(s^{t+1}\right)k\left(s^t\right)=0 \tag{17}$$

i.e. if the discounted marginal of value capital goes to zero as times goes to infinity.

Showing that this transversality condition holds in general cases is difficult. For our inherently quantitative purposes we feel nothing of substance is lost if we restrain ourselves to the situation where the intertemporal utility function has the form (up to a constant):

$$\begin{aligned}
u\left(c\left(s^t\right),l\left(s^t\right)\right) &= (1-\beta)\left(\frac{c\left(s^t\right)^{1-\sigma}}{1-\sigma}+\gamma\left(l\left(s^t\right)\right)\right) \text{ if } \sigma>1\\
&= (1-\beta)\left(\log c\left(s^t\right)+\gamma\left(l\left(s^t\right)\right)\right) \text{ if } \sigma=1
\end{aligned}$$

This specification of preferences is both compatible with the existence of a balanced growth path and, from the perspective of public finance, it satisfies a number of interesting properties related with uniform commodity taxation (see Chari and Kehoe (1999)).

Lemma 1. *If the utility function has the form proposed above, then the transversality condition (17) holds.*

Proof. First note that since $u_c\left(s^t\right)=c\left(s_t\right)^{-\sigma}$ and $r\left(s^t\right)=F_k\left(k\left(s^{t-1}\right),l\left(s^t\right),s^t\right)$ we have

$$m\left(s^{t+1}\right)=c\left(s^{t+1}\right)^{-\sigma}\left(1+\left(1-\tau_k\left(s^t\right)\right)F_k\left(k\left(s^t\right),l\left(s^{t+1}\right),s^{t+1}\right)-\delta\right)$$

and hence we need to show that

$$\lim_{t\to\infty}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi\left(s^t\right)c\left(s^{t+1}\right)^{-\sigma}\left(1+\left(1-\tau_k\left(s^t\right)\right)F_k\left(k\left(s^t\right),l\left(s^{t+1}\right),s^{t+1}\right)-\delta\right)k\left(s^t\right)=0$$

To do so...

[to be completed]

Now that we have shown the equivalence of the recursive and the sequence problem and to save on notation, when there will be no risk of ambiguity, we will use a to denote the value of a variable in this period and a' to denote its value in the next. In that way and using the equivalence result, for example we can express the value for the government $v(s)$ given the current stochastic shock as

$$v(s) = u(c(s), l(s)) + G(g(s)) + \beta \sum_{s' \succ s} \pi(s'|s) v(s')$$

Also with this notation, and for a exogenously given set of $m(s')$ for all $s' \succ s$, k, s and taxes $\tau_k(s), \tau_l(s)$, we can think of the problem of the consumer

$$\max_{c(s), k(s)} u(c(s), l(s)) + \beta \sum_{s' \succ s} \pi(s'|s) m(s') k(s)$$

such that:

$$\begin{aligned} c(s) + i(s) &= (1 - \tau_l(s)) w(s) l(s) + (1 - \tau_k(s)) r(s) k \\ k(s) &= i(s) + (1 - \delta) k, \; i(s) \geq 0 \end{aligned}$$

plus the problem of the representative firm and the government's policy, as an associated static economy $\Xi(s, k, \tau_k(s), \tau_l(s), m(s')$ for all $s' \succ s)$ indexed precisely by the marginal value of capital, the initial amount of capital, the stochastic state and the tax rates.

It is easy to see that $c(s), l(s), k(s), g(s), w(s), r(s)$ constitute a competitive equilibrium for $\Xi(s, k, \tau_k(s), \tau_l(s), m(s')$ for all $s' \succ s)$, and denoted by $(c(s), l(s), k(s), g(s), w(s), r(s)) \in CE(\Xi(s, k, \tau_k(s), \tau_l(s), m(s')$ for all $s' \succ s))$, if and only if:

$$\begin{aligned} u_c(s) &= \beta \sum_{s' \succ s} \pi(s'|s) m(s') && (18) \\ u_l(s) &= (1 - \tau_l(s)) w(s) u_c(s) && (19) \\ r(s) &= F_k(k, l(s), s) && (20) \\ w(s) &= F_l(k, l(s), s) && (21) \\ g(s) &= \tau_l(s) w(s) l(s) + \tau_k(s) r(s) k && (22) \\ k(s) &= (1 - \tau_l(s)) w(s) l(s) + (1 + (1 - \tau_k(s)) r(s) - \delta) k - c(s) && (23) \\ k(s) &\geq (1 - \delta) k && (24) \end{aligned}$$

The interpretation of these conditions is simple. The first two constraints (18) and (19) are the consumers first order conditions. Their presence assure that consumers optimize given prices and their forecast about the future value of the variables. Remember here our previous discussion of how, not matter how the government behaved, in the whole dynamic game, anonymous agents can only answer with a competitive equilibrium where they satisfy these two conditions. Analogously (20) and (21) make input prices equal to marginal productivities as implied by a competitive equilibrium. The last pieces of the definition of an equilibrium are given by the definition of government consumption, (22), and by (23), the law of motion for capital (plus the nonnegativity of investment 24), that provides market clearing.

Also appropriately linked (i.e. with the right laws of motion for k, m and s) static competitive equilibria as the one described are a competitive equilibrium of the original sequence economy. This linkage allows us to think of the original economy as one equivalent to the repetition of static economies with endogenously changing state variables and exogenous stochastic shocks.

4. Self Generation

Abreu, Pearce and Stacchetti (1990) showed a procedure to exploit a recursive formulation of a repeated game. In particular they noted that, in a repeated game, we do not need a complete specification of the whole sequence of future actions. Instead a continuation value from the next period on can summarize all relevant information about the future. In an equilibrium that continuation value is also the equilibrium payoff for the repeated game beginning next period. In our context, a vector of payoffs would be the ones generated by a sustainable plan if they are equal to the current period payoffs plus the payoffs of a continuation sustainable equilibrium.

Phelan and Stacchetti (2000) have extended that contribution to problems of optimal fiscal policy. In particular they point out two issues related with the direct application of the Abreu, Pearce and Stacchetti's framework. First, the optimal fiscal problem is a dynamic, not a repeated, game because of the presence of state variables, as in our case capital and the stochastic shock s. Second since we have a continuum of consumers we need an infinitely dimensional value correspondence.

With respect to the first issue, we only need to define a payoff correspondence in k and s. This idea has the important advantage of being rather easy to extend to more general state spaces (as those with public debt, nondegenerated income distribution or others). The second problem requires further treatment. The main idea, already hinted before, is to use the marginal value of capital, m, as the second payoff (with the first being the payment to the government v). The intuition for the procedure is that m summarizes the marginal value of wealth (a good concept of payoff for the consumer) and that it carries the information needed to make optimal intertemporal decisions.

We define a correspondence that maps all pairs of possible values of the states (k, s) into sets of payoffs (m, v) generated by a sustainable plan. For a given set of states (k, s) we will call this set the sequential equilibrium value correspondence $V(k, s)$. Because of the presence of the publicly observed random variable X_t this correspondence is convex for given values of k and s (any point in the linear combination between the payoffs to two sustainable plans in pure strategies represents the payoffs to the sustainable plan that appropriately mixes the two pure strategies). Also in most applications of interest this correspondence would not be trivial (with only one point) but it will include a large number of possible equilibrium payoffs reflecting the somehow annoying fact that most of these dynamic games place relatively little structure in the predictions regarding observables.

We also define an arbitrary value correspondence as any mapping from all pairs of possible values of the states (k, s) into sets of payoffs (m, v). We can represent one of these value correspondences as $W : \Re_+ \times S \rightarrow \Re^2$. We will denote by A the space of all value correspondences $W(\cdot, \cdot)$.

Define an operator $B : A \rightarrow A$ in the following way:

$$B(W)(k, s) = co \left\{ \begin{array}{c} (m, v) | \exists \tau_k(s), \tau_l(s), c(s), l(s), k(s), g(s), w(s), r(s), \\ \text{and } m(s'), v(s') \in W(k(s), s'), \text{ for all } s' \succ s \end{array} \right\}$$

where

$$m(s) = u_c(s) \left(1 + r(s)(1 - \tau_k(s)) - \delta\right) \tag{25}$$

$$(c(s), l(s), k(s), g(s), w(s), r(s)) \in CE \left(\Xi \left(s, k, \tau_k(s), \tau_l(s), m(s') \text{ for all } s' \succ s\right)\right) \tag{26}$$

$$v(s) = \left\{ u(c(s), l(s)) + G(g(s)) + \beta \sum_{s' \succ s} \pi(s'|s) v(s') \geq \bar{\pi}(k, s) \right\} \tag{27}$$

The interpretation of the operator and the constraints is as follows. $B(\cdot)$ is the convex hull of the payoffs $m(s)$ and $v(s)$ such that there are associated values of the taxes, consumption, labor supply, next period capital, government expenditure, input prices and next period payoffs that belong to the value correspondence for every possible realization of the shock compatible with the current state and that satisfy certain conditions. The first constrain is just our definition of $m(s)$ in recursive notation. The second constrain assures that the values of all these variables are compatible among themselves and with agent's optimization, i.e. that the constitute a competitive equilibrium in the associated static economy. Finally the incentive constrain (27) restricts the admissible payoffs $v(s)$ to those weakly bigger than the value of a certain state-dependent function $\bar{\pi}(k, s)$ that, as it will clear shortly, we call the worst punishment function.

Why do we impose this additional last constraint and why do we call it the worst punishment function? Abreu (1986 and 1990) showed that any equilibrium in a repeated game can be sustained through the use of a trigger strategy that reverts to a worst equilibrium after a deviation. Extending this logic to our context, a payoff v can only be implied by a sustainable equilibrium if it provides a higher utility than a deviation plus a reversion to the worst competitive equilibrium (worst understood as given the lowest utility). Note that we require, as mentioned in section 2, that the reversion is always to a competitive equilibrium, the only situation compatible with anonymity of the consumers. This particular type of reversion makes the worst punishment "not so bad" and it reduces substantially the set of sustainable equilibria[7].

The task is then to define this function $\overline{\kappa}(\cdot, \cdot)$ and to characterize it[8]. The worst punishment is given by:

$$\overline{\kappa}(k, s) = \max_{\tau_k(s), \tau_l(s)} \left\{ \min_{c(s), l(s), k(s), v(s'), m(s')} u(c(s), l(s)) + G(g(s)) + \beta \sum_{s' \succ s} \pi(s'|s) v(s') \right\}$$

such that

$$\{m(s'), v(s') \in W(k(s), s') \text{ for all } s'\} \tag{28}$$

$$(c(s), l(s), k(s), g(s), w(s), r(s)) \in CE\left(\Xi\left(s, k, \tau_k(s), \tau_l(s), m(s') \text{ for all } s' \succ s\right)\right) \tag{29}$$

where the first constraint, (28) limits the continuation payoffs to be those belonging to the value correspondence and the second assures that all the choices of the consumers and the government are a competitive equilibrium.

[7] If, for example, consumers could collude among themselves and, after a deviation, revert to an equilibrium with zero (or arbitrarily small) consumption, the continuation utility for the government would be so low that, rather trivially, Ramsey would be supported.

[8] If we knew the function from the beginning we could follow Benhabib and Rusticcini (1997) and solve directly for the problem of the government augmented by a constraint that forces the value for the government of an admissible policy to be greater or equal than this worst punishment.

The intuition behind Abreu's proposal is that we can manipulate the believes about future actions in such a way that the agents think that after a deviation from the government they will deviate to the worst equilibrium. To compute this function, we fixed some values for the states and then minimize over the choices of the consumers and possible continuation payoffs and maximize over taxes (the government tries to get the best possible value out of the worst equilibrium) being sure, of course, that we are always inside a competitive equilibrium.

With rather straightforward extensions of the existing results in Abreu, Pierce and Stacchetti (1990) and in Phelan and Stacchetti (2000). we can state some theorems that are quite useful to characterize the sequential equilibrium value correspondence

Theorem 2. *If $W \subseteq B(W)$ then $B(W) \subseteq V$.*

Proof: [to be completed].

This result, known as self-generation, together with the fact that the sequential equilibrium value correspondence is self-generating $V \subseteq B(V)$ imply that V is a fixed-point of the operator. Even more importantly, the next result (factorization) tells us it is the largest of such fixed points.

Theorem 3. *The sequential equilibrium value correspondence V is the largest value correspondence W such that $W = B(W)$.*

Proof: [to be completed].

Theorem 4. *$B(\cdot)$ is monotone, i.e. for W, $W' \in A$ such that $W \subseteq W'$, $B(W) \subseteq B(W')$*

Proof: See appendix.

Nearly as important as these results is that, if we define $W_{n+1} = B(W_n)$ we can show, that for an initial $W_0 \supseteq V$,

Theorem 5. $W = \lim_{n\to\infty} B(W_n)$.

Proof: [to be completed].

5. Computation

In this section we discuss how we implement in a computer the operator $B(\cdot)$ described above to find the equilibrium value correspondence and how to build strategies that support particular points in that correspondence. We cover in certain detail the logic of the procedure and how it can be easily extended to generate further results than the ones here reported. References for this section include Cronshaw (1997), Judd, Yeltekin and Conklin (2000) and Sleet and Yeltekin (2001).

5.1. Finding the Equilibrium Value Correspondence

The first task is to find the equilibrium value correspondence with all the possible payoffs. Representing a multidimensional correspondence in a computer is a challenging endeavour. A simple way to simplify this objective is to discretize the state space. Along the s dimension this is not an issue since we assumed that S is finite. For capital we define a grid $k_{grid} = [k_1, ..., k_n]$ where k_n is fixed to be sufficiently high that does not affect the computations. After this discretization, instead of a correspondence $W : \Re_+ \times S \rightarrow \Re^2$ we have a new one $W : k_{grid} \times S \rightarrow \Re^2$. It is equivalent to think about this correspondence as $n \times \# \{S\}$ convex sets $W(k_i, s_j)$, where $k_i \in k_{grid}$ and $s_j \in S$, a much simpler collection of objects than the original W.

But even after having discretized the state space we still face the problem of how to represent each of these sets and how to manipulate them to reproduce the operator $B(\cdot)$. Judd, Yeltekin and Conklin (2000) propose two alternative procedures (an outer approximation and an inner approximation algorithm) to find the equilibrium value correspondence of a repeated game. We extend their results for our case where we have a dynamic game.

5.1.1. Outer Approximation

The *Outer Approximation* approach builds on the idea that we can use a set of hyperplanes H to define a convex polytope as the intersection of the half-planes generated by H evaluated at some distances with respect to an origin of coordinates. A way to approximate an arbitrary convex set is to find the smallest convex polytope that includes it generated by H and some distances. The great advantage of this approach is that the only information required to keep track of the approximated set is precisely H and the corresponding distances, an information easily storable.

Formally we will denote by $W^o(k_i, s_j)$ the outer approximation of $W(k_i, s_j)$ defined by an outward set of hyperplanes $H \in R^{D,2}$ and a vector of distances from the origin $C_{W(k_i,s_j)}$ (note that the hyperplanes are constant across the different state values k_i and s_j while the distances may change). Then we say that a point $w(s) = [m(s), v(s)] \in W^0(k_i, s_j)$ if $H'w(s) \leq C_{W(k_i,s_j)}$. However since we are using an outward approximation we need to remember that, generically, we will have points $w'(s)$ such that $H'w'(s) \leq C_{W(k_i,s_j)}$ but $w'(s) \notin W(k_i, s_j)$.

An outline of the algorithm used to compute this outer approximation is as follows.

1. Fix H.

2. Start with an initial large $C^0_{W(k_i,s_j)}$ that defines $W^0(k_i, s_j) \supseteq V(k_i, s_j)$.

3. For $n = 1, ...$ update $C^n_{W(k_i,s_j)}$ at each row (direction) z of H. This step delivers the new sets $W^n(k_i, s_j) = B(W^{n-1}(k_i, s_j))$.

4. Iterate until convergence, $V^o(k_i, s_j) = B(V^o(k_i, s_j)) = \lim_{n\to\infty} W^n(k_i, s_j)$, where $V^o(k_i, s_j)$ is the $D-$directions outer approximation of $V(k_i, s_j)$.

Several points deserve further discussion. First, when we fix the set of hyperplanes H, we need to decide the number D of directions to use and how are we going to distribute them across the plane. With respect to the number of directions we face the standard trade-off between accuracy and efficiency. McClure and Vitale (1975) prove convergence of the outer approximation for smooth convex sets when $D \to \infty$. With respect to the directions we want to accumulate more in the parts of the plane where the set has quicker changes in shape. Step 2 is required to satisfy that the initial guess of the equilibrium correspondence $W^0(\cdot, \cdot)$ is a strict superset of the true correspondence and hence, by our results in the previous section, due to converge after repeated applications of the operator $B(\cdot)$. Note that this guess is facilitated by the fact that we know that no payoffs for the government higher than those provided by the Ramsey equilibrium can ever be sustained.

Step 3 in the outline involves two tasks: to compute the worst punishment conditional on the current guess of the value correspondence, $\overline{\kappa}^n(\cdot, \cdot)$, and to update the distances $C^n_{W(k_i,s_j)}$. To find $\overline{\kappa}^n(\cdot, \cdot)$ we solve

$$\overline{\kappa}^n(k, s) = \max_{\tau_k(s),\tau_l(s)} \left\{ \min_{c(s),l(s),k(s),v(s'),m(s')} u(c(s), l(s)) + G(g(s)) + \beta \sum_{s'\succ s} \pi(s'|s)v(s') \right\}$$

such that

$$\left\{m(s'), v(s') \in W^{n-1}(k(s), s') \text{ for all } s'\right\} \tag{30}$$

$$(c(s), l(s), k(s), g(s), w(s), r(s)) \in CE\left(\Xi\left(s, k, \tau_k(s), \tau_l(s), m(s') \text{ for all } s' \succ s\right)\right) \tag{31}$$

that is the same problem than the definition in section 4 except that now the continuation payoff values must belong to $W^{n-1}(\cdot, \cdot)$.

This last step is one of the main differences with respect to the work of Benhabib and Rusticcini (1997) since we have a constructive procedure to computationally generate $\overline{\kappa}^n(\cdot, \cdot)$ instead of relying of an exogenously given function or on some "guess and verify" method that can be difficult to implement in practice.

One important issue when solving for $\overline{\kappa}^n(\cdot, \cdot)$ is that $k(s)$ may not be one of the points in k_{grid}. To fix this problem we could linearly interpolate between points in the k_{grid} to find admissible values for $m(s'), v(s')$. Vitale (1979) shows that if the correspondence is continuous and exactly known at the points in the grid, then a linear interpolation scheme converges to the true correspondence. Unfortunately there are not joint convergence results when we do not know the exact values of the correspondence at the grid but we approximate them using a set of hyperplanes H. In fact examples can be built where the computed correspondence is not an outer approximation outside the points in the grid (see Sleet and Yeltekin (2001)). We will discuss alternative our alternative interpolation schemes in the appendix.

Also it is important to remember that through constraint (31) we always impose that investment is nonnegative. This constraint is only going to be relatively important in the first period of the game, where the government will tax capital heavily since doing that is close to levying a lump-sum tax. Nonnegativity of investment reduces the incentives of the government to do so, making the problem of the government non-trivial in the first period[9]. After the first period the constraint is unlikely to bind even if the government deviates to an equilibrium with higher tax rates. In this case the consumers know that the government will tax more heavily capital in the future and hence they want to reduce their capital holdings but they prefer to do so progressively without net disinvestment. Otherwise they

[9] Although since the government cannot carry debt, the incentive to tax heavily in the first period, invest in claims against the private sector and use the interest to reduce future taxes (and the distortions associated with them) is absent. This mechanism is, for instance, quite important in the characterization of optimal fiscal policy in Chari, Christiano and Kehoe (1994) and it accounts for most of the welfare gains of a switch from a system like that of the United States to Ramsey taxation.

will substantially reduce their marginal utility of current consumption and they could not satisfy their Euler equation between consumption today and the next period[10]. However the presence of the constraint greatly helps to avoid some numerical instabilities of the algorithm. This pragmatic consideration justifies our choice of including the constraint.

The second task of the step 3 is to update the distances $C^n_{W(k,s)}(\cdot)$. Basically we want to find, for each direction D, the maximum payoff equilibrium values, given the restrictions that we stay in a competitive equilibrium and that sequential rationality is satisfied, for the current approximation of the equilibrium value correspondence. Since we can think of each direction as a particular weighting of the payoffs, the update of the distances comes from solving the following auxiliary problem for each direction z:

$$C^n_{W(k,s)}(z) = \max_x H(z,1)m(s) + H(z,2)v(s) \tag{32}$$

such that

$$\begin{gathered}
x = (\tau_k(s), \tau_l(s), c(s), l(s), k(s), m(s'), v(s') \text{ for all possible } s' \succ s) \\
m(s) = u_c(s)\left(1 + r(s)(1 - \tau_k(s)) - \delta\right) \\
(c(s), l(s), k(s), g(s), w(s), r(s)) \in CE\left(\Xi\left(s, k, \tau_k(s), \tau_l(s), m(s') \text{ for all } s' \succ s\right)\right) \\
v(s) = u(c(s), l(s)) + G(g(s)) + \beta \sum\nolimits_{s' \succ s} \pi(s'|s)v(s') \geq \overline{\kappa}^n(k,s) \\
C^{n-1}_{W(k(s),s')}(z) \geq H'\left[m(s'), v(s')\right]
\end{gathered}$$

The interpretation of the constraints is similar to previous discussions. The first one is just a definition of the vector x and the second one the definition of the marginal value of capital. We again require that the different variables are part of a competitive equilibrium and that the government does not have an incentive to deviate with the worst punishment function as computed. The only new constraint is the last one, that says that the future payoffs search as arguments of the maximization indeed belong to the current guess of the equilibrium correspondence and hence they are admissible continuation utilities.

Finally step 4 involves the update of the guess of the equilibrium correspondence and the iteration of the previous steps until convergence is achieved up to the approximation error involved in the outer nature of our procedure.

[10] Notice that this equation needs to hold with equality since there are enough resources to finance current consumption and some capital can be stored for the future because $\tau_k(s) < 1$. The constraint may be binding, though, if the government deviates to higher tax rates for only one period.

5.1.2. Inner Approximation

The second approach to find the sequential equilibrium value correspondence is to use an *inner approximation*. Here the idea is to find q points (vertices) p in the frontier of the set and build their convex hull. Clearly the set constructed in a such a way will be always be a subset of V and generically a strict subset and hence we will have points $w'(s)$ such that $w'(s) \notin co(p)$ but $w'(s) \in V(k_i, s_j)$.

An outline of the algorithm is as follows.

1. Fix a number q of points and H.

2. Start with an initial p^0 small enough such that $p^0 \in V(k_i, s_j)$. Then we have that $co(p^0) = W^0(k_i, s_j) \subseteq V(k_i, s_j)$.

3. For $n = 1, ...$ update $C^n_{W(k_i,s_j)}$ at each row (direction) z of H. The payoffs $m(s)$ and $v(s)$ give us the new vertices p^n.

4. Make $co(p^n) = W^n(k_i, s_j)$.

5. Iterate until convergence, $V^I(k_i, s_j) = B(V^I(k_i, s_j)) = \lim_{n\to\infty} W^n(k_i, s_j)$, where $V^I(k_i, s_j)$ is the $q-$points inner approximation of $V(k_i, s_j)$.

The algorithm present a strong resemblance with the outer approximation and only needs minor comments. In addition of picking D hyperplanes now we need to select q points balancing accuracy and speed and some appropriate collocation criteria looking to assure that in fact all the chosen points are points of $V(k_i, s_j)$[11]. The role of the hyperplanes in this algorithm is only to provide the weights to be used in the optimization problem in step 3. Then we built the convex hull of these points p^0 as the initial guess $W^0(k_i, s_j)$. The building of the convex hull of a set of points is a well understood problem for which a wealth of efficient algorithms exist (see the discussion in de Berg *et al.* (2000)) that we do not discuss in detail.

The main part of the algorithm is the step 3, but since this step is exactly equivalent to step 3 in the outer approximation method (including solving for the worst punishment function) we omit details and just refer to our explanations above. The only important point to emphasize is that in the outer approximation we use the value of the maximized function $C^n_{W(k,s)}(z)$ of the problem (32) to update the distances while here we use the arguments $m(s)$ and $v(s)$ that solve the maximization.

[11] In contrast with the initial guess in the outer approximation this may not be an easy task.

5.1.3. Mixing an Outer and an Inner Approximation

A nice integration of both outer and inner approximations can be achieved as follows. We guess an initial large $W^0(k_i, s_j)$ and apply the outer approximation algorithm until we find $V^I(k_i, s_j)$. Then we can pick the equilibrium values generated in the step 3 during the last iteration and use them as the original points p^0 of an inner approximation (we know that in fact these points belong, up to the convergence criterion and floating point arithmetic) to $V(k_i, s_j)$ and then $co(p^0) = W^0(k_i, s_j) \subseteq V(k_i, s_j)$. This makes sure that the algorithm works properly. Also this initial values are very good guesses and the inner approximation converges quickly.

The main advantage of having both outer and inner approximations is that we have an upper and a lower bound on the true $V(k_i, s_j)$ so we know for sure that some points are not equilibrium payoffs and that some are. We will have a region of ambiguity where we cannot determine the nature of a point but in general that region would be quite small and, if strategies satisfy some continuity criterion with respect to payoffs (i.e. small changes in payoffs imply small changes in strategies that support them) the effect of the ambiguity would be minor[12].

5.2. Building Strategies

Even if the set of sustainable equilibria payoffs provides valuable information, in general we are interested in looking at particular strategies that implement one equilibrium. Only knowing the strategies can we make quantitative statements about how the optimal fiscal policy looks over the business cycles, which is the optimal response to some particular shock or how does the data compared with the model predictions.

We will describe here a procedure to compute a strategy that sustains the best equilibrium (one giving the highest value to the government). Why is it the best equilibria interesting? Basically because of two reasons. First it is a focal point for the players. If the government can choose an strategy and the consumers their beliefs, it is plausible to think that they may as well pick the ones that implement the best allocation. Second because we learn a lot about a set from looking at extremes and learning about the best strategies conveys a lot of information about all the other strategies.

[12] For example, in our computations, the strategies that implement the best equilibrium converge much more quickly than the sequential equilibrium value correspondence.

An outline of the procedure we propose is:

- Pick an initial point in the east frontier of the set. This point is associated with some hyperplane h.
- Compute the next period variables implied by the optimal choices in the point using the solutions x from the update of distances in step 3 of either the outer or the inner approximation in the last iteration of the approximation at the direction determined by the direction h.
- Find the new point in the value correspondence and the associated hyperplane. In the case that we fall into an intersection point of two hyperplanes in the outer approximation (or we fall in a vertice in the inner approximation) randomize with appropriate weights.
- Iterate.

As it can be seen in this outline finding strategies that support a vector of sustainable equilibrium payoffs is direct once we have solved the approximations described above. Also it is important to notice that each point in an equilibrium value correspondence may be implemented by more than one strategy, so not only we have in general a large set of equilibria payoffs but also different ways to achieve them. However in the case of the stochastic growth model, the problem has enough convexity that we have only one strategy (although it may involved randomization) to implement each particular point.

Also we can use the solutions associated with finding the worst punishment function $\overline{\kappa}^n(\cdot,\cdot)$ to recursively built the strategies associated with the worst equilibrium (the *trigger strategies*). This strategies are interesting, among other things, to evaluate how plausible each equilibrium is (the hypothesis being than less plausible equilibria require more involved triggers) and to compare reputational mechanisms with some more simpler solutions concepts as a Markov Perfect Equilibria.

6. Calibration

We will parametrize our model to match basic properties of the U.S. economy under Ramsey taxation. We take as our utility function, following Klein, Krusell, and Ríos-Rull (2001))

$$u(c,l) = (1-\beta)\left((1-\alpha_p)\alpha_c \ln c + (1-\alpha_p)(1-\alpha_c)\ln(1-l)\right)$$

and as the utility of the government from public consumption:

$$G(g) = (1-\beta)\,\alpha_p \ln g$$

These two functional forms have the advantage of clarity and in the case of government consumption the easiness with which we can control the level of desired revenue using α_p.

The production function is a standard Cobb-Douglas production function

$$A(s)\,K^{\theta}L^{1-\theta}$$

To offer further insight in the problem of optimal taxation we will offer first results for the deterministic where $A(s) = 1$ for all $s \in S$. The rest of the parameter values are fixed as following. We pick θ to match labor income share of national product, β to generate an pre-tax interest rate of around 4% (see McGrattan and Prescott (2000) for a justification of this number based on their measure of the return on capital and on the risk-free rate of inflation-protected U.S. Treasury bonds). We set δ to match a capital-output ratio of around 3, α_p to get a share of government consumption around 20% and α_c to for hours to be around one-fourth of total time. Finally we set an upper-limit of the tax rates, 0.9 that will not bind in equilibrium and and a lower-bound of zero (remember our discussion of why we do not allow negative tax rates). We summarize the calibration values in table 6.1.

Table 6.1: Parameter Values, deterministic case		
$\theta = 0.36$	$\delta = 0.08$	$T \in [0, 0.9]$
$\beta = 0.96$	$\alpha_p = 0.13$	$\alpha_c = 0.30$

Before we move on we want to highlight that there is one implicit parameter that we pick when we select β: the length of the period. This limitation is important because it determines how often the government can change the tax system. We set that period, in a first approximation, as one year since this seems a plausible value for how often the Congress can change tax rates (for instance a shorter period will greatly difficult the administration of the Income Tax) [to be completed with some empirical evidence].

For the stochastic results we allow two productivity shocks, low and high and a transition matrix Q that matches some basic characteristics of the U.S. business cycle. We present their values in table 6.2.

Table 6.2: Parameter Values, stochastic case
$A \in \{0.976, 1.024\}$
$Q = \begin{pmatrix} 0.946 & 0.054 \\ 0.054 & 0.946 \end{pmatrix}$

7. Findings

7.1. Benchmark Economy

1. Equilibrium Value Correspondence.

2. Best Equilibrium.

3. Strategies

4. Triggers: how do they look like.

5. Relation with: Ramsey and Markov.

7.2. Robustness Analysis

[to be completed]

7.3. Extensions and Further Research

1. Value of commitment.

2. Introducing Debt.

3. Other shocks.

4. Reversion to Markov.

8. Conclusions

1. Recursive Game Theory can be used to study the quantitative implications of the theory.

2. Deterministic case: strategies and worst punishment seem plausible.

3. Stochastic case: some important work remains to be done.

9. Appendix

We describe some computational details and include all the relevant proofs.

9.1. Computational Details

[to be completed]

Alternative interpolation schemes (Sleet and Yeltekin (2000)).

9.2. Proofs

[to be completed].

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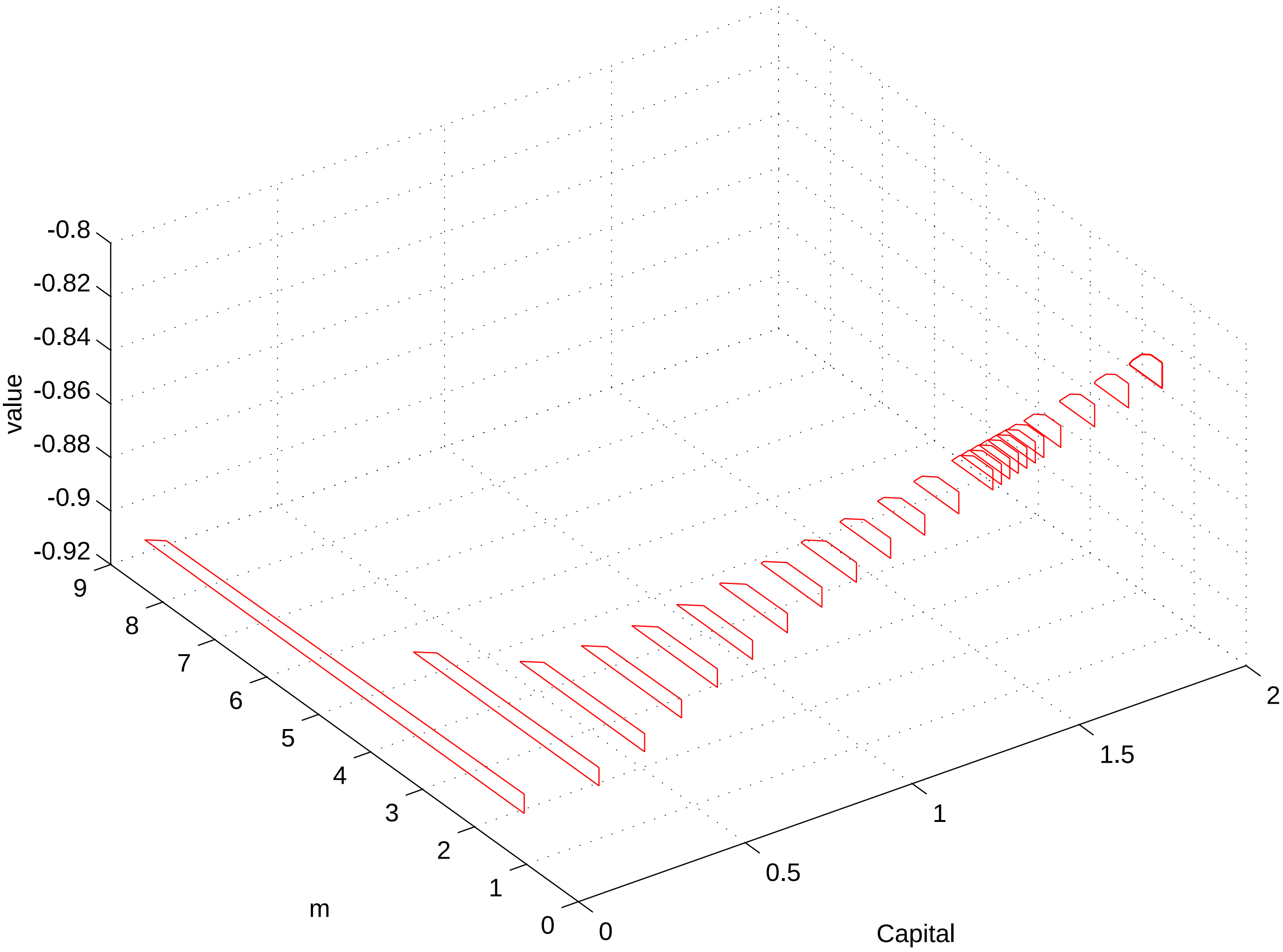
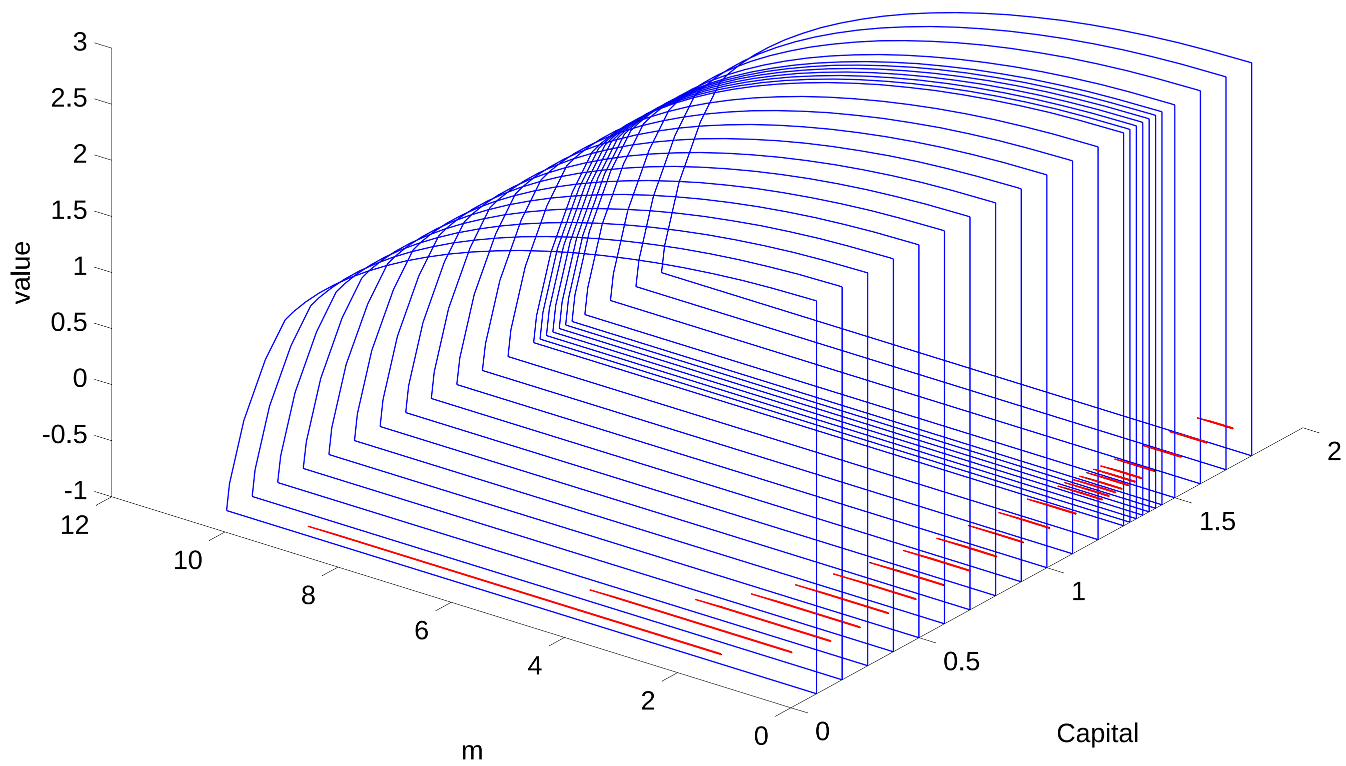
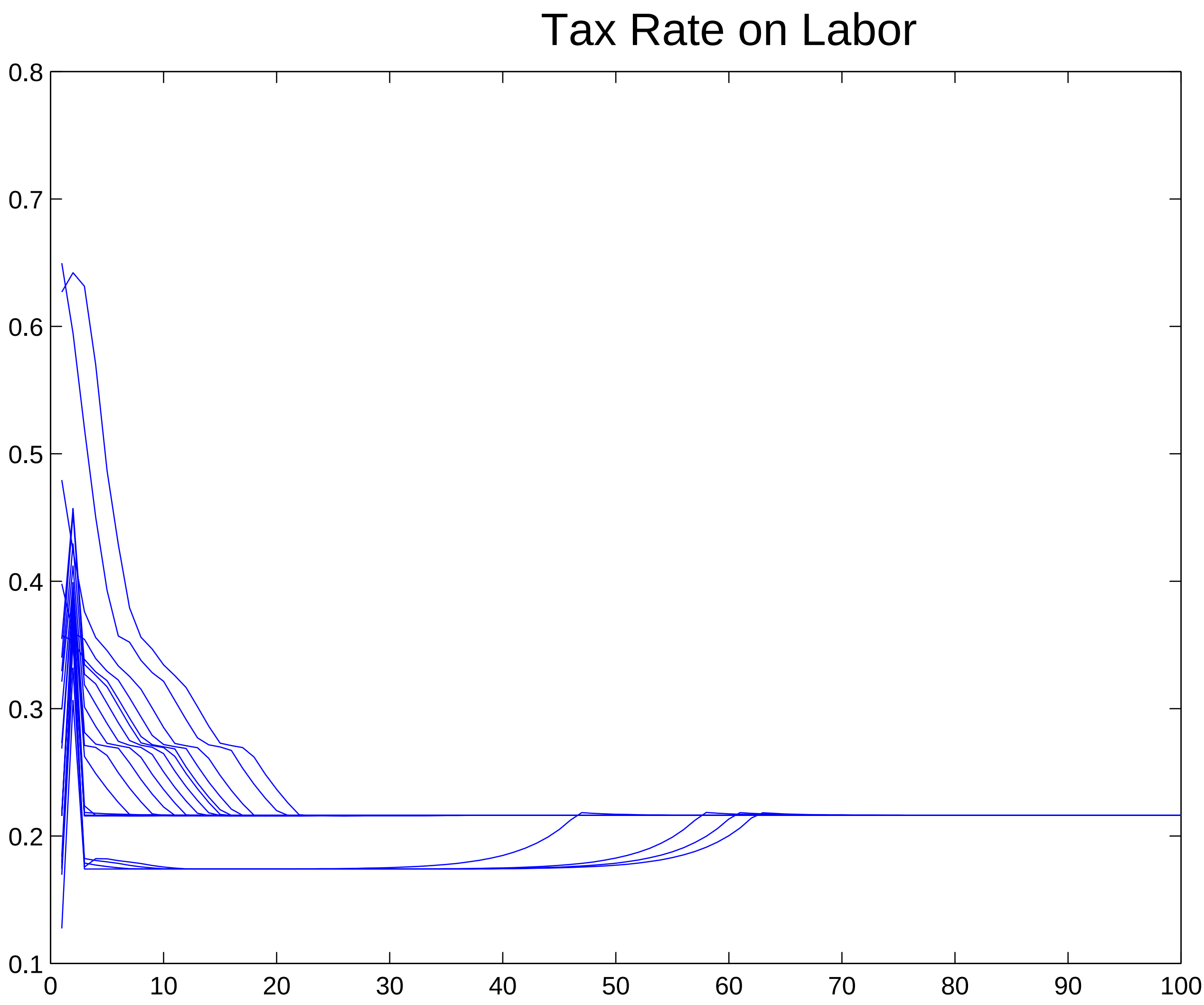
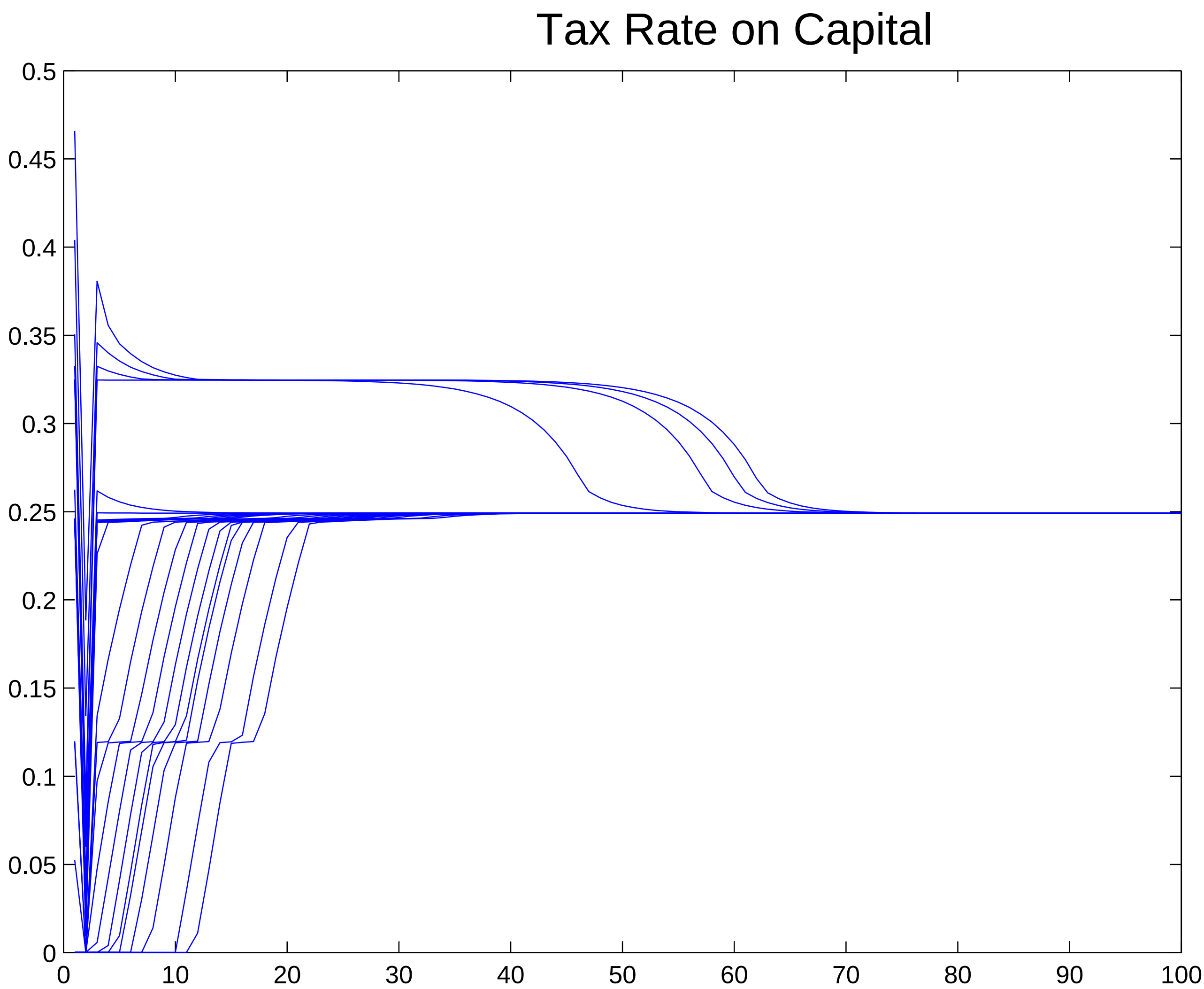
Sustainable Equilibrium Value Correspondence



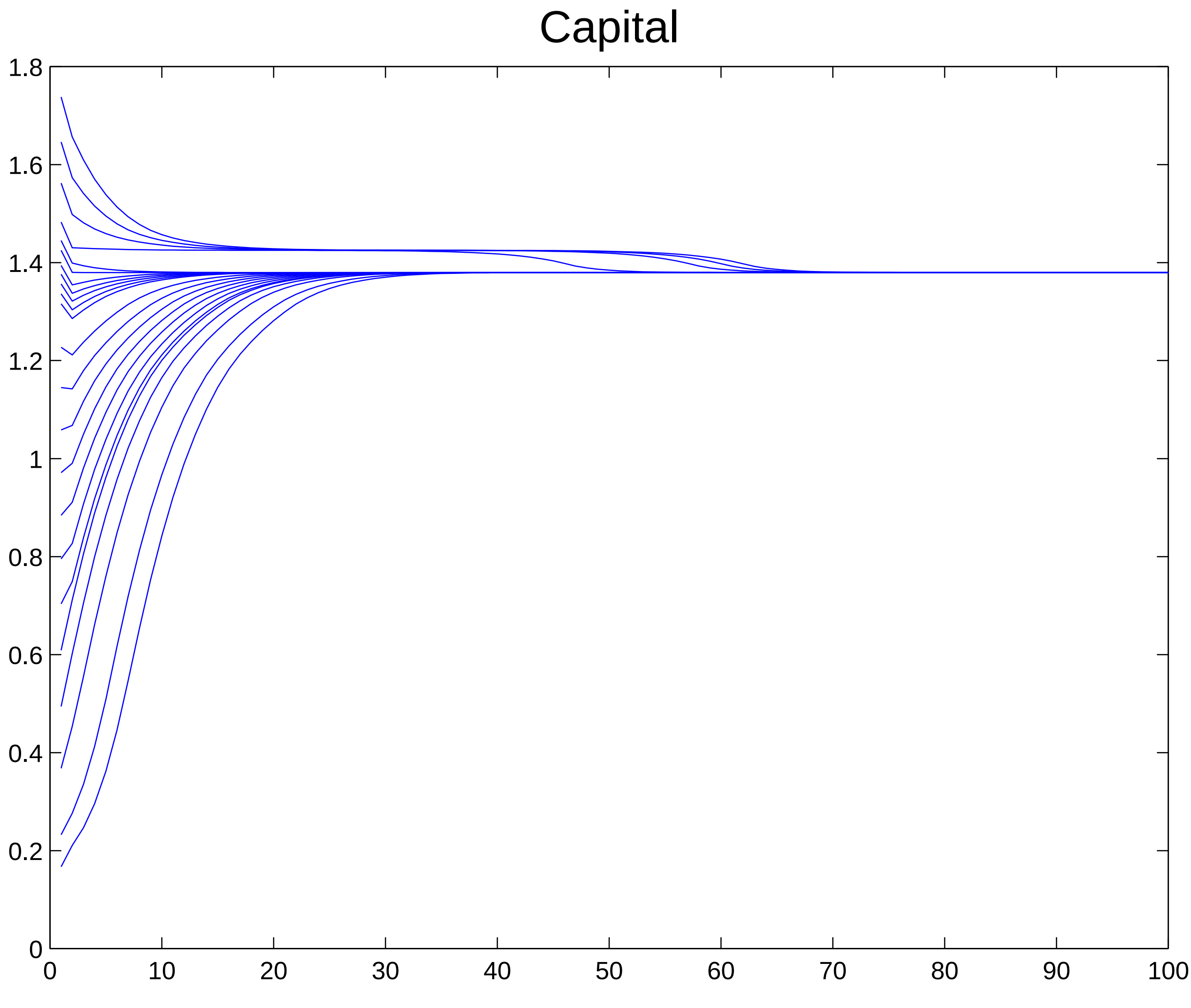
Comparison: original and final sets




Tax Rate on Labor
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
10
20
30
40
50
60
70
80
90
100



Tax Rate on Capital
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
10
20
30
40
50
60
70
80
90
100


Capital
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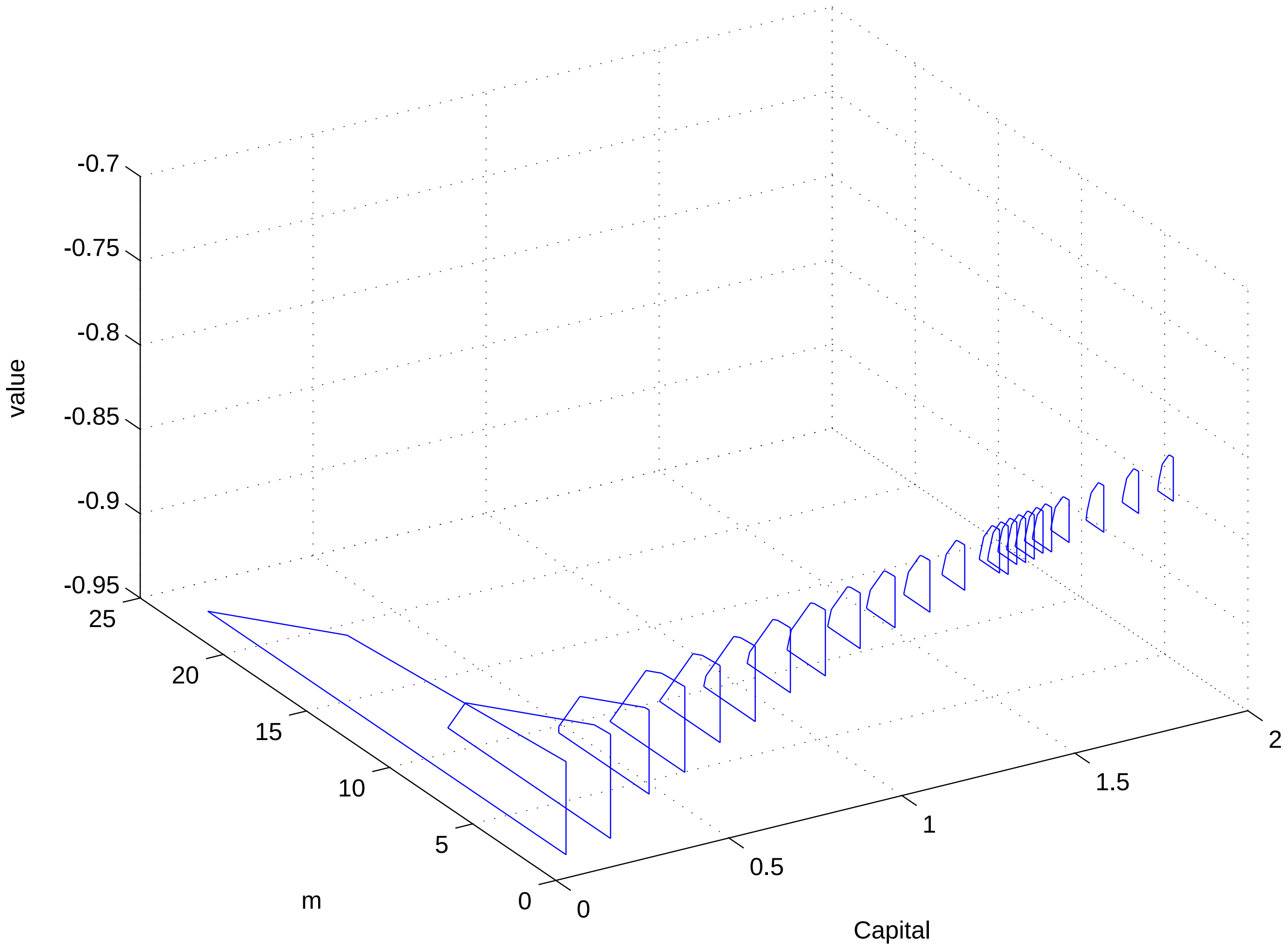
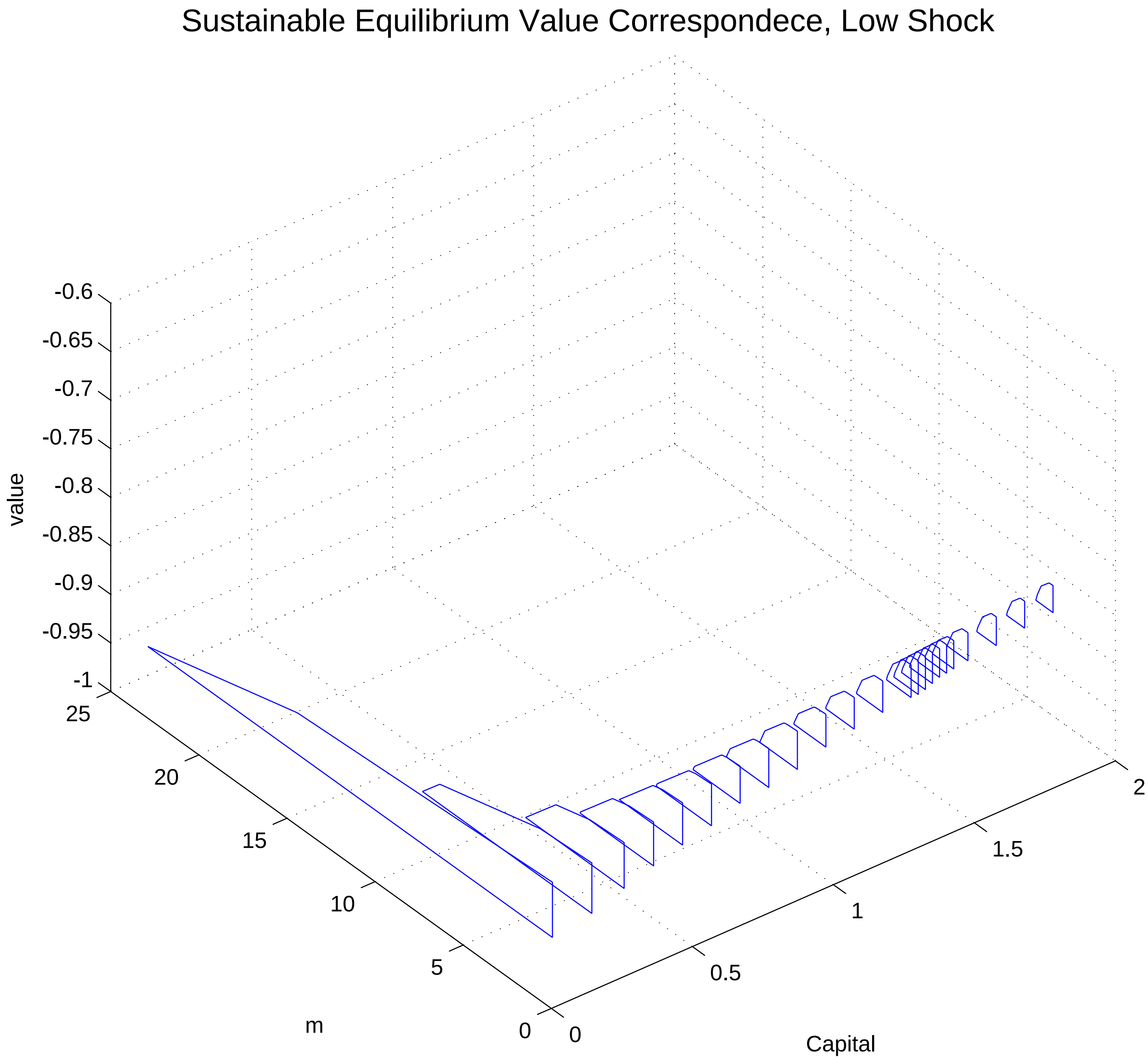
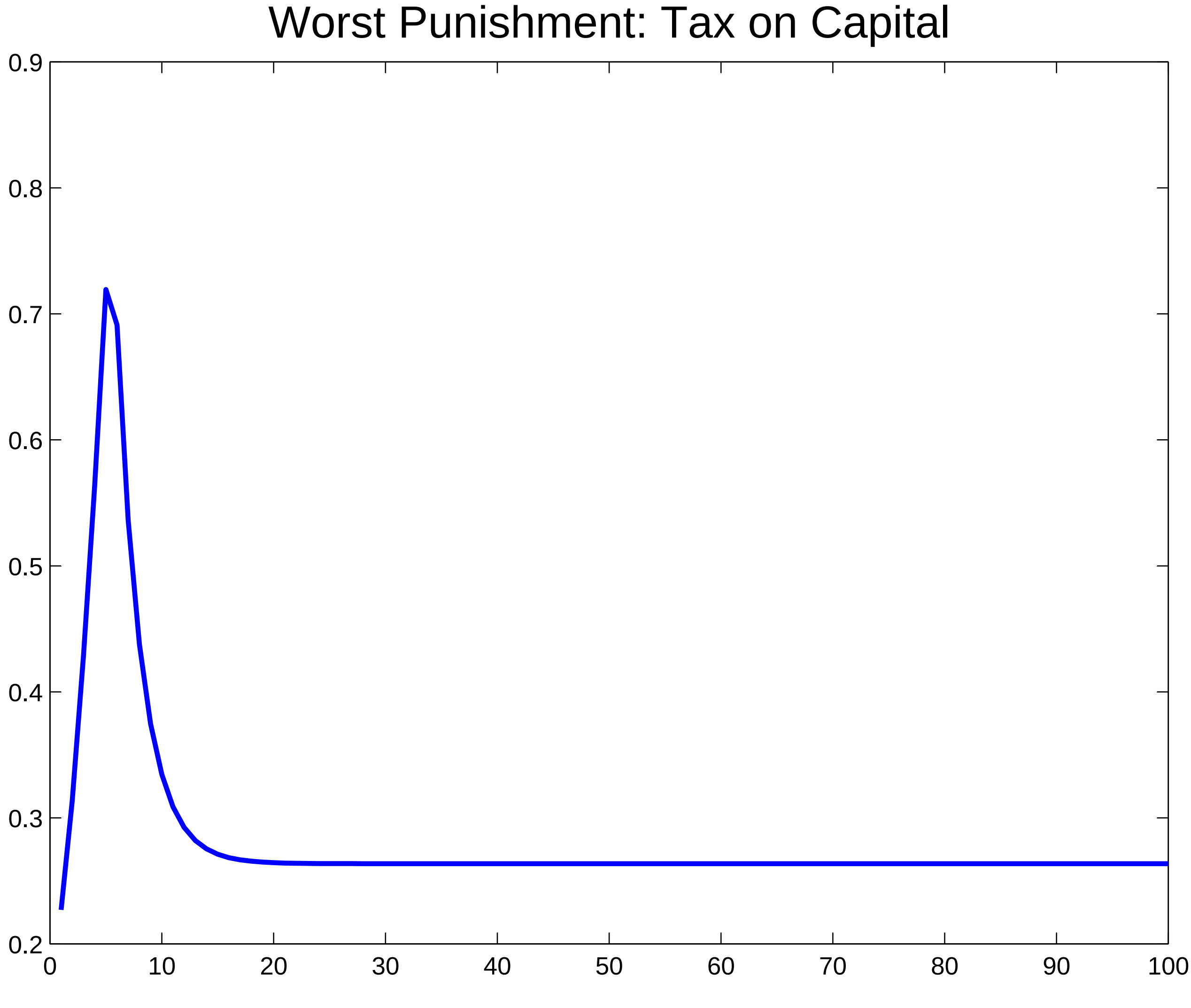
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
90
100

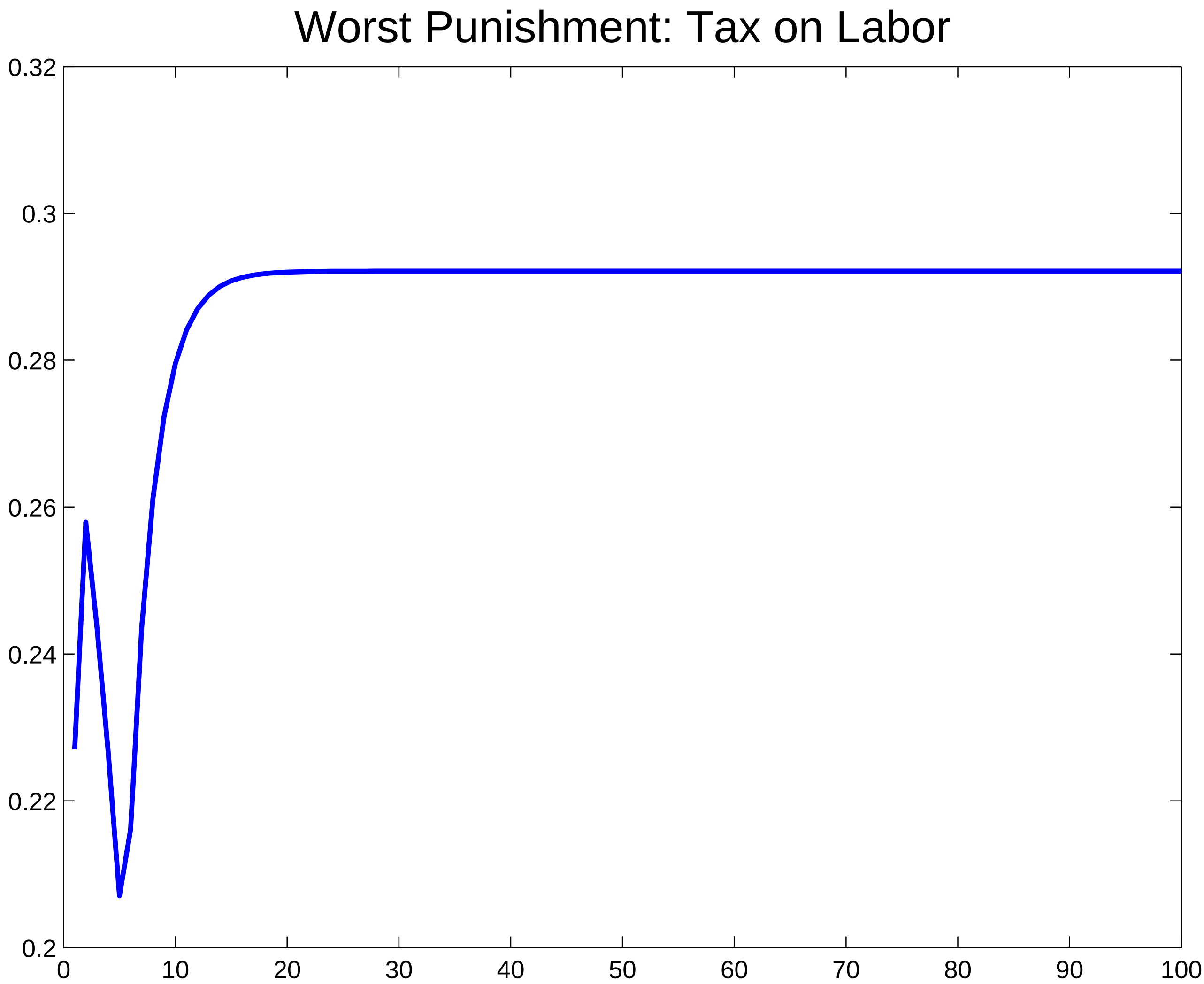
Sustainable Equilibrium Value Correspondence, High Shock





Sustainable Equilibrium Value Correspondece, Low Shock
value
-0.6
-0.65
-0.7
-0.75
-0.8
-0.85
-0.9
-0.95
-1
25
20
15
10
5
0
m
0
0.5
1
1.5
2
Capital


Worst Punishment: Tax on Capital
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
10
20
30
40
50
60
70
80
90
100


Worst Punishment: Tax on Labor
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0
10
20
30
40
50
60
70
80
90
100