Macroeconomics: an Introduction

Jesús Fernández-Villaverde University of Pennsylvania The Scope of Macroeconomics

- Microeconomics: Object of interest is a single (or small number of) household or firm.
- Macroeconomics: Object of interest is the entire economy. We care mostly about:
 - 1. Growth.
 - 2. Fluctuations.

Relation between Macro and Micro

- Micro and Macro are consistent applications of standard neoclassical theory.
- Unifying theme, EQUILIBRIUM APPROACH:
 - 1. Agents optimize given preferences and technology.
 - 2. Agents' actions are compatible with each other.
- This requires:
 - 1. Explicit about assumptions.
 - 2. Models as abstractions.

What are the Requirements of Theory?

- Well articulated models with sharp predictions.
- Good theory cannot be vague: predictions must be falsifiable.
- Internal Consistency.
- Models as measurement tools.

All this is Scientific Discipline.

Why should we care about Macroeconomics?

- Self Interest: macroeconomic aggregates affect our daily life.
- Cultural Literacy: understanding our world.
- Common Welfare: Essential for policymakers to do good policy.
- Civic Responsibility: Essential for us to understand our politicians.

A Brief Overview of the History of Macroeconomics I

- Classics (Smith, Ricardo, Marx) did not have a sharp distinction between micro and macro.
- Beginning of the XX century: Wicksell, Pigou.
- J.M. Keynes, The General Theory of Employment, Interest, and Money (1936).
- 1945-1970, heyday of *Neoclassical Synthesis*: Samuelson, Solow, Klein.
- Monetary versus Fiscal Policy: Friedman, Tobin.

A Brief Overview of the History of Macroeconomics II

- 1972, Rational Expectations Revolution: Lucas, Prescott, Sargent.
- 1982, Real Business Cycles: Kydland and Prescott.
- 1990's, Rich dynamic equilibrium models.
- Future?

Why do Macroeconomist Disagree?

- Most research macroeconomist agree on a wide set of issues.
- There is wide agreement on growth theory.
- There is less agreement on business cycle theory.
- Normative issues.
- Are economist ideologically biased? Caplan (2002).

National Income and Product Accounts

Jesús Fernández-Villaverde University of Pennsylvania

A Guide to NIPA's

- What is the goal?
- When did it begin? Role of Simon Kuznets:
 - 1. Nobel Prize in Economics 1971.
 - 2. Prof. at Penn during the key years of NIPA creation.
- Gigantic intellectual achievement.
- Elaborated by Bureau of Economic Analysis and published in the *Survey of Current Business*. http://www.bea.gov/

Question: How are macroeconomic aggregates measured?

Gross Domestic Product (GDP)

Can be measured in three different, but equivalent ways:

- 1. Production Approach.
- 2. Expenditure Approach.
- 3. Income Approach.

Nominal GDP

• For 2003, nominal GDP was:

\$11,004,000,000,000

• Population, July 2003 was:

290, 788, 976

• Nominal GDP per capita is roughly:

\$37,842

Computing GDP through Production

- Calculate nominal GDP by adding value of production of all industries: production surveys.
- Problem of double-counting: i.e. USX and GM.
- Value Added=Revenue-Intermediate Goods.
- Nominal GDP=Sum of Value Added of all Industries.

Computing GDP through Expenditure

$$C = Consumption$$

 $I = (Gross Private)$ Investment
 $G = Government Purchases$
 $X = Exports$
 $M = Imports$
 $Y = Nominal GDP$

$$Y \equiv C + I + G + (X - M)$$

Consumption (C)

- Durable Goods: 3 years rule.
- Nondurable Goods.
- Services.

Gross Private Investment (I)

- Nonresidential Fixed Investment.
- Residential Fixed Investment.
- Inventory Investment.
- Stocks vs. Flows.

Investment and the Capital Stock

- Capital Stock: total amount of physical capital in the economy
- Depreciation: the part of the capital stock that wears out during the period
- Capital Stock at end of this period=Capital Stock at end of last period+Gross Investment in this period-Depreciation in this period
- Net Investment=Gross Investment-Depreciation=Capital Stock, end this per.-Capital Stock, end of last per.

Inventory Investment

- Why included in GDP?
- Inventory Investment=Stock of Inventories at end of this year-Stock of Inventories at the end of last year
- Final Sales=Nominal GDP-Inventory Investment

Government Purchases (G)

- Sum of federal, state, and local purchases of goods and services.
- Certain government outlays do not belong to government spending: transfers (SS and Interest Payments).
- Government Investment.

Exports (E) and Imports (M)

- Exports: deliveries of US goods and services to other countries.
- Imports: deliveries of goods and services from other countries to the US.
- Trade Balance=Exports-Imports
- Trade Deficit: if trade balance negative.
- Trade Surplus: if trade balance positive

Composition of GDP - Spending	in billion \$	in % of GDP
Total Nom. GDP	11,004.0	100.0%
Consumption	7,760.0	70.5%
Durable Goods	950.7	8.6%
Nondurable Goods	2,200.1	20.0%
Services	4,610.1	41.9%
Gross Private Investment	1,667.0	15.1%
Nonresidential	1,094.7	9.9%
Residential	572.3	5.2%
Changes in Inventory	-1.2	-0.0%
Government Purchases	2,075.5	18.9%
Federal Gov.	752,2	6.8%
State & Local Gov.	1,323.3	12.2%
Net Exports	-498.1	-4.5%
Exports	1,046.2	9.5%
Imports	1,544.3	14.0%
Gross National Product	11,059.2	100.5%

Computing GDP through Income

National Income: broadest measure of the total incomes of all Americans

Gross Domestic Product (11,004.0)

+Factor Inc. from abroad (329.0) - Factor Inc. to abroad (273.9)

= Gross National Product (11,059.2)

-Depreciation (1,359.9)

= Net National Product (9,705.2)

-Statistical Discrepancy (25.6)

= National Income (9,679.6)

Distribution of National Income

- 1. Employees' Compensation: wages, salaries, and fringe benefits.
- 2. Proprietors' Income: income of noncorporate business.
- 3. Rental Income: income that landlords receive from renting, including "imputed" rent less expenses on the house, such as depreciation.
- 4. Corporate Profits: income of corporations after payments to their workers and creditors.
- 5. Net interest: interests paid by domestic businesses plus interest earned from foreigners.

Labor and Capital Share

- Labor share: the fraction of national income that goes to labor income
- Capital share: the fraction of national income that goes to capital income.
- Labor Share= Labor Income National Income
- Capital Share= Capital Income National Income
- Proprietor's Income?

Distribution of National Income

	Billion \$US	% of Nat. Inc.
National Income less Prod. Tax.	8,841.0	100.0%
Compensation of Employees	6,289.0	71.1%
Proprietors' Income	834.1	9.4%
Rental Income	153.8	1.7%
Corporate Profits	1021.1	11.6%
Net Interest	543.0	6.1%

Composition of National Income

Industries	Val. Add.	in %
National Income'	9,396.6	100.0%
Agr., Forestry, Fish.	75.8	0.8%
Mining	94.9	1.0%
Construction	476.5	5.1%
Manufacturing	1,113.1	11.8%
Public Utilities	156.0	1.7%
Transportation	259.9	2.8%
Wholesale Trade	569,6	6.1%
Retail Trade	752.8	7.7%
Fin., Insur., Real Est.	1,740.8	18.5%
Services	1,893.6	20.1%
Government	1,182.8	12.6%
Rest of the World	55.1	0.6%

Other Income Concepts: Personal Income

• Income that households and noncorporate businesses receive

National Income (9,679.6) – Corporate Profits (1021.1) –Net Taxes on Production and Imports (751.3) –Net Interest (543.0) – Contributions for Social Insurance (773.1) +Personal Interest Income (1,322.7) +Personal Current Transfer Receipts (1,335.4) = Personal Income (9,161.8) Other Income Concepts: Disposable Personal Income

• Income that households and noncorporate businesses can spend, after having satisfied their tax obligations

Personal Income (9,161.8) –Personal Tax and Nontax Payments (1,001.9) = Disposable Personal Income (8,159.9)

Investment and Saving

- Private Saving (S): gross income minus consumption and taxes plus transfers
- From income side Y = C + S + T TR + NFP
- From expenditure side Y = C + I + G + X M

$$\underbrace{I}_{\text{Private Inv. Private Saving}} \underbrace{S}_{\text{Public Saving}} + \underbrace{T - TR - G}_{\text{Public Saving}} + \underbrace{M - X + NFP}_{\text{Foreign Saving}}$$

Some Nontrivial Issues

- Releases of Information and revisions.
- Methodological Changes.
- Technological Innovation.
- Underground Economy.
- Non-market activities.
- Welfare.

Price Indices

Question: How to compute the price level?

Idea: Measure price of a particular basket of goods today versus price of same basket in some *base* period

Example: Economy with 2 goods, hamburgers and coke

$$h_t = \#$$
 of hamburgers produced, period t
 $p_{ht} =$ price of hamburgers in period t
 $c_t = \#$ of coke produced, period t
 $p_{ct} =$ price of coke in period t
 $h_0, p_{h0}, c_0, p_{c0}) =$ same variables in period 0

Laspeyres price index

$$L_t = \frac{p_{ht}h_0 + p_{ct}c_0}{p_{h0}h_0 + p_{c0}c_0}$$

Paasche price index

$$Pa_t = \frac{p_{ht}h_t + p_{ct}c_t}{p_{h0}h_t + p_{c0}c_t}$$

Problems with Price Indices

- Laspeyres index tends to overstate inflation.
- Paasche index tends to understate inflation.
- Fisher Ideal Index: geometric mean: $(L_t \times Pa_t)^{0.5}$.
- Chain Index.

From Nominal to Real GDP

- Nominal GDP: total value of goods and services produced.
- Real GDP: total production of goods and services in physical units.
- How is real GDP computed in practice, say in 2004?

- 1. Pick a base period, say 1996
- 2. Measure dollar amount spent on hamburgers.
- 3. Divide by price of hamburgers in 2004 and multiply by price in 1996. (this equals the number of hamburgers sold in 2004, multiplied by the price of hamburgers in 1996 -the base period).
- 4. Sum over all goods and services to get real GDP.
For our example ...

Nominal GDP in 2004 = $h_{2004}p_{h2004} + c_{2004}p_{c2004}$ Real GDP in 1996 = $h_{2000}p_{h1996} + c_{2004}p_{c1996}$

Note that

Measuring Inflation I

•
$$\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$
 where P_t is the "Price Level".

- GDP deflator: basket corresponds to current composition of GDP.
- Consumer Price Index (CPI): basket corresponds to what a typical household bought during a typical month in the base year

$$\mathsf{CPI} = \frac{h_{1996}p_{h2004} + c_{1996}p_{c2004}}{h_{1996}p_{h1996} + c_{1996}p_{c1996}}$$

• CPI important because of COLA's.

Measuring Inflation II

- CPI may overstate inflation: Boskin Commission, New Goods.
- How to measure new technologies? David Cutler's example:
 - 1. Average heart attack in mid-1980's costs about $_{1999}12,000$ to treat.
 - 2. Average heart attack in late 1990's costs about $_{199922,000}$ to treat.
 - 3. Average life expectancy in late 1990's is one year higher than in mid-1980's.
 - 4. Is health care more expensive now?

Inflation over History I

- How much is worth \$1 from 1789 in 2003?
 - 1. \$20.76 using the Consumer Price Index
 - 2. \$21.21 using the GDP deflator.
- How much is worth \$1 from 1861 in 2003?
 - 1. \$20.76 using the Consumer Price Index
 - 2. \$17.61 using the GDP deflator.

Inflation over History II

- How much is worth \$1 from 1929 in 2003?
 - 1. \$10.73 using the Consumer Price Index
 - 2. \$8.83 using the GDP deflator.
- How much is worth \$1 from 1985 in 2003?
 - 1. \$1.71 using the Consumer Price Index
 - 2. \$1.52 using the GDP deflator.

More on Growth Rates

• Growth rate of a variable Y (say nominal GDP) from period t - 1 to t is given by

$$g_Y(t-1,t) = \frac{Y_t - Y_{t-1}}{Y_{t-1}}$$

• Growth rate between period t-5 and period t is given by

$$g_Y(t-5,t) = rac{Y_t - Y_{t-5}}{Y_{t-5}}$$

• Suppose that GDP equals Y_{t-1} in period t-1 and it grows at rate $g_Y(t-1,t)$. How big is GDP in period t?

$$g_Y(t-1,t) = \frac{Y_t - Y_{t-1}}{Y_{t-1}}$$

$$g_Y(t-1,t) * Y_{t-1} = Y_t - Y_{t-1}$$

$$g_Y(t-1,t) * Y_{t-1} + Y_{t-1} = Y_t$$

$$(1+g_Y(t-1,t))Y_{t-1} = Y_t$$

Hence GDP in period t equals GDP in period t - 1, multiplied by 1 plus the growth rate.

• Example: If GDP is \$1000 in 2004 and grows at 3.5%, then GDP in 2005 is

$$Y_{2005} = (1 + 0.035) * 1000 = 1035$$

Suppose GDP grows at a constant rate g over time. Suppose at period
 0 GDP equals some number Y₀ and GDP grows at a constant rate of g% a year. Then in period t GDP equals

$$Y_t = (1+g)^t Y_0$$

• Example: If Octavio Augustus would have put 1 dollar in the bank at year 0AD and the bank would have paid a constant real interest rate of 1.5%, then in 2000 he would have:

$$Y_{2000} = (1.015)^{2000} * \$1 = \$8,552,330,953,000$$

which is almost the US GDP for last year.

• Reverse question: Suppose we know GDP at 0 and at t. Want to know at what constant rate GDP must have grown to reach Y_t, starting from Y₀ in t years.

$$Y_t = (1+g)^t Y_0$$

$$(1+g)^t = \frac{Y_t}{Y_0}$$

$$(1+g) = \left(\frac{Y_t}{Y_0}\right)^{\frac{1}{t}}$$

$$g = \left(\frac{Y_t}{Y_0}\right)^{\frac{1}{t}} - 1$$

 Example: In 1900 a country had GDP of \$1,000 and in 2000 it has GDP of \$15,000. Suppose that GDP has grown at constant rate g. How big must this growth rate be? Take 1900 as period 0, 2000 as period 100, then

$$= \left(\frac{\$15,000}{\$1,000}\right)^{\frac{1}{100}} - 1$$
$$= 0.027 = 2.7\%$$

 Question: We know GDP of a country in period 0 and its growth rate g. How many time periods it takes for GDP in this country to double (to triple and so forth).

$$Y_t = (1+g)^t Y_0$$
$$(1+g)^t = \frac{Y_t}{Y_0}$$
Since $\log(a^b) = b * \log(a)$
$$\log((1+g)^t) = \log\left(\frac{Y_t}{Y_0}\right)$$
$$t * \log(1+g) = \log\left(\frac{Y_t}{Y_0}\right)$$
$$t = \frac{\log\left(\frac{Y_t}{Y_0}\right)}{\log(1+g)}$$

)

• Suppose we want to find the number of years it takes for GDP to double, i.e. the t such $\frac{Y_t}{Y_0} = 2$. We get

$$t = \frac{\log(2)}{\log(1+g)}$$

• Example: with g = 1% it takes 70 years, with g = 2% it takes 35 years, with g = 5% it takes 14 years.

Transactions with the Rest of the World

Trade Balance = Exports – Imports

Current Account Balance = Trade Balance + Net Unilateral Transfers

• Unilateral transfers: include aid to poor countries, interest payments to foreigners for US government debt, and grants to foreign researchers or institutions.

- Net wealth position of the US: difference between what the US is owed and what it owes to foreign countries.
- Capital account balance: equals to the change of the net wealth position of the US

Capital Account Balance this year

- = Net wealth position at end of this year
- $-\ensuremath{\mathsf{Net}}$ wealth position at end of last year

Unemployment Rate

- Labor force: number of people, 16 or older, that are either employed or unemployed but actively looking for a job.
- Current Population Survey.
- Unemployment Rate= number of unemployed people labor force
- What is the current unemployment rate now?

Interest Rates

- Important as they determine how costly it is to borrow money
- Suppose in period t 1 you borrow the amount B_{t-1} . The loan specifies that in period t you have to repay B_t . Nominal interest rate on the loan from period t 1 to period t, i_t , is

$$i_t = \frac{B_t - B_{t-1}}{B_{t-1}}$$

• Real interest rate r_t

$$r_t = i_t - \pi_t$$

• Example: In 2003 you borrow \$15,000 and the bank asks you to repay \$16,500 exactly one year later. The yearly nominal interest rate from 2003 to 2004 is

$$i_{2004} = rac{\$16,500 - \$15,000}{\$15,000} = 0.1 = 10\%$$

Now suppose the inflation rate is 3% in 2004. Then the real interest rate equals 10% - 3% = 7%.

Introduction to Growth Theory

Jesús Fernández-Villaverde University of Pennsylvania

Growth Theory

I do not see how one can look at figures like these without seeing them as representing *possibilities*. Is there some action a government could take that would lead the Indian economy to grow like Indonesia's or Egypt's? If so, *what* exactly? If not, what is it about the "nature of India" that makes it so? The consequences for human welfare involved in questions like these are simply staggering: Once one starts to think about them, it is hard to think about anything else (Lucas 1988, p. 5).

Some Motivation

- Differences across countries:
 - Out of 6.4 billion people, 0.8 do not have access to enough food, 1 to safe drinking water, and 2.4 to sanitation.
 - 2. Life expectancy in rich countries is 77 years, 67 years in middle income countries, and 53 million in poor countries.
- Differences across time:
 - Japanese boy born in 1880 had a life expectancy of 35 years, today 81 years.
 - 2. An American worked 61 hours per week in 1870, today 34.

History of Economic Growth Theory: a Roadmap

- 1. Smith, Ricardo, Malthus and Mill had little hope for sustained growth.
- 2. Forgotten for a long while. III attempted in UK (Harrod and Domar).
- 3. Robert Solow (MIT, Nobel 1987): two main papers: 1956 and 1957.
- 4. Completed by David Cass (Penn) and Tjalling Koopmans (Nobel 1971).
- 5. 80's and 90's: Paul Romer (Stanford, Nobel 20??) and Robert Lucas (Chicago, Nobel 1995).

Growth Facts (Nicholas Kaldor)

Stylized growth facts (empirical regularities of the growth process) for the US and for most other industrialized countries

- 1. Output (real GDP) per worker $y = \frac{Y}{L}$ and capital per worker $k = \frac{K}{L}$ grow over time at relatively constant and positive rate.
- 2. They grow at similar rates, so that the ratio between capital and output, $\frac{K}{Y}$ is relatively constant over time
- 3. The real return to capital r (and the real interest rate $r-\delta$) is relatively constant over time.
- 4. The capital and labor shares are roughly constant over time.



Data

- How do incomes and growth rates vary across countries.
- Summers-Heston data set at Penn: follow 104 countries over 30 years.
- Focus on income (GDP) per worker.
- Measure income (GDP) using PPP-based exchange rates.

Development Facts I

- 1. Enormous variation of per worker income across countries.
- 2. Enormous variation in growth rates of per worker income across countries.

Growth "Miracles"	<i>9</i> 60–97
South Korea	5.9%
Taiwan	5.2%
Growth "Disasters"	
Venezuela	-0.1%
Madagascar	-1.4%









Development Facts II

- 3. Growth rates are not constant over time for a given country.
- 4. Countries change their relative position in the international income distribution.

Development Facts III

- 5. Growth in output and growth in international trade are closely related.
- 6. Demograhic transition.
- 7. International migration.
- 8. "La longue durée".

Year	Population*	GDP per Capita**	
-5000	5	\$130	
-1000	50	\$160	
1	170	\$135	
1000	265	\$165	
1500	425	\$175	
1800	900	\$250	
1900	1625	\$850	
1950	2515	\$2030	
1975	4080	\$4640	
2000	6120	\$8175	
*Millions			
**In year-2000 international dollars.			



Stylized Picture of the Demographic Transition



Growth Accounting

Jesús Fernández-Villaverde University of Pennsylvania

Growth Accounting

- Output is produced by inputs capital K and labor L, in combination with the available technology A
- Want: decompose the growth rate of output into the growth rate of capital input, the growth rate of labor input and technological progress. This exercise is called growth accounting.
- Why?
Aggregate production function

• Maps inputs into output:

$$Y = F\left(A, K, L\right)$$

A is called total factor productivity (TFP).

• Cobb-Douglas example:

$$Y = AK^{\alpha}L^{1-\alpha}$$

• Interpretation.

Discrete vs. Continuous Time

- In discrete time a variable is indexed by time: x_t .
- In continuous time a variable is a function of time: x(t).
- We observe the world only in discrete time...
- but it is often much easier to work with continuous time!

Growth Rates and Logarithms I

• Remember:

$$g_x(t-1,t) = \frac{x_t - x_{t-1}}{x_{t-1}}$$
$$1 + g_x(t-1,t) = \frac{x_t}{x_{t-1}}$$

• Take logs on both sides:

$$\log\left(1+g_x(t-1,t)\right) = \log\left(\frac{x_t}{x_{t-1}}\right)$$

Growth Rates and Logarithms II

• Taylor series expansion of $\log(1+y)$ around y = 0:

$$\log(1+y)|_{y=0} = \ln 1 + \frac{1}{1!}y + \text{higher order terms} \simeq y$$

• Then:

$$egin{aligned} \log\left(1+g_x(t-1,t)
ight) &\simeq g_x(t-1,t) \simeq \log\left(rac{x_t}{x_{t-1}}
ight) \ g_x(t-1,t) &\simeq \log x_t - \log x_{t-1} = \Delta \log x_t \end{aligned}$$

• Remember from calculus that validity of Taylor series expansion is local: *g* small!

Moving between Continuous and Discrete Time I

- Let x(t) be a variable that depends of t.
- Notation:

$$\dot{x}(t) \equiv \frac{dx(t)}{dt}$$

• Take $\log x(t)$. Then:

$$\frac{d\log((x(t)))}{dt} = \frac{\dot{x}(t)}{x(t)} = g_x(t)$$

• Why is this useful?

Moving between Continuous and Discrete Time II

• The definition of time derivative is:

$$\dot{x}(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

• Then:

$$rac{\dot{x}(t)}{x(t)} = rac{\lim_{\Delta t \to 0} rac{x(t + \Delta t) - x(t)}{\Delta t}}{x(t)}$$

• When Δt is small (let's say a year):

$$g_x(t) = rac{\dot{x}(t)}{x(t)} \simeq rac{x(t+1)-x(t)}{x(t)} = g_x(t-1,t) \simeq \Delta \log x_t$$

Growth Rates of Ratios I

• Suppose
$$k(t) = \frac{K(t)}{L(t)}$$
. What is $g_k(t)$?

• Step 1: take logs

$$\log(k(t)) = \log(K(t)) - \log(L(t))$$

• Step 2: differentiate with respect to time

$$\frac{d \log((k(t)))}{dt} = \frac{d \log(K(t))}{dt} - \frac{d \log(L(t))}{dt}$$
$$\frac{\dot{k}(t)}{\dot{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}$$
$$g_k(t) = g_K(t) - g_L(t)$$

Growth Rates of Ratios II

• Growth rate of a ratio equals the difference of the growth rates:

$$g_k(t) = g_K(t) - g_L(t)$$

 Ratio constant over time requires that both variables grow at same rate:

$$g_k(t) = 0 \Rightarrow g_K(t) = g_L(t)$$

Growth Rates of Weighted Products I

• Suppose

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}$$

What is $g_Y(t)$?

• Step 1: take logs

$$\log(Y(t)) = \alpha \log(K(t)) + (1 - \alpha) \log(L(t))$$

Growth Rates of Weighted Products II

• Step 2: differentiate

$$\frac{d \log(Y(t))}{dt} = \alpha \frac{d \log(K(t))}{dt} + (1-\alpha) \frac{d \log(L(t))}{dt}$$
$$\frac{\dot{Y}(t)}{\dot{Y}(t)} = \alpha \frac{\dot{K}(t)}{K(t)} + (1-\alpha) \frac{\dot{L}(t)}{L(t)}$$
$$g_Y(t) = \alpha g_K(t) + (1-\alpha) g_L(t)$$

• Growth rate equals weighted sum, with weights equal to the share parameters

Growth Accounting I

- Observations in discrete time.
- Production Function: Y(t) = F(A(t), K(t), L(t))
- Differentiating with respect to time and dividing by Y(t)

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{F_A A(t)}{Y(t)} \frac{\dot{A}(t)}{A(t)} + \frac{F_k K(t)}{K(t)} \frac{\dot{K}(t)}{K(t)} + \frac{F_L L(t)}{Y(t)} \frac{\dot{L}(t)}{L(t)}$$

Growth Accounting II

- Useful benchmark: Cobb-Douglas $Y(t) = A(t)K(t)^{\alpha}L(t)^{1-\alpha}$.
- Why?
- Taking logs and differentiating with respect to time gives

$$g_Y(t) = g_A(t) + \alpha g_K(t) + (1 - \alpha)g_L(t)$$

• g_A is called TFP growth or multifactor productivity growth.

Doing the Accounting

- Pick an α (we will learn that α turns out to be the capital share).
- Measure g_Y, g_K and g_L from the data.
- Compute g_A as the residual:

$$g_A(t) = g_Y(t) - \alpha g_K(t) - (1 - \alpha)g_L(t)$$

- Therefore g_A is also called the Solow residual.
- Severe problems if mismeasurement $(g_K \text{ is hard to measure})$.

 $\mathsf{Data} \text{ for the US}$

• We pick
$$\alpha = \frac{1}{3}$$

Per.	g_Y	αg_K	$(1-lpha)g_L$	TFP (g_A)	
48 – 98	2.5	0.8 (32%)	0.2 (8%)	1.4 (56%)	
48 – 73	3.3	1.0 (30%)	0.2 (6%)	2.1 (64%)	
73 – 95	1.5	0.7 (47%)	0.3 (20%)	0.6 (33%)	
95 - 98	2.5	0.8 (32%)	0.3 (12%)	1.4 (56%)	

- Key observation: Productivity Slowdown in the 70's
- Note: the late 90's look much better

Reasons for the Productivity Slowdown

- 1. Sharp increases in the price of oil in 70's
- 2. Structural changes: more services and less and less manufacturing goods produced
- 3. Slowdown in resources spent on R&D in the late 60's.
- 4. TFP was abnormally high in the 50's and 60's
- 5. Information technology (IT) revolution in the 70's

Growth Accounting for Other Countries

- One key question: was fast growth in East Asian growth miracles mostly due to technological progress or mostly due to capital accumulation?
- Why is this an important question?

Country	Per.	g_Y	α	αg_K	$(1-lpha)g_L$	g_A
Germany	60-90	3.2	0.4	59%	-8%	49%
Italy	60-90	4.1	0.38	49%	3%	48%
UK	60-90	2.5	0.39	52%	_4%	52%
Argentina	40-80	3.6	0.54	43%	26%	31%
Brazil	40-80	6.4	0.45	51%	20%	29%
Chile	40-80	3.8	0.52	34%	26%	40%
Mexico	40-80	6.3	0.63	41%	23%	36%
Japan	60-90	6.8	0.42	57%	14%	29%
Hong Kong	66-90	7.3	0.37	42%	28%	30%
Singapore	66-90	8.5	0.53	73%	31%	-4%
South Korea	66-90	10.3	0.32	46%	42%	12%
Taiwan	66-90	9.1	0.29	40%	40%	20%

Neoclassical Growth Model

Jesús Fernández-Villaverde University of Pennsylvania

Models and Assumptions I

- What is a model? A mathematical description of the economy.
- Why do we need a model? The world is too complex to describe it in every detail. A model abstracts from details to understand clearly the main forces driving the economy.
- What makes a model successful? When it is simple but effective in describing and predicting how the world works.

Models and Assumptions II

- A model relies on simplifying assumptions. These assumptions drive the conclusions of the model. When analyzing a model it is crucial to spell out the assumptions underlying the model.
- Realism may not a the property of a good assumption.
- An assumption is good when it helps us to build a model that accounts for the observations and predicts well.

Basic Assumptions of the Neoclassical Growth Model

- 1. Continuous time.
- 2. Single good in the economy produced with a constant technology.
- 3. No government or international trade.
- 4. All factors of production are fully employed.
- 5. Labor force grows at constant rate $n = \frac{\dot{L}}{L}$.
- 6. Initial values for capital, K_0 and labor, L_0 given.

Production Function I

• Neoclassical (Cobb-Douglas) aggregate production function:

$$Y(t) = F(K(t), L(t)) = K(t)^{\alpha} L(t)^{1-\alpha}$$

• To save on notation write:

$$Y = K^{\alpha} L^{1-\alpha}$$

where the dependency on t is understood implicitly.

Properties of the Technology I

• Constant returns to scale:

$$\lambda Y = (\lambda K)^{\alpha} (\lambda L)^{1-\alpha} = \lambda K^{\alpha} L^{1-\alpha}$$

• Inputs are essential:

$$F(0,0) = F(K,0) = F(0,L) = 0$$

• Marginal productivities are positive:

$$\frac{\partial F}{\partial K} = \alpha A K^{\alpha - 1} L^{1 - \alpha} > 0$$
$$\frac{\partial F}{\partial L} = (1 - \alpha) A K^{\alpha} L^{-\alpha} > 0$$

Properties of the Technology II

• Marginal productivities are decreasing,

$$\begin{aligned} \frac{\partial^2 F}{\partial K^2} &= (\alpha - 1) \, \alpha K^{\alpha - 2} L^{1 - \alpha} < 0 \\ \frac{\partial^2 F}{\partial L^2} &= (\alpha - 1) \, \alpha K^{\alpha} L^{-\alpha - 1} < 0 \end{aligned}$$

• Inada Conditions,

$$\lim_{K \to 0} \alpha K^{\alpha - 1} L^{1 - \alpha} = \infty, \quad \lim_{K \to \infty} \alpha K^{\alpha - 1} L^{1 - \alpha} = 0$$
$$\lim_{L \to 0} (1 - \alpha) K^{\alpha} L^{-\alpha} = \infty, \quad \lim_{L \to \infty} (1 - \alpha) K^{\alpha} L^{-\alpha} = 0$$

Per Worker Terms

• Define
$$x = \frac{X}{L}$$
 as a per worker variable.

• Then

$$y = \frac{Y}{L} = \frac{K^{\alpha}L^{1-\alpha}}{L} = \left(\frac{K}{L}\right)^{a} \left(\frac{L}{L}\right)^{1-\alpha} = k^{\alpha}$$

• Per worker production function has decreasing returns to scale.

Capital Accumulation I

• Capital accumulation equation:

$$\dot{K} = sY - \delta K$$

- Important additional assumptions:
 - 1. Constant saving rate
 - 2. Constant depreciation rate

Capital Accumulation II

• Dividing by K in the capital accumulation equation:

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta$$

• Some Algebra:

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta = s\frac{\frac{Y}{L}}{\frac{K}{L}} - \delta = s\frac{y}{k} - \delta$$

Capital Accumulation III

• Now remember that:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - n \Rightarrow \frac{\dot{K}}{K} = \frac{\dot{k}}{k} + n$$

• We get

$$\frac{\dot{k}}{k} + n = s\frac{y}{k} - \delta \Rightarrow \dot{k} = sy - (\delta + n)k$$

• Fundamental Differential Equation of Neoclassical Growth Model:

$$\dot{k}=sk^{lpha}-\left(\delta+n
ight)k$$

Graphical Analysis

- Change in k, k is given by difference of sk^{α} and $(\delta + n)k$
- If $sk^{\alpha} > (\delta + n)k$, then k increases.
- If $sk^{\alpha} < (\delta + n)k$, then k decreases.
- Steady state: a capital stock k^* where, when reached, $\dot{k} = 0$
- Unique positive steady state in Neoclassical Growth model.
- Positive steady state (locally) stable.

Close-Form Solution I

- $\dot{k} = sk^{\alpha} (\delta + n)k$ is a Bernoulli Equation.
- Change of variable:

$$z = k^{1-\alpha}$$

• Then:

$$\dot{z} = (1 - \alpha) k^{-\alpha} \dot{k} \Rightarrow \dot{k} = \dot{z} (1 - \alpha)^{-1} k^{\alpha}$$

Close-Form Solution II

• Some algebra

$$\begin{split} \dot{z} (1-\alpha)^{-1} k^{\alpha} &= sk^{\alpha} - (\delta+n) k\\ \dot{z} (1-\alpha)^{-1} &= s - (\delta+n) k^{1-\alpha} = s - (\delta+n) z\\ \dot{z} &= (1-\alpha) s - (1-\alpha) (\delta+n) k^{1-\alpha}\\ \hline \dot{z} + \lambda z &= (1-\alpha) (\delta+n). \end{split}$$
where $\lambda = (1-\alpha) (\delta+n).$

Close-Form Solution III

- We have a linear, first order differential equation with constant coefficients.
- Integrating with respect to $e^{\lambda t} dt$:

$$\int (\dot{z} + \lambda z) e^{\lambda t} dt = \int (1 - \alpha) s e^{\lambda t} dt$$

we get

$$ze^{\lambda t} = rac{(1-lpha)s}{\lambda}e^{\lambda t} + b$$

where b is an integrating constant.

Close-Form Solution IV

• Then:

$$z(t) = rac{(1-lpha)s}{\lambda} + be^{-\lambda t}$$

• Substituting back: $z = k^{1-\alpha}$ we get the general solution:

$$k(t) = \left(\frac{s}{\delta+n} + be^{-\lambda t}\right)^{\frac{1}{1-\alpha}}$$

• To find the particular solution note that

$$z(0) = rac{(1-lpha)s}{\lambda} + be^{-\lambda 0} = rac{s}{\delta+n} + b = z_0 \Rightarrow b = z_0 - rac{s}{\delta+n}$$

Close-Form Solution V

• Then:

$$z(t) = \frac{s}{\delta + n} + \left(z_0 - \frac{s}{\delta + n}\right)e^{-\lambda t}$$

- Interpretation of λ .
- Substituting back $z = k^{1-\alpha}$ we get:

$$k(t) = \left(\frac{s}{\delta+n} + \left(k_0^{1-\alpha} - \frac{s}{\delta+n}\right)e^{-\lambda t}\right)^{\frac{1}{1-\alpha}}$$
$$y(t) = \left(\frac{s}{\delta+n} + \left(k_0^{1-\alpha} - \frac{s}{\delta+n}\right)e^{-\lambda t}\right)^{\frac{\alpha}{1-\alpha}}$$

Steady State Analysis

- Steady State: $\dot{k} = 0$
- Solve for steady state

$$0 = s \left(k^*\right)^{\alpha} - (n+\delta)k^* \Rightarrow k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- Steady state output per worker $y^* = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$
- Steady state output per worker depends positively on the saving (investment) rate and negatively on the population growth rate and depreciation rate.

Comparative Statics

- Suppose that of all a sudden saving rate s increases to s' > s. Suppose that at period 0 the economy was at its old steady state with saving rate s.
- $(n+\delta)k$ curve does not change.
- $sk^{\alpha} = sy$ shifts up to s'y.
- New steady state has higher capital per worker and output per worker.
- Monotonic transition path from old to new steady state.
Evaluating the Basic Neoclassical Growth Model: the Good

- Why are some countries rich (have high per worker GDP) and others are poor (have low per worker GDP)?
- Neoclassical Growth model: if all countries are in their steady states, then:
 - 1. Rich countries have higher saving (investment) rates than poor countries.
 - 2. Rich countries have lower population growth rates than poor countries.
- Data seem to support this prediction of the Neoclassical Growth model.



GDP per Worker 1990 as Function of Investment Rate





Evaluating the Basic Neoclassical Growth Model: the Bad

- Are saving and population growth rates exogenous?
- Are the magnitude of differences created by the model right?

$$y_{ss}^{us} = \left(\frac{0.2}{0.01 + 0.06}\right)^{\frac{1}{2}} = 1.69$$
$$y_{ss}^{chad} = \left(\frac{0.05}{0.02 + 0.06}\right)^{\frac{1}{2}} = 0.79$$

• No growth in the steady state: only transitional dynamics.

The Neoclassical Growth Model and Growth

- We can take the absence of growth as a positive lesson.
- Illuminates why capital accumulation has an inherit limitation as a source of economic growth:
 - 1. Soviet Union.
 - 2. Development theory of the 50's and 60's.
 - 3. East Asian countries today?
- Tells us we need to look some place else: technology.

Introducing Technological Progress

• Aggregate production function becomes

$$Y = K^{\alpha} \left(AL \right)^{1-\alpha}$$

- A: Level of technology in period t.
- Key assumption: constant positive rate of technological progress:

$$rac{\dot{A}}{A}=g>$$
 0

• Growth is exogenous.

Balanced Growth Path

• Situation in which output per worker, capital per worker, and consumption per worker grow at constant (but potentially different) rates

• Steady state is just a balanced growth path with zero growth rate

• For Neoclassical Growth model, in BGP: $g_y = g_k = g_c$

Proof

- Capital Accumulation Equation $\dot{K} = sY \delta K$
- Dividing both sides by K yields $g_K \equiv \frac{\dot{K}}{K} = s \frac{Y}{K} \delta$

• Remember that
$$g_k\equiv rac{\dot{k}}{k}=rac{\dot{K}}{K}-n$$

• Hence

$$g_k \equiv rac{\dot{k}}{k} = srac{Y}{K} - (n+\delta)$$

• In BGP g_k constant. Hence $\frac{Y}{K}$ constant. It follows that $g_Y = g_K$. Therefore $g_y = g_k$

What is the Growth Rate?

• Output per worker

$$y = \frac{Y}{L} = \frac{K^{\alpha} (AL)^{1-\alpha}}{L} = \frac{K^{\alpha} (AL)^{1-\alpha}}{L^{\alpha} L^{1-\alpha}} = k^{\alpha} A^{1-\alpha}$$

- Take logs and differentiate $g_y = \alpha g_k + (1 \alpha)g_A$
- We proved $g_k = g_y$ and we use $g_A = g$ to get

$$g_k = \alpha g_k + (1 - \alpha)g = g = g_y$$

• BGP growth rate equals rate of technological progress. No TP, no growth in the economy.

Analysis of Extended Model

• In BGP variables grow at rate g. Want to work with variables that are constant in long run. Define:

$$egin{array}{rcl} ilde{y} &=& rac{y}{A} = rac{Y}{AL} \ ilde{k} &=& rac{k}{A} = rac{K}{AL} \end{array}$$

• Repeat the analysis with new variables:

$$egin{array}{rcl} ilde{y} &=& ilde{k}^lpha \ ilde{k} &=& s ilde{y} - (n+g+\delta) ilde{k} \ ilde{k} &=& s ilde{k}^lpha - (n+g+\delta) ilde{k} \end{array}$$

Close-Form Solution

• Repeating all the steps than in the basic model we get:

$$\begin{split} \widetilde{k}(t) &= \left(\frac{s}{\delta + n + g} + \left(\widetilde{k}_0^{1 - \alpha} - \frac{s}{\delta + n + g}\right)e^{-\lambda t}\right)^{\frac{1}{1 - \alpha}} \\ \widetilde{y}(t) &= \left(\frac{s}{\delta + n + g} + \left(\widetilde{k}_0^{1 - \alpha} - \frac{s}{\delta + n + g}\right)e^{-\lambda t}\right)^{\frac{\alpha}{1 - \alpha}} \end{split}$$

• Interpretation.

Balanced Growth Path Analysis I

• Solve for \tilde{k}^* analytically

$$egin{array}{rcl} 0&=&s ilde{k}^{st^lpha}-(n+g+\delta) ilde{k}^st \ ilde{k}^st&=&\left(rac{s}{n+g+\delta}
ight)^{rac{1}{1-lpha}} \end{array}$$

• Therefore

$$\tilde{y}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Balanced Growth Path Analysis II

$$k(t) = A(t) \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$y(t) = A(t) \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$K(t) = L(t)A(t) \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$Y(t) = L(t)A(t) \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Evaluation of the Model: Growth Facts

- 1. Output and capital per worker grow at the same constant, positive rate in BGP of model. In long run model reaches BGP.
- 2. Capital-output ratio $\frac{K}{V}$ constant along BGP
- 3. Interest rate constant in balanced growth path
- 4. Capital share equals α , labor share equals 1α in the model (always, not only along BGP)
- 5. Success of the model along these dimensions, but source of growth, technological progress, is left unexplained.

Evaluation of the Model: Development Facts

- 1. Differences in income levels across countries explained in the model by differences in s, n and δ .
- 2. Variation in growth rates: in the model *permanent* differences can only be due to differences in rate of technological progress g. *Temporary* differences can be explained by transition dynamics.
- 3. That growth rates are not constant over time for a given country can be explained by transition dynamics and/or shocks to n, s and δ .
- 4. Changes in relative position: in the model countries whose s moves up, relative to other countries, move up in income distribution. Reverse with n.

Interest Rates and the Capital Share

- Output produced by price-taking firms
- Hire workers L for wage w and rent capital K from households for r
- Normalization of price of output to 1.
- Real interest rate equals $r \delta$

Profit Maximization of Firms

$$\max_{K,L} K^{\alpha} (AL)^{1-\alpha} - wL - rK$$

• First order condition with respect to capital K

$$\alpha K^{\alpha-1} (AL)^{1-\alpha} - r = 0$$
$$\alpha \left(\frac{K}{AL}\right)^{\alpha-1} = r$$
$$\alpha \tilde{k}^{\alpha-1} = r$$

• In balanced growth path $\tilde{k} = \tilde{k}^*$, constant over time. Hence in BGP rconstant over time, hence $r - \delta$ (real interest rate) constant over time.

Capital Share

- Total income = Y, total capital income = rK
- Capital share

capital share =
$$\frac{rK}{Y}$$

= $\frac{\alpha K^{\alpha-1} (AL)^{1-\alpha} K}{K^{\alpha} (AL)^{1-\alpha}}$
= α

• Labor share $= 1 - \alpha$.

Wages

• First order condition with respect to labor L

$$(1-\alpha)K^{\alpha}(LA)^{-\alpha}A = w$$
$$(1-\alpha)\tilde{k}^{\alpha}A = w$$

• Along BGP $\tilde{k} = \tilde{k}^*$, constant over time. Since A is growing at rate g, the wage is growing at rate g along a BGP.

Human Capital and Growth

Jesús Fernández-Villaverde University of Pennsylvania Introduction to Human Capital

- Education levels are very different across countries.
- Rich countries tend to have higher educational levels than poor countries.
- We have the intuition that education (learning skills) is an important factor in economic growth.

Production Function

• Cobb-Douglas aggregate production function:

$$Y = K^{\alpha} H^{\beta} \left(AL \right)^{1 - \alpha - \beta}$$

- Again we have constant returns to scale.
- Human capital and labor enter with a different coefficient.

Inputs Accumulation

• Society accumulates human capital according to:

$$\dot{H} = s_h Y - \delta H$$

• Capital accumulation equation:

$$\dot{K} = s_k Y - \delta K$$

- Technology progress: $\frac{\dot{A}}{A} = g > 0$.
- Labor force grows at constant rate: $\frac{\dot{L}}{L} = n > 0$.

Rewriting the Model in Efficiency Units

• Redefine the variables in efficiency units:

$$\tilde{x} \equiv \frac{X}{AL}$$

• Then, diving the production function by *AL*:

$$\tilde{y} = \tilde{k}^{lpha} \tilde{h}^{eta}$$

• Decreasing returns to scale in per efficiency units.

Human Capital Accumulation

• The evolution of inputs is determined by:

$$egin{array}{rcl} ec{k} &=& s_k ilde{k}^lpha ilde{h}^eta - (n+g+\delta) ilde{k} \ \dot{ ilde{h}} &=& s_h ilde{k}^lpha ilde{h}^eta - (n+g+\delta) ilde{h} \end{array}$$

- System of two differential equations.
- Solving it analytically it is bit tricky so we will only look at the BGP.

Phase Diagram

- Solving the system analytically it is bit tricky.
- Alternatives:
 - 1. Use numerical methods.
 - 2. Linearize the system.
 - 3. Phase diagram.



Balanced Growth Path Analysis I

• To find the BGP equate both equations to zero:

$$egin{array}{rcl} s_k ilde{k}^{stlpha lpha} ilde{h}^{st eta} &- (n+g+\delta) ilde{k}^st &=& 0 \ s_h ilde{k}^{stlpha lpha} ilde{h}^{st eta} - (n+g+\delta) ilde{h}^st &=& 0 \end{array}$$

• From first equation:

$$\tilde{h}^* = \left(\frac{(n+g+\delta)}{s_k}\tilde{k}^{*1-\alpha}\right)^{\frac{1}{\beta}}$$

Balanced Growth Path Analysis II

• Plugging it in the second equation

$$egin{aligned} &s_h ilde{k}^{stlpha} rac{(n+g+\delta)}{s_k} ilde{k}^{st 1-lpha} - (n+g+\delta) \left(rac{n+g+\delta}{s_k} ilde{k}^{st 1-lpha}
ight)^{rac{1}{eta}} = \mathbf{0} \Rightarrow \ & rac{s_h}{s_k} ilde{k}^st = \left(rac{n+g+\delta}{s_k} ilde{k}^{st 1-lpha}
ight)^{rac{1}{eta}} \end{aligned}$$

• Work with the expression.

Some Algebra

$$\frac{s_h}{s_k} \tilde{k}^* = \left(\frac{n+g+\delta}{s_k} \tilde{k}^{*1-\alpha}\right)^{\frac{1}{\beta}} \Rightarrow$$
$$\tilde{k}^{*1-\frac{1-\alpha}{\beta}} = \tilde{k}^{*-\frac{1-\alpha-\beta}{\beta}} = \frac{s_k}{s_h} \left(\frac{(n+g+\delta)}{s_k}\right)^{\frac{1}{\beta}} \Rightarrow$$

$$\boxed{ \begin{split} \tilde{k}^* &= \left(\frac{s_k^{1-\beta} s_h^{\beta}}{n+g+\delta} \right)^{\frac{1}{1-\alpha-\beta}} \\ \tilde{h}^* &= \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{n+g+\delta} \right)^{\frac{1}{1-\alpha-\beta}} \end{split} }$$

Evaluating the Model I

• Using the production function:

$$\begin{split} \tilde{y} &= \frac{Y}{AL} = \tilde{k}^{\alpha} \tilde{h}^{\beta} = \left(\frac{s_{k}^{1-\beta} s_{h}^{\beta}}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{s_{k}^{\alpha} s_{h}^{1-\alpha}}{n+g+\delta}\right)^{\frac{\beta}{1-\alpha-\beta}} \Rightarrow \\ y &= \frac{Y}{L} = \left(\frac{s_{k}^{1-\beta} s_{h}^{\beta}}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{s_{k}^{\alpha} s_{h}^{1-\alpha}}{n+g+\delta}\right)^{\frac{\beta}{1-\alpha-\beta}} A \end{split}$$

• Given some initial value of technology A_0 we have:

$$y = \left(\frac{s_k^{1-\beta}s_h^{\beta}}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{s_k^{\alpha}s_h^{1-\alpha}}{n+g+\delta}\right)^{\frac{\beta}{1-\alpha-\beta}} A_0 e^{gt}$$

Evaluating the Model II

• Taking logs:

$$egin{aligned} \log y &= & \log A_0 + gt - rac{lpha + eta}{1 - lpha - eta} \log \left(n + g + \delta
ight) + \ & + rac{lpha}{1 - lpha - eta} \log s_k + rac{eta}{1 - lpha - eta} \log s_h \end{aligned}$$

- What if we have a lot of countries i = 1, ..., n?
- We can assume that $\log A_0 = a + \varepsilon_i$
- Also assume that g and δ are constant across countries.

Evaluating the Model III

• Then we have:

$$egin{array}{rcl} \log y_i &=& a+gt-rac{lpha+eta}{1-lpha-eta}\log\left(n_i+g+\delta
ight)+ \ &+rac{lpha}{1-lpha-eta}\log s_{ki}+rac{eta}{1-lpha-eta}\log s_{hi}+arepsilon_i \end{array}$$

• This is a functional form that can be taken to the data.



Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	6.89	7.81	8.63
	(1.17)	(1.19)	(2.19)
ln(I/GDP)	0.69	0.70	0.28
	(0.13)	(0.15)	(0.39)
$\ln(n+g+\delta)$	-1.73	-1.50	-1.07
	(0.41)	(0.40)	(0.75)
ln(SCHOOL)	0.66	0.73	0.76
	(0.07)	(0.10)	(0.29)
\overline{R}^2	0.78	0.77	0.24
s.e.e.	0.51	0.45	0.33
Restricted regression:			
CONSTANT	7.86	7.97	8.71
	(0.14)	(0.15)	(0.47)
$\ln(\mathrm{I}/\mathrm{GDP}) - \ln(n + g + \delta)$	0.73	0.71	0.29
	(0.12)	(0.14)	(0.33)
$\ln(\text{SCHOOL}) - \ln(n + g + \delta)$	0.67	0.74	0.76
	(0.07)	(0.09)	(0.28)
\overline{R}^2	0.78	0.77	0.28
s.e.e.	0.51	0.45	0.32
Test of restriction:			
<i>p</i> -value	0.41	0.89	0.97
Implied α	0.31	0.29	0.14
	(0.04)	(0.05)	(0.15)
Implied β	0.28	0.30	0.37
	(0.03)	(0.04)	(0.12)

ESTIMATION OF THE AUGMENTED SOLOW MODEL

1.1.1

1 000

n

л

- 1005

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.
Convergence and World Income Distribution

Jesús Fernández-Villaverde University of Pennsylvania The Convergence Hypothesis

- Fact: Enormous variation in incomes per worker across countries
- Question: Do poor countries eventually catch up?
- Convergence hypothesis: They do, in the right sense!
- Main prediction of convergence hypothesis: Poor countries should grow faster than rich countries.
- Let us look at the data.

Neoclassical Growth Model and Convergence

Countries with same s, n, δ, α, g

- eventually same growth rate of output per worker and same level of output per worker (*absolute* convergence).
- countries starting further below the balanced growth path (poorer countries) should grow faster than countires closer to balanced growth path.
- seems to be the case for the sample of now industrialized countries.

Countries with same g, but potentially differing s,n,δ,α

- countries have different balanced growth path.
- countries that start further below *their* balanced growth path (countries that are poor relative to their BGP) should grow faster than rich countries (relative to their BGP). This is called *conditional* convergence.
- data for full sample lend support to conditional convergence.

World Income Distribution

What is happening with the distribution of world income?

Look at the data again.

Conclusion: The Neoclassical Growth Model

- Offers a simple and elegant account of a number of growth facts.
- However:
 - 1. leaves unexplained factors that make countries leave (or not attain) their BGP.
 - 2. leaves unexplained why certain countries have higher s, n than others.
 - 3. leaves unexplained technological progress, the source of growth.



Figure 1.a: Growth Rate Versus Initial Per Capita GDP



Figure 1.b: Growth Rate Versus Initial Per Capita GDP



Figure 1.c: Growth Rate Versus Initial Per Capita GDP

Figure 2 Relative Y/L, 1960 vs. 1988 (log scale)















log(income)

1980 📥 1990 💳 1998























Figure 5 Steady State Incomes, Based on Current Policies



Figure 6 Steady States Implied by Transition Dynamics



Table 1

Frequency of Growth Miracles and Growth Disasters

Interval	Number of Countries	Fast Growth	Intermediate Growth	Slow Growth
All Countries	(121)	40	45	15
$\tilde{\gamma} \leq .05$	(18)	22	61	17
$0.05 < \tilde{y} \le 0.10$	(23)	22	35	43
$.10 < \tilde{\tilde{y}} \le .20$	(31)	65	32	3
$.20 < \tilde{\tilde{y}} \le .40$	(24)	42	50	8
$.40 < \tilde{\tilde{y}} \le .80$	(21)	43	52	5
$\tilde{y} > .80$	(4)	0	75	25

Notes: Entries in the main part of the table reflect the percentage of countries in each interval exhibiting fast, intermediate and slow growth. Fast growth is defined to be one percentage point faster than U.S. growth (1.4 percent), and slow growth is defined to be one percentage point slower.

Table 2 World Income Distributions, Using Markov Transition Method

Interval	1960	1988		Predicted	
			2010	2050	Long-Run
$\tilde{y} \le .05$	15	17	15	12	8
$.05 < \tilde{y} \le .10$	19	13	13	11	8
$.10 < \tilde{y} \le .20$	26	17	14	13	11
$.20 < \tilde{y} \le .40$	20	22	23	23	24
$.40 < \tilde{y} \le .80$	17	22	23	26	30
$\tilde{y} > .80$	3	9	12	15	19

Note. Entries in the table reflect the percentage of countries with relative incomes in each interval.

Endogenous Growth Theory

Jesús Fernández-Villaverde University of Pennsylvania

New Growth Theory

• Remember from Solow Model:

$$g=\frac{\dot{A}}{A}=\frac{\dot{y}}{y}$$

Growth depends on technological progress.

- Good thing: now we know where to look at.
- Challenge: we need a theory of technological progress.
- Paul Romer's big contribution to economics.

Ideas as Engine of Growth

- Technology: the way inputs to the production process are transformed into output.
- Without technological progress:

$$Y = K^{\alpha} L^{1-\alpha}$$

With technological progress:

$$Y = K^{\alpha} \left(AL \right)^{1-\alpha}$$

- Technological progress due to new ideas: very different examples.
- Why (and under what circumstances) are resources are spent on the development of new ideas?

Ideas

- What is an idea?
- What are the basic characteristics of an idea?
 - 1. Ideas are *nonrivalrous* goods.
 - 2. Ideas are, at least partially, *excludable*.

Different Types of Goods

- 1. Rivalrous goods that are excludable: almost all private consumption goods, such as food, apparel, consumer durables fall into this group.
- 2. Rivalrous goods that have a low degree of excludability: tragedy of the commons.
- 3. Nonrivalrous goods that are excludable: most of what we call ideas falls under this point.
- 4. Nonrivalrous and nonexcludable goods: these goods are often called public goods.

Nonrivalrousness and Excludability of Ideas

- Nonrivalrousness: implies that cost of providing the good to one more consumer, the *marginal cost* of this good, is constant at zero. Production process for ideas is usually characterized by substantial fixed costs and low marginal costs. Think about software.
- Excludability: required so that firm can recover fixed costs of development. Existence of intellectual property rights like patent or copyright laws are crucial for the private development of new ideas.

Intellectual Property Rights and the Industrial Revolution

- Ideas engine of growth.
- Intellectual property rights needed for development of ideas.
- Sustained growth recent phenomenon.
- Coincides with establishment of intellectual property rights.

Data on Ideas

- Measure technological progress directly through ideas
- Measure ideas via measuring patents
- Measure ideas indirectly by measuring resources devoted to development of ideas

Important Facts from Data

- Number of patents issued has increased: in 1880 roughly 13,000 patents issued in the US, in 1999 150,000
- More and more patents issued in the US are issued to foreigners. The number of patents issued to US firms or individuals constant at 40,000 per year between 1915 and 1991.
- Number of researchers engaged in research and development (R&D) in the US increased from 200,000 in 1950 to 1,000,000 in 1990.
- Fraction of the labor force in R&D increased from 0.25% in 1950 to 0.75% in 1990.

Infrastructure or Institutions

- Question: why does investment rate *s* differ across countries?
- Answer: some countries have political institutions that make investing more profitable than others.
- Investment has costs and benefits: some countries invest more than others because either the costs of investment are lower or the benefits are higher.
Cost of Investment

- Cost of investment: resources to develop idea, purchase of buildings and equipment.
- Cost of obtaining all legal permissions.
- Hernando de Soto "The Other Path" (1989).
- Deficient or corrupt bureaucracy can impede profitable investment activities.

Benefits of Investment

- 1. The size of the market. Depends on openness of the economy
- 2. The extent to which the benefits from the investment accrue to the investor. Diversion of benefits due to high taxes, theft, corruption, the need to bribe government officials or the payment of protection fees to the Mafia or Mafia-like organizations.
- 3. Rapid changes in the economic environment in which firms and individuals operate: increase uncertainty of investors.
- 4. Data show that these considerations may be important

A Basic Model of Endogenous Growth

- Can we built a model that puts all this ideas together?
- Yes, Romer 1990. You can get a copy of the paper at the Class Web Page.
- A bit of work but we can deal with it.

Basic Set-up of the Model I

- Model of Research and Growth.
- Three sectors: final-goods sector, intermediate-goods sector and research sector.
- Those that invent a new product and those that sell it do not need to be the same: Holmes and Schmitz, Jr. (1990).
- Why? comparative advantage.

Basic Set-up of the Model II

- Total labor: L.
- Use for production of final goods, L_Y , or to undertake research, $L_A = L L_Y$.
- Total capital: K.
- Used for production of intermediate goods.

Final-Goods Sector I

- Competitive producers.
- Production Function:

$$Y = L_Y^{1-\alpha} \int_0^A x(i)^\alpha di$$

• Optimization problem:

$$\Pi = L_Y^{1-\alpha} \int_0^A x(i)^{\alpha} di - w_Y L_Y - \int_0^A p(i) x(i) di$$

Final-Goods Sector II

• First Order Conditions:

$$w_Y = (1 - lpha) L_Y^{-lpha} \int_0^A x(i)^{lpha} di = (1 - lpha) rac{Y}{L_Y}$$

 $p(i) = lpha L_Y^{1-lpha} x(i)^{lpha - 1} ext{ for } orall i \in [0, A]$

• Interpretation.

Intermediate-Goods Sector I

- Continuum of monopolist.
- Only use capital for production.
- Optimization problem:

$$\pi\left(i\right) = \max_{x(i)} p\left(i\right) x\left(i\right) - rx\left(i\right)$$

• Since
$$p(i) = \alpha L_Y^{1-\alpha} x(i)^{\alpha-1}$$
 we have:

$$\pi(i) = \max_{x(i)} \alpha L_Y^{1-\alpha} x(i)^{\alpha} - rx(i)$$

Intermediate-Goods Sector II

• First Order Conditions:

$$\alpha^{2} L_{Y}^{1-\alpha} x(i)^{\alpha-1} = r \Rightarrow \alpha L_{Y}^{1-\alpha} x(i)^{\alpha-1} = \frac{1}{\alpha} r$$
$$p(i) = \frac{1}{\alpha} r$$

• Interpretation: mark-up of a monopolist.

Intermediate-Goods Sector III

• Total demand:

$$p(i) = \frac{1}{\alpha}r = \alpha L_Y^{1-\alpha}x(i)^{\alpha-1} \Rightarrow x(i) = \left(\frac{1}{\alpha^2}r\right)^{\frac{1}{\alpha-1}}L_Y$$

• The profit of the monopolist:

$$\pi\left(i
ight)=rac{1}{lpha}rx\left(i
ight)-rx\left(i
ight)=\left(rac{1}{lpha}-1
ight)rx\left(i
ight)$$

Aggregation I

• Solution of monopolist is independent of *i*:

$$x\left(i
ight)=x ext{ and } \pi\left(i
ight)=\pi ext{ for } orall i\in\left[0,A
ight]$$

• Then:

$$Y = L_Y^{1-\alpha} \int_0^A x(i)^{\alpha} di = L_Y^{1-\alpha} \int_0^A x^{\alpha} di = L_Y^{1-\alpha} x^{\alpha} \int_0^A di = A x^{\alpha} L_Y^{1-\alpha}$$

Aggregation II

• Since the total amount of capital in the economy is given:

$$\int_{0}^{A} x(i) \, di = K$$

• Then:

$$\int_{0}^{A} x(i) \, di = x \int_{0}^{A} di = Ax = K \Rightarrow x = \frac{K}{A}$$

• Plugging it back:

$$Y = Ax^{\alpha}L_Y^{1-\alpha} = A\left(\frac{K}{A}\right)^{\alpha}L_Y^{1-\alpha} = K^{\alpha}\left(AL_Y\right)^{1-\alpha}$$

Aggregation III

• Taking logs and derivatives:

$$rac{\dot{Y}}{Y} = rac{\dot{A}}{A} + lpha rac{\dot{x}}{x} + (1-lpha) rac{\dot{L_Y}}{L_Y}$$

• Then, in a balance growth path, since $\frac{L_Y}{L_Y} = 0$ and $x = \left(\frac{1}{\alpha^2}r\right)^{\frac{1}{\alpha-1}}L_Y$ are constant:

$$g=\frac{\overset{\cdot}{Y}}{\overset{}{Y}}=\frac{\overset{\cdot}{A}}{\overset{}{A}}$$

Research Sector I

• Production function for ideas:

$$\dot{A} = BAL_A$$

• Then:

$$g = \frac{\dot{A}}{A} = BL_A$$

Research Sector II

- P_A is the price of the new design A.
- Arbitrage idea.
- By arbitrage:

$$rP_A = \pi + \dot{P_A}$$

Research Sector III

• Then
$$r = \frac{\pi}{P_A} + \frac{P_A}{P_A}$$

• In a BGP, r and $\frac{P_A}{P_A}$ are constant and then also $\frac{\pi}{P_A}$. But since π is also constant:

$$P_A \text{ is constant} \Rightarrow \frac{P_A}{P_A} = 0$$

•

• And:

$$r = \frac{\pi}{P_A} = \frac{\left(\frac{1}{\alpha} - 1\right)rx}{P_A} \Rightarrow P_A = \left(\frac{1}{\alpha} - 1\right)x$$

Research Sector IV

• Each unit of labor in the research sector then gets:

$$w_R = BAP_A = BA\left(\frac{1}{\alpha} - 1\right)x$$

• Remember that the wage in the final-goods sector was:

$$w_Y = (1 - lpha) rac{Y}{L_Y}$$

• By free entry into the research sector both wages must be equal:

$$w = w_R = w_Y$$

Research Sector V

$$BA\left(\frac{1}{\alpha}-1\right)x = (1-\alpha)\frac{Y}{L_Y}$$

• and with some algebra:

$$\frac{BA}{\alpha} = \frac{Y}{xL_Y} = \frac{Ax^{\alpha}L_Y^{1-\alpha}}{xL_Y} = Ax^{\alpha-1}L_Y^{-\alpha} \Rightarrow \frac{B}{\alpha} = x^{\alpha-1}L_Y^{-\alpha}$$

Balanced Growth Path I

• As in the Solow's model:

$$\dot{K} = sY - \delta K = sK^{\alpha} \left(AL_Y\right)^{1-\alpha} - \delta K$$

• Dividing by *K*:

$$g = \frac{\dot{K}}{K} = s \left(\frac{K}{A}\right)^{\alpha - 1} L_Y^{1 - \alpha} - \delta = s x^{\alpha - 1} L_Y^{1 - \alpha} - \delta$$

Balanced Growth Path II

• Then:

$$g = BL_A = B(L - L_Y) = s\frac{B}{\alpha}L_Y - \delta \Rightarrow$$
$$BL + \delta = \left(1 + \frac{s}{\alpha}\right)BL_Y \Rightarrow$$
$$L_Y = \frac{BL + \delta}{\left(1 + \frac{s}{\alpha}\right)B} = \frac{1}{1 + \frac{s}{\alpha}}L + \frac{\delta}{\left(1 + \frac{s}{\alpha}\right)B}$$

• And we can compute all the remaining variables in the model.

Is the Level of R&D Optimal?

- Sources of inefficiency:
 - 1. Monopoly power.
 - 2. Externalities
- Possible remedies.
- Implications for Antitrust policy.

The Very Long Run

Jesús Fernández-Villaverde University of Pennsylvania

The Very Long Run

- Economist want to understand the growth experience of ALL human history (*Big History Movement*).
- What are the big puzzles:
 - 1. Why are there so big differences in income today?
 - 2. Why did the West develop first? or Why not China? or India?
- Data Considerations.

Where Does Data Come From?

- Statistics: Customs, Tax Collection, Census, Parish Records.
- Archeological Remains: Farms, Skeletons.
- Literary Sources: Memories, Diaries, Travel Books.

Some Basic Facts I

- For most human history, income per capita growth was glacially slow.
- Before 1500 little or no economic growth.

Paul Bairoch (*Economics and World History: Myths and Paradoxes*)

- Living standards were roughly equivalent in Rome (1st century A.D.), Arab Caliphates (10th Century), China (11th Century), India (17th century), Western Europe (early 18th century).
- 2. Cross-sectional differences in income were a factor of 1.5 or 2.

Some Basic Facts II

- Angus Maddison (*The World Economy: A Millenial Perspective*) calculates 1500-1820 growth rates:
 - 1. World GDP per capita: 0.05%.
 - 2. Europe GDP per capita: 0.14%.
- After 1820: great divergence in income per capita.

FIGURE 1.8 World Inequality and Its Components, 1820–1992



Source: Bourguignon and Morrison (2002).

Europe Becomes Dominant

- From 1492 to 1770, different human populations come into contact. European countries expand until early 20th century:
 - 1. American and Australia: previous cultures were nearly wiped out.
 - 2. Asia: partial control.
 - 3. Africa: somehow in the middle.
- Proximate causes: weaponry and social organization of Europeans was more complex.
- Ultimate causes: why?

Possible Explanations

- Geography.
- Colonies.
- Culture.

Geography

- How can Geography be important?
- Examples:
 - 1. Europe is 1/8 of the size of Africa but coastline is 50% longer.
 - 2. Wheat versus Rice, Braudel (*The Structures of Everyday Life: Civilization and Capitalism, 15th-18th Century*).
- Let's look at a map.



FIGURE 15.2 Regional Variation in Income and Access to the Sea



Source: Gallup, Sachs, and Mellinger (1998).

Jared Diamond (*Guns, Germs, and Steel*): geography.

- Euroasia is bigger (50% than America, 250% as Sub-saharian Africa, 800% than Australia):
 - More plants i.e. out of 56 food grains, 39 are native to Euroasia, 11 to America, 4 to Sub-saharian Africa, and 2 to Australia.
 - 2. More animals to domesticate: cows/pigs/horses/sheeps/goats versus llamas and alpacas.
- Euroasia is horizontal: transmission of technology, plants, and animals.
- Consequence: higher population density—guns and germs.

Why Not China?

- But, how can Diamond explain China?
- Between the 8th and the 12th century, China experienced a burst of economic activity: gunpowder, printing, water-powered spinning wheel
- Voyages of exploration by admiral Zheng: Louise Levathes (*When China Ruled the Seas: The Treasure Fleet of the Dragon Throne, 1405-1433*).
- With the arrival of the Ming dynasty (1368), China stagnates.
- Europe gets ahead.



Eric Jones (*The European Miracle*): geography, hypothesis 1.

- China was first unified around 221 B.C. Since then, except for relatively short periods, unified state (last partition ended with arrival of Mongols in 13th century).
- Europe has never been unified since the fall of Roman Empire (476 a.d.).
- Why? Dispersion of core areas.
FIGURE 15.3 Core Areas in Preindustrial Europe



Source: Pounds and Ball (1964).

FIGURE 15.4 Core Areas in PreIndustrial China



Source: Stover (1974).

Kenneth Pomeranz (*The Great Divergence: China, Europe, and the Mak-ing of the Modern World Economy*): geography, hypothesis 2.

• Coal:

- 1. Far away from production centers
- 2. Steam engine versus ventilations.
- Environmental limits.

Colonies

Immanuel Wallerstein (*The Modern World System*).

- Small initial differences in income.
- Patterns of labor control and trade policies created "plantation" economies.
- Trade: primary goods for manufacturing.
- Forward and Backward linkages.

Differences across Colonies

Daron Acemoglu, Simon Johnson, and James Robinson (*The Colonial Origins of Comparative Development: An Empirical Investigation*).

- Differences in settlers mortality.
- Differences in outcomes:
 - 1. British America: 9 universities for 2.5 million people.
 - 2. Spanish and Portuguese America: 2 universities for 17 million people.



FIGURE 1. REDUCED-FORM RELATIONSHIP BETWEEN INCOME AND SETTLER MORTALITY

Cultural Differences: Yes

- Max Weber (*The Protestant Ethic and the Spirit of Capitalism*)
- Letter from the Chinese emperor Qian Long to King George III of England:

"Our dynasty's majestic virtue has penetrated unto every country under Heaven...As your Ambassador can see for himself, we possess all things. I set no value on objects strange or ingenious, and have no use for your country's manufactures".

• Leibniz's Instructions to a European traveler to China:

"Not too worry so much about getting things European to the Chinese, but rather about getting remarkable Chinese inventions to us". Cultural Differences: No

• A Western traveler, 1881

"The Japanese are a happy race, and being content with little, are not likely to achieve much".

• Karen Kupperman (*Providence Island*, 1630-1641 : The Other Puritan Colony):

Documents differences between Providence Island and New England.

Philip Benedict (*The Faith and Fortunes of France's Huguenots*):
 Differences between Catholics and Huguenots in France.

An Empirical Application:

Population Growth and Technological Change since 1 Million B.C.

- Basic lesson so far: growth depends on technology progress.
- Intuition: more people probably must imply higher knowledge accumulation.
- Growth and population may be closely link.
- Empirical evidence.

A Simple Model

• Production function:

$$Y = T^{\alpha} \left(AL \right)^{1-\alpha}$$

• Technology progress:

$$\dot{A} = BAL$$

• Malthusian assumption:

$$\frac{Y}{L} = y^*$$

Solving the Model

• We find the level of population allowed by a technology:

$$\frac{T^{\alpha} (AL)^{1-\alpha}}{L} = y^* \Rightarrow L^* = \left(\frac{1}{y^*}\right)^{\frac{1}{\alpha}} A^{\frac{1-\alpha}{\alpha}} T$$

• Growth rate of population:

$$\frac{\dot{L^*}}{L^*} = \frac{1 - \alpha}{\alpha} \frac{\dot{A}}{A}$$

• Then:

$$n_t = \frac{\dot{L^*}}{L^*} = \frac{1 - \alpha}{\alpha} \frac{BAL}{A} = \frac{1 - \alpha}{\alpha} BL$$

Time Series Evidence

- A first look at the data.
- Regression:

$$n_t = -0.0026 + 0.524 L_t$$

(0.0355) (0.0258)
 $R^2 = 0.92, D.W = 1.10$

• Robust to different data sets and specifications.

Cross-Section Evidence

- World population was separated from 10,000 BC to circa 1500 AD
- Population and Population Density circa 1500:

	Land Area	Population	Pop/km^2
"Old World"	83.98	407	4.85
Americas	38.43	14	0.36
Australia	7.69	0.2	0.026
Tasmania	0.068	0.0012-0.005	0.018-0.074
Flinders Islands	0.0068	0.0	0.0

• England vs. Europe and Japan vs. Asia.



Population

Introduction to General Equilibrium I: Households

Jesús Fernández-Villaverde University of Pennsylvania The Representative Household

- Who is the representative household?
- Robinson Crusoe in a desert island.
- Justification: aggregation.
- When does aggregation work and when does it not?

What are we going to do?

- Think about the goods existing in the economy.
- Think about what does Robinson prefer.
- Think about his constraints.
- Think about what will Robinson do given his preferences and his constraints

Commodity Space

- 2 goods, consumption c and leisure 1 l.
- Each goods set:
 - **1.** $c \in \Re_+$
 - 2. $l \in [0, 1]$
- Then $(c,l)\in \Re_+ imes [0,1].$

Preferences

• Preferences: binary relation \succeq defined over pairs (c, l):

$$(c_i, l_i) \succeq (c_j, l_j)$$

- Assumptions on preferences:
 - 1. Complete: for $\forall (c_i, l_i), (c_j, l_j) \in \Re_+ \times [0, 1]$ either $(c_i, l_i) \succeq (c_j, l_j)$ or $(c_j, l_j) \succeq (c_i, l_i)$.
 - 2. Reflexive: for $\forall (c_i, l_i) \in \Re_+ \times [0, 1] (c_i, l_i) \succeq (c_i, l_i)$.
 - 3. Transitive: for $\forall (c_i, l_i), (c_j, l_j), (c_k, l_k) \in \Re_+ \times [0, 1]$, if $(c_i, l_i) \succeq (c_j, l_j)$ and $(c_j, l_j) \succeq (c_k, l_k) \Rightarrow (c_i, l_i) \succeq (c_k, l_k)$.

Indifference Curves

• Loci of pairs such that:

$$egin{array}{rcl} (c_i,l_i) &\succeq & \left(c_j,l_j
ight) \ \left(c_j,l_j
ight) &\succeq & \left(c_i,l_i
ight) \end{array}$$

- If we assume that preferences are strictly monotonic, convex and normal, the indifference curves are:
 - 1. Negative sloped in 1 l.
 - 2. Convex.

Utility Function I

- Working directly with binary relations difficult.
- Can we transform them into a function?
- Why is this useful?

Utility Function II

- Definition: a real-value function $u : \Re^2 \to \Re$ is called a utility function representing the binary relation \succeq defined over pairs (c, l) if for $\forall (c_i, l_i), (c_j, l_j) \in \Re_+ \times [0, 1], (c_i, l_i) \succeq (c_j, l_j) \Leftrightarrow u(c_i, l_i) \geq u(c_j, l_j).$
- Theorem: if the binary relation
 <u>≻</u> is complete, reflexive, transitive, strictly monotone and continuous, there exist a continuous real-value function u that represents
 <u>≻</u>.
- Proof (Debreu, 1954): intuition.

Utility Function III

- Utility function and monotone transformations.
- Interpretation of u.
- Differentiability of *u*.

Budget Constraint

- Leisure $1 l \Rightarrow$ labor supply l.
- Wage w.
- Then

$$c = lw$$

• Interpretation for Robinson.

Household's Problem

• Problem for Robinson is then

$$egin{array}{l}{\max_{c,l} u\left(c,1-l
ight)} \\ {s.t.} \; c = lw \end{array}$$

• First order condition:

$$-\frac{u_l}{u_c} = w$$

• Interpretation: marginal rate of substitution equal to relative price of leisure.

A Parametric Example

•
$$u(c,l) = \log c + \gamma \log (1-l)$$

• FOC+Budget constraint:

$$\gamma \frac{c^*}{1-l^*} = w$$
$$c^* = l^* w$$

• Then:

$$l^* = \frac{1}{1+\gamma}$$

Income and Substitution Effect

- We will follow the Hicksian decomposition.
- Substitution Effect: changes in w make leisure change its relative price with total utility constant.
- Income Effect: changes in w induce changes in total income even if l* stays constant.
- For u (c, l) = log c+γ log (1 l) income and substitution effect cancel each other!

Theory and Data

- Can we use the theory to account for the data?
- What are the trends in labor supply?

Chart 1 Two Aggregate Facts

Average Weekly Hours Worked per Person and Real Compensation per Hour Worked* in the United States, 1950–90



*The compensation series is an index of hourly compensation in the business sector, deflated by the consumer price index for all urban consumers.

Sources: Tables 1 and 14

Charts 2-4

Possible Shifts in Hours Worked

Extrapolated Average Weekly Hours Worked per Person by Cohorts at Various Ages in the United States





Table 1

A Look Behind an Aggregate Fact

In the United States, 1950–90

Average Weekly	Hours Worked	Employment_to_
Per Person	Per Worker	Population Ratio
22.03	40.71	.52
20.97	37.83	.52
20.55	36.37	.53
22.00	35.97	.58
23.62	36.64	.61
, 7.2	-10.0	17.3
	Average Weekly Per Person 22.03 20.97 20.55 22.00 23.62 7.2	Average Weekly Hours Worked Per Person Per Worker 22.03 40.71 20.97 37.83 20.55 36.37 22.00 35.97 23.62 36.64

Source: U.S. Department of Commerce, Bureau of the Census

Tables 2–4

A Distribution of Hours Worked

Average Weekly Hours Worked per Person for Demographic Categories in the United States, 1950–90

Table 2 By Sex

Table 4 By Marital Status*

	Weekly H		Weekly Hours Worked per Person by Marital Status						
	Total		Sex		Married W	/ith Spouse			
Year	Population	Males	Females	Year	Present	Absent	Single	Widowed	Divorced
1950	22.03	33.46	10.95	1950	23.89	23.11	28.10	11.82	28.65
1960	20.97	30.70	11.82	1960	23.86	20.43	25.72	10.37	26.31
1970	20.55	28.54	13.29	1970	24.31	20.50	24.19	9.41	26.17
1980	22.00	28.30	16.24	1980	24.15	22.71	25.42	6.86	27.22
1990	23.62	28.53	19.09	1990	26.26	22.22	27.73	5.98	28.41
% Change 1950–90	7.2	-14.7	74.3	% Char 1950–9	nge O 9.9	-3.9	-1.3	-49.4	8

*This excludes individuals less than 25 years old.

Table 3 By Age

	Weekly Hours Worked per Person by Age (in Years)								
Year	15–24	25–34	35–44	45–54	55-64	65–74	75–84		
1950	17.47	24.92	27.09	26.31	22.19	12.03	3.93		
1960	14.15	24.73	27.00	27.63	22.58	8.43	2.97		
1970	14.05	26.16	28.03	28.27	23.28	6.91	2.17		
1980	19.64	28.80	29.89	28.16	20.68	5.11	1.39		
1990	19.13	30.83	32.62	31.47	20.75	5.15	1.18		
% Change 1950–90	9.5	23.7	20.4	19.6	-6.5	-57.2	-70.0		

Source: U.S. Department of Commerce, Bureau of the Census

Tables 5–6

A More Comprehensive Distribution of Hours Worked

Average Weekly Hours Worked per Person for Sets of Demographic Categories in the United States, 1950–90

Table 5 Married . . .

				Weekly Hours Worked per Person by Age (in Years)							
Status		Sex	Year	15–24	25–34	35-44	45–54	55–64	65–74	75–84	
Spouse Present	Total	Males	1950 1960 1970 1980 1990	38.69 36.58 34.19 33.63 34.18	41.14 40.67 40.30 38.70 40.25	43.06 41.79 41.52 40.22 41.34	41.95 40.99 40.65 38.89 40.03	37.58 35.74 34.74 29.83 28.39	23.39 14.74 11.51 8.20 7.71	9.77 6.22 4.08 2.95 2.50	
		% Change	1950—90	-11.7	-2.2	-4.0	-4.6	-24.5	-67.0	-74.4	
		Females	1950 1960 1970 1980 1990	9.17 10.00 14.65 18.36 21.13	8.09 9.10 12.21 18.77 23.90	9.60 12.35 14.95 19.64 25.41	8.61 13.55 16.18 18.16 24.04	4.60 8.66 11.75 11.95 13.83	1.79 2.27 2.55 2.48 2.79	.56 .94 1.03 .70 .65	
		% Change	1950–90	130.4	195.4	164.7	179.2	200.7	55.9	16.1	
	Youngest Child Under 6 Years Old	Females	1950 1960 1970 1980 1990	3.40 5.71 9.08 11.72 15.49	4.60 5.75 8.33 13.47 19.48	6.49 6.36 9.04 13.00 19.62	6.41 9.17 12.11 11.77 18.55	4.24 7.25 10.13 9.32 13.11	3.93 2.10 3.70 1.34 6.61	6.82 2.25 6.02 .30 7.86	
		% Change	1950–90	355.6	323.5	202.3	189.4	209.2	68.2	15.2	
	Youngest Child 6–17 Years Old	Females	1950 1960 1970 1980 1990	3.89 13.27 16.23 15.46 23.43	5.57 13.44 15.90 20.79 24.85	7.64 13.75 15.85 20.01 25.70	6.81 11.99 14.49 16.76 23.01	4.50 8.75 11.41 11.91 15.09	2.28 2.53 4.15 3.90 5.98	10.08 1.45 6.81 3.41 11.01	
		% Change	1950—90	502.3	346.1	236.4	237.9	235.3	162.3	9.2	
Spouse Absent	Total	Males	1950 1960 1970 1980 1990	24.17 17.13 16.49 25.27 21.03	27.54 25.80 27.12 30.64 27.31	31.56 27.83 29.67 31.99 28.80	30.48 29.49 30.48 29.18 29.84	26.62 24.69 25.06 20.58 21.63	16.54 9.66 9.13 5.99 6.23	5.93 3.47 2.94 2.40 1.48	
		% Change	1950—90	-13.0	8	-8.7	-2.1	-18.7	-62.3	-75.0	
		Females	1950 1960 1970 1980 1990	15.37 14.24 16.05 17.12 15.89	20.00 17.52 18.03 21.77 21.95	22.26 20.51 20.17 22.78 25.26	19.74 20.74 21.43 21.32 24.22	13.82 15.58 17.25 15.79 15.72	4.42 4.30 5.43 3.75 4.01	1.04 1.57 2.16 1.50 .84	
		% Change	1950—90	3.4	9.8	13.5	22.7	13.7	-9.3	-19.2	

				Weekly Hours Worked per Person by Age (in Years)								
Status	Sex	Year	15–24	25–34	35–44	45–54	55–64	65–74	75–84			
Single	Males	1950 1960 1970 1980 1990	18.29 12.67 11.37 19.23 18.76	31.58 30.61 29.78 30.55 31.50	33.82 30.35 29.82 29.01 30.17	31.97 28.98 28.03 26.24 26.64	27.18 24.30 22.60 19.60 17.87	15.47 9.74 8.58 6.26 5.83	6.12 5.01 4.17 2.06 2.03			
		% Change 1950–90	2.6	3	-10.8	-16.7	-34.3	-62.3	-66.8			
	Females	1950 1960 1970 1980 1990	14.33 10.70 10.43 17.23 17.35	30.58 29.33 28.82 29.15 29.73	30.51 29.37 27.65 28.24 30.21	28.61 28.94 27.62 25.76 27.59	22.77 24.40 24.23 20.68 18.55	10.36 10.63 8.41 4.93 4.98	3.14 3.35 3.07 1.19 1.02			
		% Change 1950–90	21.1	-2.8	-1.0	-3.6	-18.5	-51.9	-67.5			
Widowed	Males	1950 1960 1970 1980 1990	19.65 19.74 19.68 18.64 15.20	33.50 32.00 29.63 28.31 26.62	35.76 31.33 32.08 29.66 28.70	34.12 31.97 31.93 29.10 29.06	29.15 25.95 25.36 20.89 18.32	14.99 9.24 7.24 5.24 4.90	4.67 3.56 2.34 1.70 1.38			
		% Change 1950–90	-22.6	-20.5	-19.7	-14.8	-37.2	-67.3	-70.4			
	Females	1950 1960 1970 1980 1990	17.02 15.64 17.66 17.12 10.56	21.75 17.61 21.00 17.25 18.50	23.90 22.82 21.85 21.13 24.06	20.11 23.35 23.52 20.71 24.41	12.96 15.71 17.82 15.68 15.16	4.31 4.72 4.20 3.30 3.59	.83 1.18 1.04 .66 .57			
		% Change 1950–90	-38.0	-14.9	.7	21.4	17.0	-16.7	-31.3			
Divorced	Males	1950 1960 1970 1980 1990	29.53 24.08 25.56 29.16 29.17	32.82 30.54 33.14 33.73 33.94	34.93 31.51 33.35 34.39 34.23	32.71 29.50 31.16 30.90 32.90	28.77 25.75 24.62 22.34 22.23	15.76 9.75 8.63 6.09 6.76	11.67 4.95 4.12 2.65 2.46			
		% Change 1950–90	-1.2	3.4	-2.0	.6	-22.7	-57.1	-78.9			
	Females	1950 1960 1970 1980 1990	25.27 24.01 25.16 24.42 23.26	28.72 27.69 27.59 29.38 29.13	30.68 29.87 29.73 30.38 32.82	27.32 29.51 29.71 28.73 31.86	21.99 24.05 25.04 22.65 23.73	10.07 9.19 7.99 5.53 6.68	1.96 2.48 3.26 1.48 1.49			
		% Change 1950–90	-8.0	1.4	7.0	16.6	7.9	-33.7	-24.0			

Table 6 ... And Not Married

Source: U.S. Department of Commerce, Bureau of the Census

Table 7 Partial Life-Cycle Profiles of Hours Worked by Males

Based on U.S. Census Data

	Average Weekly Hours Worked per Person at Age (in Years)										
Year Born	15–24	25–34	35–44	45–54	55–64	65–74	75–84				
1866–75	_	_	_		_	_	7.46				
1876–85	_	_	_	_	_	20.75	5.12				
1886–95	_	_	_	_	35.34	13.31	3.53				
1896–1905	_	_	_	40.06	33.60	10.65	2.57				
1906–15	_	_	41.41	39.15	32.84	7.71	2.16				
1916–25	_	38.60	39.98	38.95	28.38	7.28	_				
1926–35	22.65	38.20	39.79	37.20	26.73	—	_				
1936–45	17.65	37.89	38.59	37.75	_	_	_				
1946–55	15.96	36.15	38.40	_	_	_	_				
1956–65	21.59	36.00	_	_	_	_	_				
1966–75	20.23	—	—	—	—	_	—				

Source: U.S. Department of Commerce, Bureau of the Census
Tables 8–10

Extrapolated Life-Cycle Profiles of Hours Worked

U.S. Census Data Extrapolated as Explained in Appendix C^{\star}

Table 8 By Males

		Average	Weekly Hours	Worked per Pe	erson at Age (i	n Years)	
Year Born	15–24	25–34	35–44	45–54	55–64	65-74	75–84
1866–75	34.08	41.71	45.23	42.55	39.27	25.10	7.46
1876–85	31.90	41.13	44.28	41.70	37.66	20.75	5.12
1886–95	29.67	40.50	43.20	41.01	35.34	13.31	3.53
1896—1905	27.92	39.93	42.35	40.06	33.60	10.65	2.57
1906–15	25.35	39.42	41.41	39.15	32.84	7.71	2.16
1916–25	23.00	38.60	39.98	38.95	28.38	7.28	1.17
1926–35	22.65	38.20	39.79	37.20	26.73	5.27	.39
1936–45	17.65	37.89	38.59	37.75	24.44	3.48	.00
1946–55	15.96	36.15	38.40	37.28	21.64	2.07	.00
1956–65	21.59	36.00	37.87	36.73	19.39	.33	.00
1966–75	20.23	35.27	37.23	36.57	16.95	.00	.00

Table 9 By Females

Average Weekly Hours Worked per Person at Age (in Years)							
Year Born	15–24	25–34	35–44	45–54	55–64	65-74	75–84
1866–75	8.19	.08	3.63	4.44	4.65	4.12	1.01
1876–85	8.69	2.20	6.02	7.45	7.16	3.87	1.36
1886–95	9.19	4.61	8.19	10.18	8.85	4.23	1.29
1896-1905	10.00	6.43	10.38	12.58	12.30	3.98	.74
1906–15	10.18	8.58	13.17	16.48	14.71	3.13	.66
1916–25	10.68	11.84	14.70	18.38	13.91	3.48	.43
1926–35	12.43	11.87	16.97	19.78	15.41	3.23	.14
1936–45	10.73	15.03	21.53	25.48	16.45	2.98	.00
1946–55	12.18	21.63	26.96	28.48	17.03	2.93	.00
1956–65	17.68	25.67	31.05	31.85	18.06	2.75	.00
1966–75	17.99	30.27	35.74	35.87	18.95	2.59	.00

*Highlighted areas indicate actual U.S. census data. The other data are extrapolations.

Table 10	By	Total	Population
----------	----	-------	------------

		Average	Weekly Hours	Worked per Pe	erson at Age (i	n Years)	
Year Born	15–24	25–34	35–44	45–54	55-64	65-74	75–84
1866–75	20.91	20.40	24.36	23.90	22.50	14.34	3.93
1876–85	20.09	21.20	25.04	24.83	22.69	12.03	2.97
1886–95	19.26	22.13	25.53	25.69	22.19	8.43	2.17
1896–1905	18.81	22.78	26.16	26.31	22.58	6.91	1.39
1906–15	17.65	23.63	27.09	27.63	23.28	5.11	1.18
1916–25	16.75	24.92	27.00	28.27	20.68	5.15	.58
1926–35	17.47	24.73	28.03	28.16	20.75	4.06	.05
1936–45	14.15	26.16	29.89	31.47	20.14	3.11	.00
1946–55	14.05	28.80	32.62	32.75	19.09	2.44	.00
1956–65	19.64	30.83	34.49	34.24	18.56	1.53	.00
1966–75	19.13	32.86	36.65	36.27	17.84	.69	.00

Source: U.S. Department of Commerce, Bureau of the Census

Lifetime Hours Worked

Average Weekly Hours Worked Between Ages 15 and 84 by Cohorts Born Between 1896 and 1945 in the United States

	Weekly	Weekly Hours Worked per Person by				
	Tatal	(Sex			
Year Born	Population	Males	Females			
1896–1905	17.85	28.15	8.06			
1906–15	17.94	26.86	9.56			
1916–25	17.62	25.34	10.49			
1926–35	17.61	24.32	11.40			
1936—45	17.85	22.83	13.17			
% Change 1896–1945	0	-18.9	63.4			

Sources: Tables 8-10

Partial Life-Cycle Profiles for the Portion of the Population Employed . . .

			Empl	loyment-to-Pop	oulation Ratio a	at Age (in Yea	rs)	
Sex	Year Born	15–24	25–34	35–44	45–54	55-64	65-74	75–84
Males	1866–75	—	_			_		.19
Sex Males Females	1876–85	—	—	—	—	—	.49	.15
	1886—95	—	—	—	—	.80	.36	.11
	1896–1905	—	—	—	.89	.79	.31	.09
	1906–15	—	—	.91	.89	.78	.23	.08
	1916–25	—	.87	.90	.90	.68	.22	—
	1926–35	.55	.87	.90	.86	.64		—
	1936–45	.50	.87	.88	.86	_		—
	1946–55	.47	.85	.87	—	_		_
	1956–65	.61	.83	—	_	—		—
	1966–75	.60	—	—	—	—	—	—
	% Change	9.1	-4.6	-4.4	-3.4	-20.0	-55.1	-57.9
Fomalos	1866_75							03
T UITIAIUS	1876-85	_	_		_	_	10	.00
	1886-95	_	_			23	.10	.04 04
	1896-1905				32	.20	13	.04
	1906-15	_	_	34	.02	.01	11	.00
	1916-25	_	31	.01	51	40	12	
	1926-35	33	.33	48	.56	44		_
	1936-45	32	43	62	68			_
	1946-55	.38	61	73		_		_
	1956-65	.56	.69					
	1966–75	.59		—	—	—	—	—
	% Change	78.8	122.6	114.7	112.5	91.3	20.0	0

Source: U.S. Department of Commerce, Bureau of the Census

Table 13... And for the Hours Worked per Worker

			Average V	Veekly Hours V	Vorked per We	orker at Age (in Years)	
Sex	Year Born	15–24	25–34	35–44	45–54	55–64	65–74	75–84
Males	1866_75	_	_	_	_	_	_	38 62
IVIAI63	1876 85			_			42.07	2/ 12
Sex Males Females	1886_05					/13.05	42.07	32.56
	1896-1905	_	_		11 96	43.33	34.88	20 50
	1000 1000	_		45 22	13 02	42.00	33.06	28.64
	1900-15	_	AA A7	4J.22 ЛЛ ЛЛ	43.32	42.02	32.00	20.04
	1976-35	10 /0	13.87	11.TT 11.TT	43.06	41.50 //1.60	02.00	_
	1926-45	33.94	43.19	43 58	43.00			
	1946-55	32 10	42.46	43.30	-0.00			
	1956-65	34.80	43.00		_		_	
	1966-75	33.46		_	_	_	_	
		00.10						
	% Change	-17.4	-3.1	-2.3	-2.3	-5.1	-21.8	-25.8
Freedor	1000 75							00.04
Females	1866-75	_	_	_	_	_		30.34
	1870-83	_		_	_	20.10	37.30	32.37
	1000-90				 20.50	30.10	32.17	31.00 05.00
	1890-1905	_			30.30	30.04	30.30	20.08
	1900-13		20.14	30.32 25.70	30.0Z	01.70	27.04	24.01
	1910-23	07.71	30.14 DE 4E	30.79	30.00	34.73	27.94	
	1920-33	37.71	30.40	34.94	33.30	34.98	_	_
	1930-45	33.23	34.7Z	34.79	37.12	_	_	_
	1940-00	31.48	33.47	30.91	_		_	
	1900-00	31.37	37.14	_	_		_	
	1900-70	30.53		_	—		—	
	% Change	-19.0	-2.6	-3.7	-3.8	-8.2	-25.6	-32.6

Source: U.S. Department of Commerce, Bureau of the Census

Table 14 Possible Factors Behind Work Reallocations

In the United States, 1950–90

		Average		% of Population in Each Marital Status Category				
	Index of Real	Monthly	Total	Married W	ith Spouse			
Year	(1982=100)	Social Security Benefit (1990 \$)	Fertility Rate**	Present	Absent	Single	Widowed	Divorced
1950	49.4	238	3,337	64.45	4.01	21.10	8.25	2.24
1960	68.8	327	3,449	65.45	3.87	20.12	8.01	2.55
1970	91.3	397	2,480	61.33	3.86	23.23	8.17	3.41
1980	99.5	541	1,840	57.95	2.25	25.98	7.62	6.20
1990	103.8	603	2,081	53.56	4.29	26.42	7.37	8.35
% Change 1950–90	110.1	153.4	-37.6	-16.9	7.0	25.2	-10.7	272.8

* This is an index of hourly compensation in the business sector, deflated by the consumer price index for all urban consumers.

** The *fertility rate* for any year is the number of births that 1,000 females would have in their lifetime if, at each age, they experienced that year's birthrate.

Sources: See Appendix A.

A Decomposition of Average Weekly Hours Worked per Person

		Hours per Person Recalculated With				
Year	Actual Hours per Person	1950 Weights	1950 Hours			
1950	22.03	22.03	22.03			
1960	20.97	21.40	21.57			
1970	20.55	21.93	20.92			
1980	22.00	23.30	20.99			
1990	23.62	25.00	21.50			
% Change 1950–90	7.2	13.5	-2.4			

Source of basic data: See Appendix A.

Introduction to General Equilibrium II: Firms

Jesús Fernández-Villaverde University of Pennsylvania

What is a firm?

• A technology:

$$y = F(k,l) = Ak^{\alpha}l^{1-\alpha}$$

for $\alpha \in (0, 1)$.

- Operational definition.
- We are in a static world: we will assume k to be constant.

Properties of the Technology I

From lectures in growth we know that:

- 1. Constant returns to scale.
- 2. Inputs are essential.
- 3. Marginal productivities are positive and decreasing.
- 4. Inada Conditions.

Problem of the Firm I

• Wants to maximize profits given r and w

(we are taking the consumption good as the numeraire!):

$$\pi = Ak^{\alpha}l^{1-\alpha} - rk - wl$$

• We take first order conditions:

$$\alpha A k^{\alpha - 1} l^{1 - \alpha} = r \tag{1}$$

$$(1-\alpha)Ak^{\alpha}l^{-\alpha} = w$$
 (2)

• We want to solve for k and l.

Problem of the Firm II

• We begin dividing (1) by (2):

$$\frac{\alpha A k^{\alpha - 1} l^{1 - \alpha}}{(1 - \alpha) A k^{\alpha} l^{-\alpha}} = \frac{r}{w}$$

or

$$\frac{\alpha}{1-\alpha}\frac{l}{k} = \frac{r}{w}$$

or

$$k = \frac{w}{r} \frac{\alpha}{1 - \alpha} l \tag{3}$$

Problem of the Firm III

• but if we substitute (3) in (2):

$$(1 - \alpha) A k^{\alpha} l^{-\alpha} = w$$

$$(1 - \alpha) A \left(\frac{w}{r} \frac{\alpha}{1 - \alpha} l\right)^{\alpha} l^{-\alpha} = w$$

$$(1 - \alpha) A \left(\frac{w}{r} \frac{\alpha}{1 - \alpha}\right)^{\alpha} = w$$

l disappears!

- You can check that the same happens with k if we substitute (3) in (1).
- What is wrong?

Problem of the Firm IV

- We have constant returns to scale.
- The size of the firm is indeterminate: we can have just one!
- To see that remember that profits are always zero if firm maximizes:

$$\pi = Ak^{\alpha}l^{1-\alpha} - rk - wl$$

= $Ak^{\alpha}l^{1-\alpha} - \alpha Ak^{\alpha-1}l^{1-\alpha}k - (1-\alpha)Ak^{\alpha}l^{-\alpha}l = 0$

• So the firms really only pick the labor-capital ratio given relative prices:

$$\frac{l}{k} = \frac{\alpha}{1 - \alpha} \frac{r}{w}$$

Problem of the Firm V

• In equilibrium, markets clear so:

$$l(r, w) = l^{s}(w)$$

$$k(r, w) = k^{f}$$

$$r = \alpha A k(r, w)^{\alpha - 1} l(r, w)^{1 - \alpha}$$

$$w = (1 - \alpha) A k(r, w)^{\alpha} l(r, w)^{-\alpha}$$

• We have a system of four equations in four unknowns.

What are we missing?

- A lot.
- Wages are a lot of time different from marginal productivites.
- Reasons for that:
 - 1. Efficiency Wages: Shapiro-Stiglitz (1984).
 - 2. Bargaining: Nash (1950).
 - 3. Monopoly rents: Holmes and Schmitz (2001).
 - 4. Sticky wages: Taylor (1980).

Table 7 Annual Turnover Rates, 1913–15

	1913	1914	1915
Average force employed	13,623	12,115	18,028
Total leaving	50,448	6,508	2,931
Turnover rate (%)	370	54	16
Resignations	39,575	5,199	2,871
Layoffs	2,383	385	23
Discharges	8,490	926	27

SOURCE,—Slichter (1921, p. 244).

Table 8 Absenteeism at Ford

	Total Workers	Number Absent	Percent Absent
October 6, 1913	12,548	1,250	10
October 6, 1914	12,645	311	2.5

SOURCE.—Abell (1915, p. 37).

Dock Workers Were Paid More Than Most Other Workers in New Orleans . . .

Hourly Wage Rates (Cents) in Union Agreements and in Manufacturing Establishments, 1904–5



Introduction to General Equilibrium III: Government

Jesús Fernández-Villaverde University of Pennsylvania What is the Government?

• Operational definition: takes taxes and spends them.

$$G = T$$
$$T = \tau^l w$$

- No public debt.
- Why?

New Problem of the Household

• Problem for Robinson is now

$$egin{aligned} \max u\left(c,1-l
ight)\ s.t. \ c &= \left(1- au^l
ight)wl+rk \end{aligned}$$

- Why the new last term?
- FOC:

$$-\frac{u_l}{u_c} = \left(1 - \tau^l\right) w$$

A Parametric Example

•
$$u(c,l) = \log c + \gamma \log (1-l)$$

• FOC+Budget constraint:

$$egin{aligned} &\gammarac{c^{*}}{1-l^{*}}=\left(1- au^{l}
ight)w\ &c^{*}=\left(1- au^{l}
ight)wl^{*}+rk \end{aligned}$$

• Then:

$$l^* = \frac{\left(1 - \tau^l\right)w - rk}{\left(1 + \gamma\right)\left(1 - \tau^l\right)w}$$

- Taxes affect labor supply!!!
- How important is the effect?
- Two Examples:
 - 1. Tax reform of 1986.
 - 2. Why do Americans work so much more than Europeans?

Table 1Average and Marginal Federal Tax Rates (Percent)at Alternative Levels of Family Income, 1980–88°

		Combined tax rate ^b						
	Median income	One-half Median income		Median income		Twice Median income		
Year	(dollars)	Avg.	Mrg.	Avg.	Mrg.	Avg.	Mrg.	
1980	24,332	18.3	30.3	23.7	36.3	24.8	43.0	
1982	27,619	19.9	29.4	24.5	38.4	25.9	39.0	
1984	31,097	19.9	27.4	23.7	35.4	24.8	38.0	
1986	34,716	20.9	28.3	24.8	36.3	25.7	38.0	
1988	37,482°	19.8	30.0	24.2	30.0	23.9	28.0	

Source: U.S. Department of the Treasury.

^aTax rates for a four-person family. It is assumed that each family contains only a single-earner and that all taxable income consists of wage or self-employment earnings.

^bCombined tax rate includes federal income tax and FICA payroll tax.

^cEstimate.

Change in Annual Hours of Work by Income Quintile, 1989 (Percent^a)

Demographic	Bottom	Second	Middle	Fourth	Тор	All
group	quintile	quintile	quintile	quintile	quintile	quintiles
Men						
Aged 25-64	31.0*	3.6	4.1*	2.5*	3.2*	6.0*
Aged 25-44	10.5*	1.5	2.5	2.7	1.8	3.3*
Aged 45-64	96.4*	6.3	4.0*	3.5	4.3*	9.2*
Married,						
aged 25–64	19.4*	2.2	2.3	3.9*	3.6*	5.1*
Women						
Aged 25-64	16.7*	-6.9*	6.4	10.5*	11.8*	5.4
Aged 25-44	-14.0*	-11.1*	4.5	4.8	2.2	-1.6
Aged 45–64 Married,	141.4*	0.1	6.3	19.3*	27.4*	18.1*
aged 25-64	20.9*	1.2	4.9	3.9	13.8*	7.1*

Output, Labor Supply, and Productivity

In Selected Countries in 1993-96 and 1970-74

		Relative to United States (U.S. = 100)				
Period	Country	Output per Person*	Hours Worked per Person*	Output per Hour Worked		
1993–96	Germany	74	75	99		
	France	74	68	110		
	Italy	5/	64	90		
	Canada	79	88	89		
	United Kingdom	67	88	76		
	Japan	78	104	74		
	United States	100	100	100		
1970–74	Germany	75	105	72		
	France	77	105	74		
	Italy	53	82	65		
	Canada	86	94	91		
	United Kingdom	68	110	62		
	Japan	62	127	49		
	United States	100	100	100		

*These data are for persons aged 15–64. Sources: See Appendix.

Actual and Predicted Labor Supply

In Selected Countries in 1993-96 and 1970-74

		Labor Supply*		Differences (Predicted	Prediction Factors	
						Consumption/
Period	Country	Actual	Predicted	Less Actual)	Tax Rate $ au$	Output (c/y)
1993–96	Germany	19.3	19.5	.2	.59	.74
	France	17.5	19.5	2.0	.59	.74
	Italy	16.5	18.8	2.3	.64	.69
	Canada	22.9	21.3	-1.6	.52	.77
	United Kingdom	22.8	22.8	0	.44	.83
	Japan	27.0	29.0	2.0	.37	.68
	United States	25.9	24.6	-1.3	.40	.81
1970–74	Germany	24.6	24.6	0	.52	.66
	France	24.4	25.4	1.0	.49	.66
	Italy	19.2	28.3	9.1	.41	.66
	Canada	22.2	25.6	3.4	.44	.72
	United Kingdom	25.9	24.0	-1.9	.45	.77
	Japan	29.8	35.8	6.0	.25	.60
	United States	23.5	26.4	2.9	.40	.74

*Labor supply is measured in hours worked per person aged 15–64 per week. Sources: See Appendix. How Does the Government Behave?

- Economist use their tools to understand how governments behave.
- Political Economics (different than Political Economy).
- Elements:
 - 1. Rational Agents.
 - 2. Optimization.
 - 3. Equilibrium outcomes.

Overall Questions

- How do we explain differences and similarities in observed economic policy over time?
 - 1. Why do countries limit free trade?
 - 2. Why do countries levy inefficient taxes?
- Can we predict the effects of changing political arrangements:
 - 1. What would happen if we abandon the electoral college?
 - 2. What would happen if we introduce proportional representation?

Some Basic Results

- Arrow's Impossibility Theorem.
- Median Voter's Theorem.
- Probabilistic Voting.

Political Economics in Macro

- How taxes are fixed?
- Time-Consistency Problems.
- Redistribution.

General Equilibrium

Jesús Fernández-Villaverde University of Pennsylvania Now we are going to put everything together

- We have a household that decides how much to work, *l*, and how much to consume, *c* to maximize utility. It takes as given the wage, *w* and the interest rate *r*.
- We have a firm that decides how much to produce, y and how much capital, k, and labor, l to hire. It takes as given the wage, w and the interest rate r.
- We have a government (maybe not) that raises taxes, T, and spends G.
- We are in a static world: we will assume k to be constant.

Allocations, Feasible Allocations, Government Policy and Price Systems

- An allocation is a set of value for production, y, work, l, capital, k and consumption c.
- A feasible allocation is an allocation that is possible:

$$c + G = y = Ak^{\alpha}l^{1-\alpha}$$

- A government policy is a set of taxes τ^l and government spending G.
- A price system is a set of prices w and r.

A Competitive Equilibrium

A Competitive Equilibrium is an allocation $\{y, l, k, c\}$, a price system $\{w, r\}$ and a government policy $\{\tau^l, T\}$ such that:

- 1. Given the price system and the government policy, households choose l and c to maximize their utility.
- 2. Given the price system and the government policy, firms maximize profits, i.e. they $\alpha Ak^{\alpha-1}l^{1-\alpha} = r$ and $(1-\alpha)Ak^{\alpha}l^{-\alpha} = w$.
- 3. Markets clear:

$$c + G = y = Ak^{\alpha}l^{1-\alpha}$$
Existence of an Equilibrium

- Does it exist an equilibrium?
- Tough problem.
- Shown formally by Arrow (Nobel Prize Winner 1972) and Debreu (Nobel Prize Winner 1983).
- Uniqueness?

What do we get out of the concept of an Equilibrium?

- Consistency: we are sure that everyone is doing things that are compatible. Economics is only social science that is fully aware of this big issue.
- We talk about Competitive Equilibrium but there are other concepts of equilibrium: Ramsey Equilibrium, Nash Equilibrium, etc...
- It is a prediction about the behavior of the model. Theory CAN and SHOULD be tested against the data. Some theories are thrown away.

Application I: WWII and the Increase in G

- During WWII government spending to finance the war effort increased to levels unseen previously in the US.
- What are the predictions of the model for this increase in spending?
- The assumption that government spending is a pure loss of output arguably makes sense here. Pure spending/diversion of resources in short run. Positive effects more long-run and harder to measure.

Figure 1.06 U.S. Federal government spending and tax collections, 1869-1999



Abel/Bernanke, Macroeconomics, © 2001 Addison Wesley Longman, Inc. All rights reserved

- Preferences $u(c, l) = \log c + \gamma (1 l)$.
- Cobb-Douglas technology.
- Production possibilities (goods market):

$$c = y - G = Ak^{\alpha}l^{1-\alpha} - G$$

• To simplify
$$g = G/y$$
. So:

$$c = (1 - g)Ak^{\alpha}l^{1 - \alpha}.$$

• Household utility maximization:

$$\frac{u_l(c,l)}{u_c(c,l)} = \frac{\gamma}{1/c} = w \Rightarrow \gamma c = w$$

Choice under Uncertainty

Jesús Fernández-Villaverde University of Pennsylvania

- In the previous chapter, we studied intertemporal choices.
- However, there is a second important dimension of choice theory: uncertainty.
- Life is also full of uncertainty: Will it rain tomorrow? Who will win the next election? What will the stock market do next period?
- How do economist think about uncertainty?
- Von Neumann, Morgenstern, Debreu, Arrow, and Savage.

Simple Example I

- Flip a coin.
- Two events: $s_1 = \{Heads\}$ and $s_2 = \{Tails\}$.
- Set of possible events $S = \{s_1, s_2\}$.
- Heads with probability $\pi(s_1) = p$, tails with $\pi(s_2) = 1 p$.
- If heads, consumption is $c(s_1)$, if tails consumption is $c(s_2)$.
- Utility function is $u(c(s_i))$ for $i = \{1, 2\}$.

Simple Example II

• Under certain technical conditions, there is a Expected or Von Neumann-Morgenstern utility function:

$$U(c(s_i)) = Eu(c(s_i)) = \pi(s_1)u(c(s_1)) + \pi(s_2)u(c(s_2))$$

- Linear in probabilities.
- Should you gamble? Role of the shape of the utility function.
- Risk-neutrality, risk-loving, risk-aversion.

More General Case

• We have *n* different events:

$$Eu(c(s_i)) = \sum_{i=1}^n \pi(s_i) u(c(s_i))$$

- Note that now consumption is a function mapping events into quantities.
- $\pi(s_i)$ can be objective or subjective.

Time and Uncertainty

- We can also add a time dimension.
- An event history $s^t = (s_0, s_1, ..., s_t)$.

• Then
$$s^t \in S^t = S \times S \times ... \times S$$

- Probabilities $\pi\left(s^{t}\right)$.
- Utility function:

$$\sum_{t=0}^{\infty} \sum_{s^{t} \in S^{t}} \beta^{t} \pi \left(s^{t} \right) u \left(c \left(s^{t} \right) \right)$$

A Simpler Example: a Two Period World

- First period, only one event s_0 with $\pi(s^0) = 1$.
- Second period, n events, with probabilities $\pi\left(s^{1}
 ight)$.
- Then, utility is:

$$u(c(s_0)) + \beta \sum_{s^1 \in S^1} \pi(s^1) u(c(s^1))$$

Markets

- Goods are also indexed by events.
- One good: endowment process $e\left(s^{t}
 ight)$.
- We introduce a set of one-period contingent securities $b(s^t, s_{t+1})$ for all $s^t \in S^t$ and s_{t+1} .
- Interpretation: $b(s^t, s_{t+1})$ pays one unit of good if and only if the current history is s^t and tomorrow's event is s_{t+1} .
- What are these securities in the real world?

Price of Securities

• Price of
$$b(s^t, s_{t+1})$$
: $q(s^t, s_{t+1})$.

- Quantity of $b(s^t, s_{t+1})$: $a(s^t, s_{t+1})$.
- Budget constraint:

$$c\left(s^{t}\right) + \sum_{s_{t+1} \in S} q\left(s^{t}, s_{t+1}\right) a\left(s^{t}, s_{t+1}\right) = e\left(s^{t}\right) + a\left(s^{t-1}, s_{t}\right)$$

• $\left\{q\left(s^{t}, s_{t+1}\right)a\left(s^{t}, s_{t+1}\right)\right\}_{s_{t+1} \in S}$ is the portfolio of the household.

Equilibrium

A Sequential Markets equilibrium is an allocation $\left\{c^*\left(s^t\right), a^*\left(s^t, s_{t+1}\right)\right\}_{t=0, s^t \in S^t}^{\infty}$ and prices $\left\{q^*\left(s^t, s_{t+1}\right)\right\}_{t=0, s^t \in S^t}^{\infty}$ such that:

1. Given prices, the allocation solves:

$$\max \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi \left(s^t \right) u \left(c \left(s^t \right) \right)$$

s.t. $c \left(s^t \right) + \sum_{s_{t+1} \in S} q \left(s^t, s_{t+1} \right) a \left(s^t, s_{t+1} \right) = e \left(s^t \right) + a \left(s^{t-1}, s_t \right)$

2. Markets clear
$$c\left(s^{t}
ight)=e\left(s^{t}
ight).$$

Characterization of the Equilibrium

- 1. We can prove existence of a Sequential Markets equilibrium.
- 2. There are other, equivalent, market structures.
- 3. The two fundamental welfare theorems hold.
- 4. Note, however, how we need markets for all goods under all possible events!

Is Our Representation of Uncertainty a Good One?

- Uncertainty aversion: Ellsberg's paradox.
- Different models of the world.
- Robustness of our decisions.
- Unawareness.

Putting Theory to Work: Asset Pricing

- We revisit our two periods example.
- Preferences:

$$u(c(s_0)) + \beta \sum_{s^1 \in S^1} \pi(s^1) u(c(s^1))$$

• Budget constraints:

$$c(s_0) + \sum_{s_1 \in S} q(s^0, s_1) a(s^0, s_1) = e(s^0)$$
$$c(s_1) = e(s_1) + a(s^0, s_1) \text{ for all } s_1 \in S$$

Problem of the Household

• We write the Lagrangian:

$$\begin{aligned} u(c(s_0)) &+ \beta \sum_{s^1 \in S^1} \pi(s^1) u(c(s^1)) \\ &+ \lambda(s_0) \left(e(s^0) - c(s_0) + \sum_{s_1 \in S} q(s^0, s_1) a(s^0, s_1) \right) \\ &+ \sum_{s^1 \in S^1} \lambda(s^1) \left(e(s_1) + a(s^0, s_1) - c(s_1) \right) \end{aligned}$$

• We take first order conditions with respect to $c(s_0)$, $c(s_1)$, and $a(s^0, s_1)$.

Solving the Problem

• FOCs

$$u'(c(s_0)) = \lambda(s_0)$$

$$\beta \pi(s^1) u'(c(s^1)) = \lambda(s^1) \text{ for all } s_1 \in S$$

$$\lambda(s_0) q(s^0, s_1) = \lambda(s^1) \text{ for all } s_1 \in S$$

• Then:

$$q\left(s^{0}, s_{1}\right) = \pi\left(s^{1}\right)\beta\frac{u'\left(c\left(s^{1}\right)\right)}{u'\left(c\left(s_{0}\right)\right)}$$

• Fundamental equation of Asset Pricing.

The Stochastic Discount Factor

• Stochastic discount factor (or pricing kernel):

$$m\left(s^{1}\right) = \beta \frac{u'\left(c\left(s^{1}\right)\right)}{u'\left(c\left(s_{0}\right)\right)}$$

• Note that:

$$Em\left(s^{1}\right) = \sum_{s^{1} \in S^{1}} \pi\left(s^{1}\right) m\left(s^{1}\right) = \beta \sum_{s^{1} \in S^{1}} \pi\left(s^{1}\right) \frac{u'\left(c\left(s^{1}\right)\right)}{u'\left(c\left(s_{0}\right)\right)}$$

Pricing Redundant Securities

- With our framework we can price any security.
- For example, an uncontingent bond:

$$q\left(s^{0}\right) = \sum_{s^{1} \in S^{1}} q\left(s^{0}, s_{1}\right) = \beta \sum_{s^{1} \in S^{1}} \pi\left(s^{1}\right) \frac{u'\left(c\left(s^{1}\right)\right)}{u'\left(c\left(s_{0}\right)\right)}$$

• Generalize to very general financial contracts:

$$p\left(s^{0}, s_{1}\right) = \beta \pi\left(s^{1}\right) x\left(s^{1}\right) \frac{u'\left(c\left(s^{1}\right)\right)}{u'\left(c\left(s_{0}\right)\right)}$$

Risk-Free Rate

• Note that:

$$q\left(s^{0}\right) = Em\left(s^{1}\right)$$

• Then, the risk-free rate:

$$R^{f}\left(s^{1}\right)=\frac{1}{q\left(s^{0}\right)}=\frac{1}{Em\left(s^{1}\right)}$$
 or $ER^{f}\left(s^{1}\right)m\left(s^{1}\right)=1.$

Example of Financial Contracts

- 1. Stock: buy at price $p(s_0)$, delivers a dividend $d(s^1)$, sells at $p(s_1)$ $p(s^0) = \beta \sum_{s^1 \in S^1} \pi(s^1) \left(p(s_1) + d(s^1) \right) \frac{u'(c(s^1))}{u'(c(s_0))}$
- 2. Call option: buy at price $o(s_0)$ the right to buy an asset at price K_1 . Price of asset $J(s^1)$

$$o\left(s^{0}\right) = \beta \sum_{s^{1} \in S^{1}} \pi\left(s^{1}\right) \max\left(\left(J\left(s^{1}\right) - K_{1}\right) \frac{u'\left(c\left(s^{1}\right)\right)}{u'\left(c\left(s_{0}\right)\right)}, 0\right)$$

Non Arbitrage

- A lot of financial contracts are equivalent.
- From previous results, we derive a powerful idea: absence of arbitrage.
- Empirical evidence regarding non arbitrage.
- Possible limitations to non arbitrage conditions.
- Related idea: spanning of non-traded assets.

Simple Example

•
$$u(c) = \log c, \ \beta = 0.99$$

•
$$e(s^0) = 1$$
, $e(s_1 = high) = 1.1$, $e(s_1 = low) = 0.9$.

•
$$\pi$$
 ($s_1 = high$) = 0.5, π ($s_2 = low$) = 0.5.

• Equilibrium prices:

$$q(s^{0}, s_{1} = high) = 0.99 * 0.5 * \frac{\frac{1}{1.1}}{\frac{1}{1}} = 0.45$$
$$q(s^{0}, s_{1} = low) = 0.99 * 0.5 * \frac{\frac{1}{0.9}}{\frac{1}{1}} = 0.55$$
$$q(s^{0}) = 0.45 + 0.55 = 1$$

• Note how the price is different from a naive adjustment by expectation and discounting:

$$\begin{array}{rcl} q_{naive}\left(s^{0},s_{1}=high\right) &=& 0.99*0.5*1=0.495\\ q_{naive}\left(s^{0},s_{1}=low\right) &=& 0.99*0.5*1=0.495\\ q_{naive}\left(s^{0}\right) &=& 0.495+0.495=0.99 \end{array}$$

• Why is
$$q\left(s^{0}, s_{1} = high\right) < q\left(s^{0}, s_{1} = low\right)$$
?

- Two forces:
 - 1. Discounting β .
 - 2. Ratio of marginal utilities: $\frac{u'(c(s^1))}{u'(c(s_0))}$.
- Covariance is key.

Risk Correction I

We recall three facts:

1.
$$q(s^{0}) = Em(s^{1})$$
.
2. $p(s^{0}, s_{1}) = Em(s^{1})x(s^{1})$.

3.
$$cov(xy) = E(xy) - E(x)E(y)$$
.

Risk Correction II

Then:

$$p\left(s^{0}, s_{1}\right) = Em\left(s^{1}\right) Ex\left(s^{1}\right) + cov\left(m\left(s^{1}\right)x\left(s^{1}\right)\right)$$

or

$$p(s^{0}, s_{1}) = \frac{Ex(s^{1})}{R^{f}(s^{1})} + cov(m(s^{1})x(s^{1}))$$

$$= \frac{Ex(s^{1})}{R^{f}(s^{1})} + cov\left(\beta\frac{u'(c(s^{1}))}{u'(c(s_{0}))}x(s^{1})\right)$$

$$= \frac{Ex(s^{1})}{R^{f}(s^{1})} + \beta\frac{cov(u'(c(s^{1}))x(s^{1}))}{u'(c(s_{0}))}$$

Risk Correction III

Now we can see how if:

1. If
$$cov\left(m\left(s^{1}\right)x\left(s^{1}\right)\right) = 0 \Rightarrow p\left(s^{0}, s_{1}\right) = \frac{Ex(s^{1})}{R^{f}(s^{1})}$$
, not adjustment for risk.

- 2. If $cov\left(m\left(s^{1}\right)x\left(s^{1}\right)\right) > 0 \Rightarrow p\left(s^{0}, s_{1}\right) > \frac{Ex(s^{1})}{R^{f}(s^{1})}$, premium for risk (insurance).
- 3. If $cov\left(m\left(s^{1}\right)x\left(s^{1}\right)\right) < 0 \Rightarrow p\left(s^{0}, s_{1}\right) < \frac{Ex(s^{1})}{R^{f}(s^{1})}$, discount for risk (speculation).

Utility Function and the Risk Premium

- We see how risk depends of marginal utilities:
 - 1. Risk-neutrality: if utility function is linear, you do not care about var(c).
 - 2. Risk-loving: if utility function is convex you want to increase var(c).
 - 3. Risk-averse: if utility function is concave you want to reduce var(c).
- It is plausible that household are (basically) risk-averse.

CRRA Utility Functions I

- Market price of risk has been roughly constant over the last two centuries.
- This observation suggests that risk aversion should be relatively constant over the wealth levels.
- This is delivered by constant relative risk aversion utility function:

$$\frac{c^{1-\sigma}}{1-\sigma}$$

• Note that when $\sigma = 1$, the function is $\log c_t$ (you need to take limits and apply L'Hôpital's rule).

CRRA Utility Functions II

- σ plays a dual role controlling risk-aversion and intertemporal substitution.
- Coefficient of Relative Risk-aversion:

$$-\frac{u''(c)}{u'(c)}c = \sigma$$

• Elasticity of Intertemporal Substitution:

$$-\frac{u(c_2)/u(c_1)}{c_2/c_1}\frac{d(c_2/c_1)}{d(u(c_2)/u(c_1))} = \frac{1}{\sigma}$$

• Advantages and disadvantages.

Size of σ

- Most evidence suggests that σ is low, between 1 and 3. At most 10.
- Types of evidence:
 - 1. Questionnaires.
 - 2. Experiments.
 - 3. Econometric estimates from observed behavior.
- A powerful arguments from international comparisons.

A Small Detour

- Note that all we have said can be applied to the trivial case without uncertainty.
- In that situation, there is only one security, a bond, with price:

$$q_1 = \beta \frac{u'(c_1)}{u'(c_0)}$$

• And the interest rate is:

$$R_1 = \frac{1}{q_1} = \frac{1}{\beta} \frac{u'(c_0)}{u'(c_1)}$$
Pricing Securities in the Solow Model

- Assume that the utility is CRRA and that we are in a BGP with $\gamma = g$.
- Then:

$$R = \frac{1}{\beta} \left(\frac{c}{(1+g)c} \right)^{-\sigma} = \frac{(1+g)^{\sigma}}{\beta}$$

- Or in logs: $r \simeq 1 + \sigma g \beta$, i.e., the real interest rate depends on the rate of growth of technology, the readiness of households to substitute intertemporally, and on the discount factor.
- Also, σ must be low to reconcile small international differences in the interest rate and big differences in g.

Mean-Variance Frontier

• The pricing condition for a contract i with price 1 and yield $R^{i}\left(s^{1}\right)$ is:

$$1 = Em\left(s^1\right)R^i\left(s^1\right)$$

• Then:

$$1 = Em\left(s^{1}\right)ER^{i}\left(s^{1}\right) + cov\left(m\left(s^{1}\right)R^{i}\left(s^{1}\right)\right)$$

or:

$$1 = Em\left(s^{1}\right)ER^{i}\left(s^{1}\right) + \frac{cov\left(m\left(s^{1}\right)R^{i}\left(s^{1}\right)\right)}{sd\left(m\left(s^{1}\right)\right)sd\left(R^{i}\left(s^{1}\right)\right)}sd\left(m\left(s^{1}\right)\right)sd\left(R^{i}\left(s^{1}\right)\right)$$

• The coefficient of correlation between to random variables is:

$$\rho_{m,R_{i}} = \frac{cov\left(m\left(s^{1}\right)R^{i}\left(s^{1}\right)\right)}{sd\left(m\left(s^{1}\right)\right)sd\left(R^{i}\left(s^{1}\right)\right)}$$

• Then, we have:

$$1 = Em\left(s^{1}\right)ER^{i}\left(s^{1}\right) + \rho_{m,R_{i}}sd\left(m\left(s^{1}\right)\right)sd\left(R^{i}\left(s^{1}\right)\right)$$

• Or:

$$ER^{i}\left(s^{1}\right) = R^{f} - \rho_{m,R_{i}} \frac{sd\left(m\left(s^{1}\right)\right)}{Em\left(s^{1}\right)} sd\left(R^{i}\left(s^{1}\right)\right)$$

• Since
$$ho_{m,R_i} \in [-1,1]$$
 :

$$\left| ER^{i}\left(s^{1}\right) - R^{f} \right| \leq \frac{sd\left(m\left(s^{1}\right) \right)}{Em\left(s^{1}\right)} sd\left(R^{i}\left(s^{1}\right) \right)$$

- This relation is known as the *Mean-Variance frontier*.
- Relation between mean and variance of an asset: "How much return can you get for a given level of variance?"
- $\frac{sd(m(s^1))}{Em(s^1)}$ can be interpreted as the market price of risk.
- Any investor would hold assets within the mean-variance region.

The Sharpe Ratio

• Another way to represent the Mean-Variance frontier is:

$$\left|\frac{ER^{i}\left(s^{1}\right)-R^{f}}{sd\left(R^{i}\left(s^{1}\right)\right)}\right| \leq \frac{sd\left(m\left(s^{1}\right)\right)}{Em\left(s^{1}\right)}$$

- This relation is known as the *Sharpe Ratio*.
- It answers the question: "How much more mean return can I get by shouldering a bit more volatility in my portfolio?"

The Equity Premium Puzzle I

• Assume a CRRA utility function.

• Then,
$$m\left(s^{1}\right) = \left(\frac{c\left(s^{1}\right)}{c\left(s^{0}\right)}\right)^{-\sigma}$$

• A good approximation of $\frac{sd(m(s^1))}{Em(s^1)}$ is (forget about the algebra details):

$$\sigma sd\left(\Delta\ln c\left(s^{t}
ight)
ight)$$

The Equity Premium Puzzle II

- Let us go to the data and think about the stock market (i.e. Rⁱ (s¹) is the yield of an index) versus the risk free asset (the U.S. treasury bill).
- Average return from equities in XXth century: 6.7%. From bills 0.9%.
- Standard deviation of equities: 16%.
- Standard deviation of $\Delta \ln c \left(s^t\right)$: 1%.

The Equity Premium Puzzle III

• Then:

$$\left|rac{6.7\%-0.9\%}{16\%}
ight|=0.36\leq\sigma1\%$$

that implies a σ of at least 36!

- But we argued before that σ is at most 10.
- This observation is known as the Equity Premium Puzzle (Mehra and Prescott, 1985)

Answers to Equity Premium Puzzle

- 1. Returns from the market have been odd. For example, if return from bills had been around 4% and returns from equity 5%, you would only need a σ of 6.25. Some evidence related with the impact of inflation.
- 2. There were important distortions on the market. For example regulations and taxes.
- 3. People is an order of magnitude more risk averse that we think. Epstein-Zin preferences.
- 4. The model is deeply wrong.

Random Walks I

- Can we predict the market?
- Remember that the price of a share was

$$p\left(s^{0}\right) = \beta \sum_{s^{1} \in S^{1}} \pi\left(s^{1}\right) \left(p\left(s_{1}\right) + d\left(s^{1}\right)\right) \frac{u'\left(c\left(s^{1}\right)\right)}{u'\left(c\left(s_{0}\right)\right)}$$

or:

$$p(s^{0}) = \beta E(p(s_{1}) + d(s^{1})) \frac{u'(c(s^{1}))}{u'(c(s_{0}))}$$

Random Walks II

• Now, suppose that we are thinking about a short period of time, i.e. $\beta \approx 1$ and that firms do not distribute dividends (not a bad approximation because of tax reasons):

$$p(s^{0}) = Ep(s_{1}) \frac{u'((c(s^{1})))}{u'(c(s_{0}))}$$

• If in addition $\frac{u'(c(s^1))}{u'(c(s_0))}$ does not change (either because utility is linear or because of low volatility of consumption):

$$p(s^{0}) = Ep(s_{1}) = p(s^{0}) + \varepsilon_{0}$$

Random Walks III

- $p(s^{0}) = p(s^{0}) + \varepsilon_{0}$ is called a Random Walk.
- The best forecast of the price of a share tomorrow is today's price.
- Can we forecast future movements of the market? No!
- We can generalize the idea to other assets.
- Empirical evidence.

Main Ideas of Asset Pricing

- 1. Non-arbitrage.
- 2. Risk-free rate is $r \simeq 1 + \sigma g \beta$.
- 3. Risk is not important by itself: the key is covariance.
- 4. Mean-Variance frontier.
- 5. Equity Premium Puzzle.
- 6. Random walk of asset prices.

• Firm profit maximization:

$$(1-\alpha)Ak^{\alpha}l^{-\alpha} = w$$

• Equate and impose goods market clearing:

$$egin{aligned} &\gamma c = \left(1-lpha
ight)Ak^{lpha}l^{-lpha} \ &\Rightarrow \gamma(1-g)Ak^{lpha}l^{1-lpha} = \left(1-lpha
ight)Ak^{lpha}l^{-lpha} \ &\Rightarrow l = rac{1-lpha}{\gamma(1-g)} \end{aligned}$$

• Government spending has a pure income effect here (since financed by lump sum taxes). Increases labor supply.

• Solve for rest of allocation:

$$y = Ak^{\alpha} \left[\frac{1-\alpha}{\gamma(1-g)} \right]^{1-\alpha}$$
$$c = (1-g)y = (1-g)^{\alpha}k^{\alpha} \left[\frac{1-\alpha}{\gamma} \right]^{1-\alpha}$$

• Output increases with g, consumption decreases.

• Solve for wages and interest rates:

$$w = (1 - lpha) A k^{lpha} \left[rac{1 - lpha}{\gamma(1 - g)}
ight]^{-lpha}$$

 $r = lpha A k^{lpha - 1} \left[rac{1 - lpha}{\gamma(1 - g)}
ight]^{1 - lpha}$

• Wages decrease with g, interest rates increase.

Summing Up

- Following increase in g = G/Y, the model predicts an increase in (y, l, r), decrease in (c, w).
- Private consumption spending is "crowded out" by increased government spending.
- Output increases but loss of welfare as both c, 1 l fall.
- These predictions match US experience of WWII.

Figure 5.7 GDP, Consumption, and Government Expenditures



Figure 15.04 Deficits and primary deficits: Federal, state, and local, 1940-1998



Abel/Bernanke, Macroeconomics, © 2001 Addison Wesley Longman, Inc. All rights reserved

Figure 1.02 Average labor productivity in the United States, 1900-1998



Abel/Bernanke, Macroeconomics, © 2001 Addison Wesley Longman, Inc. All rights reserved

What Does This Analysis Miss?

- Government debt. A large fraction of the wartime spending was financed by government debt. Deficit/GDP ratio hit 24% by 1944.
- Debt allows for intertemporal substitution of resources and smoothing burden of taxation. If needed to increase (distortionary) taxes to finance full war spending, production would have been less.
- Increased productivity. Wartime mobilization of production increased labor productivity dramatically.
- Led to larger increase in production than our model suggests.

Application II: Skill Biased Technical Change and Inequality

- Large literature documenting increase in income and wealth inequality in the US.
- Started in the 1970s and continues today.
- At same time, has been a large increase in the returns to education:
 - 1. Average wages of college graduates from increased by 60% for males and 90% for females from 1963 to 2002.
 - 2. Average wages of high school graduates only increased by 20% for males and 50% for females over same period.

Figure 4 Men's Earnings by Quantiles

Index of Real Wages of Full-Time Full-Year Men Ages 22-65 by Specific Percentiles



Figure 5 Women's Earnings by Quantiles

Index of Real Wages of Full-Time Full-Year Women Ages 22-65 by Specific Precentiles



Figure 6 Men's Earnings by Education

Index of Mean of Real Wages of Full-Time Full-Year Men Ages 22-65 by Education Group



Figure 7 Women's Earnings by Education

Index of Mean of Real Wages of Full-Time Full-Year Women Ages 22-65 by Education Group

Index, 1963=100



Main explanation:

- Skill-biased technical change.
- Skilled and unskilled labor are effectively different labor markets.
- Productivity changes have increased the relative demand for skilled labor.

Analysis of Skill-Biased Change

- Extend the previous to two types of households: skilled and unskilled.
- Households: Assume both skilled and unskilled workers have same preferences:

$$u(c,l)=c-\frac{l^2}{2}$$

- Assume skilled workers own a share β of the capital stock, unskilled a share (1β) .
- Wages w_s for skilled w_u for unskilled.

Skilled Household Problem

$$\max_{l_s} \left\{ l_s w_s + \beta r K - \frac{{l_s}^2}{2} \right\}$$

Optimality conditions:

$$-\frac{u_l}{u_c} = l_s = w_s$$

• So
$$l_s = w_s$$
, $c_s = w_s^2 + \beta r K$.

• Unskilled household problem is equivalent:

$$l_u = w_u, \, c_u = w_u^2 + (1 - \beta) \, rK.$$

Firms

- Assume representative firm hires both skilled and unskilled labor. Each has different productivity (z_s, z_u) .
- Firm substitutes between skilled and unskilled for total labor input.

$$l = (z_s l_s^\rho + z_u l_u^\rho)^{1/\rho}$$

where $0 < \rho < 1$.

• Thus production is:

$$y = k^{\alpha} l^{1-\alpha}$$

• Firms maximize profits:

$$k^{lpha}l(l_s,l_u)^{1-lpha}-rk-w_sl_s-w_ul_u$$

• FOC's – each type paid its marginal product:

$$\begin{array}{lll} (1-\alpha)k^{\alpha}l^{-\alpha}\left(z_{s}l_{s}^{\rho}+z_{u}l_{u}^{\rho}\right)^{1/\rho-1}z_{s}l_{s}^{\rho-1} &=& w_{s}\\ (1-\alpha)k^{\alpha}l^{-\alpha}\left(z_{s}l_{s}^{\rho}+z_{u}l_{u}^{\rho}\right)^{1/\rho-1}z_{u}l_{u}^{\rho-1} &=& w_{u} \end{array}$$

Characterize Equilibrium

• Divide firm FOC's:

$$egin{array}{ll} rac{w_s}{w_u}&=&rac{z_s l_s^{
ho-1}}{z_u l_u^{
ho-1}}\Rightarrow\ \log rac{w_s}{w_u}&=&\log rac{z_s}{z_u}+(
ho-1)\log rac{l_s}{l_u} \end{array}$$

- $\frac{w_s}{w_u}$, the skilled premium depends on:
 - 1. Relative productivities: $\frac{z_s}{zu}$.
 - 2. Relative supplies: $\frac{l_s}{l_u}$

• Then:

$$\gamma_{rac{w_s}{w_u}} = \gamma_{rac{z_s}{z_u}} + \left(
ho - 1
ight)\gamma_{rac{l_s}{l_u}}$$

- Changes in skill-premium depend on:
 - 1. Relative changes in productivities.
 - 2. Relative changes in abundance of factors.

Why Did it Happen?

- Nature of modern science.
- Size of the Market.
- Economics of Superstars.

What Does This Analysis Miss?

- Captures broad aggregate facts. Misses on some dimensions.
- Consumption Inequality. Evidence that inequality in consumption was less than inequality in income (Krueger and Perri, 2003). Here we have greater consumption inequality (since $\beta \approx 1$):

$$\frac{c_s}{c_u} = \frac{w_s^2 + \beta rK}{w_u^2 + (1 - \beta)rK} \approx \left(\frac{w_s}{w_u}\right)^2 + \frac{rK}{w_u^2}$$

 Changes by Gender. Most dramatic effects have been increase in female labor supply, especially in skilled labor. Hard to argue this was all from skill-biased technical change. Composition effects may be more important. (Eckstein and Nagypal, 2004)

Welfare Theorems

Jesús Fernández-Villaverde University of Pennsylvania
Pareto Optimality

- An allocation is *Pareto Optimal* if there is no way to rearrange production or reallocate goods so that someone is made better off without making someone else worse off.
- Pareto Optimality \neq perfect state of the world or any concept like that.

The Social Planner

- Let us imagine we have a powerful dictator, the Social Planner, that can decide how much the households consume and work and how much the firms produce.
- The Social Planner does not follow prices. But it understands opportunity cost.
- The Social Planner is benevolent. It searches for the best possible allocation.

Social Planner's Problem I

• Maximizes utility household given a level of government purchases G^*

$$\max_{c,l} u\left(c,1-l\right)$$

such that

$$c + G = Ak^{\alpha}l^{1-\alpha}$$
$$G = G^*$$
$$k = k^*$$

- Note: we do not have prices in the budget constraint!!!
- Standard Maximization problem.

Social Planner's Problem II

• We can rewrite the problem as:

$$\max_{l} u\left(Ak^{*\alpha}l^{1-\alpha}-G^*,1-l\right)$$

• First Order Condition with respect to *l*:

$$u_{c}^{\prime} \left(Ak^{*\alpha} l^{1-\alpha} - G^{*}, 1-l \right) \left(1-\alpha \right) Ak^{*\alpha} l^{-\alpha}$$
$$-u_{1-l}^{\prime} \left(Ak^{*\alpha} l^{1-\alpha} - G^{*}, 1-l \right) = 0$$

Social Planner's Problem III

• We rearrange as:

$$\frac{u_{1-l}'\left(Ak^{*\alpha}l^{1-\alpha}-G^*,1-l\right)}{u_c'\left(Ak^{*\alpha}l^{1-\alpha}-G^*,1-l\right)} = (1-\alpha)Ak^{*\alpha}l^{-\alpha}$$

- The lhs is the Marginal Rate of Substitution, MRS while the rhs is the Marginal Rate of Transformation, MRT.
- Thus, optimality implies:

$$MRS = MRT$$

• Let's look at it graphically.

The Big Question

- What is the relation between the solution to the Planners Problem and the Competitive Equilibrium?
- Or equivalently, is the Competitive Equilibrium Pareto-Optimal?
- Why do we care about this question?
 - 1. Positive reasons
 - 2. Normative reasons.

The Intuition

- First think about the case when $G^* = \tau^l = \mathbf{0}$
- Look again at the Social Planner's optimality condition

$$\frac{u_{1-l}'\left(Ak^{*\alpha}l^{1-\alpha},1-l\right)}{u_c'\left(Ak^{*\alpha}l^{1-\alpha},1-l\right)} = (1-\alpha)Ak^{*\alpha}l^{-\alpha}$$

• Remember that the Household first order condition was:

$$\frac{u_{1-l}'\left(Ak^{*\alpha}l^{1-\alpha} - G^*, 1-l\right)}{u_c'\left(Ak^{*\alpha}l^{1-\alpha} - G^*, 1-l\right)} = w$$

• And that firms profit maximization implied:

$$w = (1 - \alpha) A k^{*\alpha} l^{-\alpha}$$

• First order conditions are equivalent!!!

The Formal Statement

- First Fundamental Welfare Theorem: under certain conditions, the Competitive Equilibrium is Pareto Optimal.
- We have the converse.
- Second Fundamental Welfare Theorem: under certain conditions, a Pareto optimum is a Competitive Equilibrium.

Some consequences

- First Fundamental Welfare Theorem states that, under certain conditions, an allocation achieved by a market economy is Pareto-Optimal.
- Formalization of Adam Smith's "invisible hand" idea.
- Strong theoretical point in favour of decentralized allocation mechanisms: prices direct agents to do what is needed to get a Pareto optimum.
- Second Fundamental Welfare Theorem states what is the best way to change allocations: redistribute income. Do not mess with prices!!!

How robust is the First Welfare theorem?

- Not too much.
- Plenty of reasons that deviate the allocation from a Pareto optimum:
 - 1. Taxes.
 - 2. Externalities.
 - 3. Asymmetric Information.
 - 4. Market Incompleteness.
 - 5. Bounded Rationality of Agents.

What if taxes are not zero?

- Now think about the case when $G^* \neq 0, \tau^l \neq 0$
- Look again at the Social Planner's optimality condition

$$\frac{u_{1-l}'\left(Ak^{*\alpha}l^{1-\alpha},1-l\right)}{u_c'\left(Ak^{*\alpha}l^{1-\alpha},1-l\right)} = (1-\alpha)Ak^{*\alpha}l^{-\alpha}$$

• But now the Household first order condition is:

$$\frac{u_{1-l}'\left(Ak^{*\alpha}l^{1-\alpha}-G^{*},1-l\right)}{u_{c}'\left(Ak^{*\alpha}l^{1-\alpha}-G^{*},1-l\right)} = \left(1-\tau^{l}\right)w$$

• And since that firms profit maximization implied:

$$w = (1 - \alpha) A k^{*\alpha} l^{-\alpha}$$

• First order conditions are NOT equivalent!!!

Externalities

- What is an externality? When an agents consumption or production decision changes the production or consumption possibilities of other agents.
- Externalities can be good or bad.
- Example:
 - 1. Cities
 - 2. Environment

Asymmetric Information

- Information is dispersed in society.
- We may want to change our behavior based on the information we have.
- Akerlof-Spence-Stiglitz, Nobel Prize Winners 2001.
- Townsend and Prescott (1985).

Market Incompleteness

- We have assumed that we have complete markets.
- Every good can be traded.
- Is that a good representation of the world?
- Closely related with the problem of asymmetric information.

Bounded Rationality

- We have assumed that agents are "rational".
- Is this a good hypothesis?
- In some sense, yes:
 - 1. it is very powerful and more general that sometimes claimed.
 - 2. it is simple.
- Are we rational? Can we process information accurately?

A rationality quiz

- Assume:
 - 1. 1 in 100 people in the world are rational.
 - 2. We have a test for rationality.
 - 3. If someone is rational, it has a 99% chance of passing the test. If someone is irrational, it has a 99% chance of failing.
 - 4. My brother just passed the test.
 - 5. My brother was selected randomly from the population.
- What is the probability that my brother is rational?





Ask yourself (honestly): which answer I thought it was right?

Bounded rationality models

- Basic insight from Evolution Theory: we are not perfect machines, but frozen DNA accidents.
- Intersection between Evolutionary Psychology and Economics.
- Models with agents have problems to compute and they do not really know what they want.
- Problem: There is ONE way to be rational. There are MANY ways to be irrational.
- Which one is a better modelling choice?

What happens when we deviate from the assumptions of the theorem?

- It is hard to say.
- Second-Best Theorem (Tony Lancaster).
- Basic implications for reforms.
- How do we think about the real world?

Putting our theory to work

- How do we allocate resources in society?
- Why is this important? a little bit of history
- Could Central Planning work? Mises, Hayek in the 20's: NO
- Experience is rather clear that it did not, but maybe they just did not apply the recipe properly.
- That was the idea behind Lange-Lerner proposals for a Market-based socialism. Modern defenders of the idea: Roemer.

The intuition behind the idea

- What really matters is the use of prices, not private ownership.
- If somehow we can replicate the behavior of prices we'd be home free.
- Never really tried, but we have strong theoretical predictions against it.

The problem of information

"The problem of rational economic order is determined precisely by the fact that the knowledge of the circumstances of which we must make use never exist in concentrated or integrated form, but solely as the dispersed bits of incomplete knowledge which all the separate individuals possess...

The problem is thus in no way solved if we can show that all the facts, *if* they were known to a single mind (as we hypothetically assume them to be given to the observing economist) would uniquely determine the solution; instead we must show how a solution is produced by the interactions of people, each of whom possesses only partial knowledge".

Friedrich von Hayek

The market as a way to process information

- At the end of the day the Welfare Theorems do not capture all the advantages of market economies.
- They do not talk for instance about experimentation.
- Market economies are robust: they allow experimentation.
- Intuition from Biology.
- Example: Minitel in France versus Internet in the U.S.

Intertemporal Choice

Jesús Fernández-Villaverde University of Pennsylvania

- So far we have only studied static choices.
- Life is full of intertemporal choices: Should I study for my test today or tomorrow? Should I save or should I consume now? Should I marry this year or the next one?
- We will present a simple model: the Life-Cycle/Permanent Income Model of Consumption.
- Developed by Modigliani (Nobel winner 1985) and Friedman (Nobel winner 1976).

The Model

- Household, lives 2 periods.
- Utility function

$$u(c_1, c_2) = U(c_1) + \beta U(c_2)$$

where c_1 is consumption in first period of his life, c_2 is consumption in second period of his life, and β is between zero and one and measures household's degree of impatience.

• Income $y_1 > 0$ in the first period of life and $y_2 \ge 0$ in the second period of his life.

Budget Constraint I

- Household can save some of his income in the first period, or he can borrow against his future income y_2 .
- Interest rate on both savings and on loans is equal to r. Let s denote saving.
- Budget constraint in first period of life:

$$c_1 + s = y_1$$

• Budget constraint in second period of his life:

$$c_2 = y_2 + (1+r)s$$

Budget Constraint II

• Summing both budget constraints:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} = I$$

- We have normalized the price of the consumption good in the first period to 1. Price of the consumption good in period 2 is ¹/_{1+r}, which is also the relative price of consumption in period 2, relative to consumption in period 1.
- Gross interest rate 1 + r is the relative price of consumption goods today to consumption goods tomorrow.

Household's Problem

$$\max_{c_1, c_2} U(c_1) + \beta U(c_2)$$

s.t. $c_1 + \frac{c_2}{1+r} = I$

• FOC:

$$U'(c_1) = \lambda$$

$$\beta U'(c_2) = \lambda \frac{1}{1+r}$$

• Then we get Euler Equation: $U'(c_1) = \beta (1+r) U'(c_2)$.

A Parametric Example

• If $U(c) = \log c$, Euler Equation:

$$\frac{1}{c_1} = \beta \left(1+r\right) \frac{1}{c_2} \Rightarrow c_2 = \beta \left(1+r\right) c_1$$

• Note that:

$$c_1=I-\frac{c_2}{1+r}=I-\beta c_1$$
 and then $c_1=\frac{1}{1+\beta}I$ and $c_2=\frac{\beta(1+r)}{1+\beta}I.$

• Then
$$s = y_1 - c_1 = \frac{\beta}{1+\beta}y_1 - \frac{1}{1+\beta}\left(\frac{y_2}{1+r}\right)$$

Key Results

• Optimal consumption choice today: eat a fraction $\frac{1}{1+\beta}$ of total lifetime income *I* today and save the rest for the second period of your life.

• What variables does current consumption depend on? y_1, y_2, r .

Comparative Statics: Income Changes

- What happens to consumption if y_1 or y_2 increases?
- Both c_1 and c_2 increase.
- Marginal propensity to consume out of current income or wealth

$$\frac{dc_1}{dy_1} = \frac{1}{1+\beta} > 0$$

• Marginal propensity to consume out of tomorrows income

$$\frac{dc_1}{dy_2} = \frac{1}{(1+\beta)(1+r)} > 0$$

Comparative Statics: Changes in the Interest Rate

- Income effect: if a saver, then higher interest rate increases income for given amount of saving. Increases consumption in first and second period. If borrower, then income effect negative for c_1 and c_2 .
- Substitution effect: gross interest rate 1 + r is relative price of consumption in period 1 to consumption in period 2. c₁ becomes more expensive relative to c₂. This increases c₂ and reduces c₁.
- Hence: for a saver an increase in r increases c_2 and may increase or decrease c_1 . For a borrower an increase in r reduces r_1 and may increase or decrease c_2 .

Borrowing Constraints

- So far: household could borrow freely at interest rate r.
- Now: assume borrowing constraints $s \ge 0$.
- If household is a saver, nothing changes.
- If household would be a borrower without the constraint, then c₁ = y₁, c₂ = y₂. He would like to have bigger c₁, but he can't bring any of his second period income forward by taking out a loan. In this situation first period consumption does not depend on second period income or the interest rate.
Extension of the Basic Model: Life Cycle Hypothesis

- \bullet We can extend to T periods: Franco Modigliani' life-cycle hypothesis of consumption
- Individuals want smooth consumption profile over their life. Labor income varies substantially over lifetime, starting out low, increasing until the 50'th year of a person's life and then declining until 65, with no labor income after 65.
- Life-cycle hypothesis: by saving (and borrowing) individuals turn a very nonsmooth labor income profile into a very smooth consumption profile.

- Main predictions:
 - 1. current consumption depends on total lifetime income.
 - 2. Saving should follow a very pronounced life-cycle pattern with borrowing in the early periods of an economic life, significant saving in the high earning years from 35-50 and dissaving in retirement years.
- Do we observe these predictions in the data?

Empirical puzzles

- Hump in consumption. Role of demographics and uncertainty.
- Excess sensitivity of consumption to income.
- Older household do not dissave to the extent predicted by the theory. Several explanations:
 - 1. Individuals are altruistic and want to leave bequests to their children.
 - 2. Uncertainty with respect to length of life and health status.







Application of the Theory I: Social Security in the Life-cycle model

- Social Security is an important ongoing debate.
- Two classes of questions:
 - 1. Positive questions: What are the effects of social security and its possible reforms? What is the forecasted evolution of the current system?
 - 2. Normative questions: How should we organize social security?
- How does social security work in the U.S.?

- We want to distinguish:
 - 1. Between sustainability of the current system and the optimal reforms.
 - 2. Between pay-as-you-go versus fully funded system and private versus public systems.
 - 3. Role of system as saving, insurance, and redistribution.
- Use simple life-cycle model to analyze some of these issues.

Social Security in the Life-cycle model

- Assume $y_2 = 0$.
- Without social security (or a fully funded system).

$$c_1 = \frac{y_1}{1+\beta}$$

$$c_2 = \frac{\beta(1+r)y_1}{1+\beta}$$

$$s = \frac{\beta y_1}{1+\beta}$$

Pay As-You-Go Social Security System

- Introduce a pay as-you-go social security system: currently working generation pays payroll taxes, whose proceeds are used to pay the pensions of the currently retired generation
- Payroll taxes at rate au in first period. After tax wage is $(1 \tau)y_1$. Currently in US au = 12.4%
- Social security payments SS in second period: assume that population grows at rate n and pre-tax-income grows at rate g.
- Social security system balances its budget:

$$SS = (1+g)(1+n)\tau y_1$$

• Household's budget constraints:

$$c_1 + s = (1 - \tau)y_1$$

 $c_2 = (1 + r)s + SS$

• Intertemporal budget constraint:

$$c_1 + \frac{c_2}{1+r} = (1-\tau)y_1 + \frac{SS}{1+r} = I$$

• Maximizing utility subject to the budget constraint yields:

$$c_{1} = \frac{I}{1+\beta}$$

$$c_{2} = \frac{\beta}{1+\beta}(1+r)I$$

• Since $SS = (1+g)(1+n)\tau y_1$:

$$I = (1 - \tau)y_1 + \frac{SS}{1 + r}$$

= $(1 - \tau)y_1 + \frac{(1 + g)(1 + n)\tau y_1}{1 + r}$
= $y_1 - \left(1 - \frac{(1 + g)(1 + n)}{1 + r}\right)\tau y_1$
= \tilde{y}_1

• Hence:

$$egin{array}{rcl} c_1 &=& rac{ ilde y_1}{1+eta} \ c_2 &=& rac{eta}{1+eta}(1+r) ilde y_1 \end{array}$$

• Consumption in both periods is higher with social security than without if and only if $\tilde{y}_1 > y_1$, i.e. if and only if $\frac{(1+g)(1+n)}{1+r} > 1$. People are better off with social security if

$$(1+g)(1+n) > 1+r$$

Intuition: If people save by themselves for retirement, return on their savings equals 1 + r. If they save via a social security system, return equals (1 + n)(1 + g).

Numbers

- n = 1%, g = 2%.
- What is a good estimate r (in real terms)?
- Historical Record:

	1900-2000	1970-2000
Equities	6.7%	7.2%
Bonds	1.6%	4.1%
Bills	0.9%	1.5%

- These numbers suggest that a fully funded system is better than a pay-as-you-go system.
- Note that a fully funded system can either be public or private.
- Potential drawbacks:
 - 1. Management costs.
 - 2. Distribution of intergenerational risk.
 - 3. Bad choices of households.
 - 4. Lack of redistribution.

Possibilities of Reform

- Should we reform the system? Transition!
- Problem: one missing generation: at the introduction of the system there was one generation that received social security but never paid taxes.
- Dilemma:
 - 1. Currently young pay double, or
 - 2. Default on the promises for the old, or
 - 3. Increase government debt, financed by higher taxes in the future, i.e. by currently young and future generations.

Does Pay-as-you-go Social Security Decrease Saving?

• Without social security saving was given as $s = \frac{\beta y_1}{1+\beta}$.

• With social security saving it is given by
$$s = rac{eta(1- au)y_1 - rac{SS}{1+r}}{1+eta}$$
.

- Obviously private saving falls. The social security system as part of the government does not save, it pays all the tax receipts out immediately as pensions.
- Hence saving unambiguously goes down with pay-as-you go social security.

Application of the Theory II: Ricardian Equivalence

- What are the effects of government deficits in the economy?
- A first answer: none (Ricardo, 1817, and Barro, 1974).
- How can this be?
- The answer outside our small model is tricky.

- Lump-sum taxes.
- Government budget constraints:

$$G_1 = T_1 + B$$

 $G_2 + (1+r)B = T_2$

• Consolidating:

$$G_1 + \frac{G_2}{1+r} = T_1 + \frac{T_2}{1+r}$$

• Note that r is constant (you should not worry too much about this).

Household's Problem

• Original problem:

$$\max_{c_1, c_2} U(c_1) + \beta U(c_2)$$

s.t. $c_1 + \frac{c_2}{1+r} + T_1 + \frac{T_2}{1+r} = I$

• Now suppose that the government changes timing of taxes T'_1, T'_2 and government consumption G'_1, G'_2 .

• Then the problem of the household is:

$$\max_{c_1, c_2} U(c_1) + \beta U(c_2)$$

s.t. $c_1 + \frac{c_2}{1+r} + T_1' + \frac{T_2'}{1+r} = I$

• Since these new taxes must satisfy:

$$T_1 + \frac{T_2}{1+r} = T_1' + \frac{T_2'}{1+r}$$

problem of the consumer is equivalent!!!

Empirical Evidence

- Taxes in the world are not lump-sum.
- Does the Ricardian Equivalence hold?
- Important debate.

Dynamic General Equilibrium

Jesús Fernández-Villaverde University of Pennsylvania

The Basic Model

- We learned how to think about a household that makes dynamic decisions.
- We learned how to think about a household that makes decisions under uncertainty.
- We learned how to think about the intertemporal choice of government.
- Now we want to introduce investment and put everything together in a definition of Dynamic General Equilibrium.

Basic Model

- Simple model:
 - 1. Two periods
 - 2. No uncertainty.
- Households can invest in capital.
- Capital tomorrow is:

$$k_2 = (1-\delta) k_1 + i_1$$

Budget Constraints

• The household budget constrain for the first period, given T_1 , is:

$$c_1 + i_1 + b + T_1 = w_1 l_1 + r_1 k_1$$

• Using the fact that $k_2 = (1 - \delta) k_1 + i_1$:

$$c_1 + k_2 + b + T_1 = w_1 l_1 + r_1 k_1 + (1 - \delta) k_1$$

• Budget Constraint in the second period:

$$c_2 + T_2 = w_2 l_2 + r_2 k_2 + R_2 b + (1 - \delta) k_2$$

Household Problem

$$\begin{array}{rl} \max u\left(c_{1},1-l_{1}\right)+\beta u\left(c_{2},1-l_{2}\right)\\ \\ c_{1}+k_{2}+b+T_{1}&=&w_{1}l_{1}+r_{1}k_{1}+\left(1-\delta\right)k_{1}\\ \\ c_{2}+T_{2}&=&w_{2}l_{2}+r_{2}k_{2}+R_{2}b+\left(1-\delta\right)k_{2}\\ \\ k_{2}&>&0 \end{array}$$

Two Questions

- Why $k_2 > 0$?
 - We have a model with just one agent.
 - Capital at the end of second period.
- Why are the households taking the investment decisions and not the firms? Role of complete markets.

Solving the Household Problem I

- We want to solve for c_1 , l_1 , c_2 , l_2 , k_2 and b given T_1 , T_2 , k_1 , w_1 , w_2 , r_1 , r_2 and R_2 .
- This was your homework for a parametric example.

Solving the Household Problem II

• We build a Lagrangian function:

$$\begin{aligned} \mathcal{L} &= u\left(c_{1},1-l_{1}\right)+\beta u\left(c_{2},1-l_{2}\right) \\ &+\lambda_{1}\left(w_{1}l_{1}+r_{1}k_{1}+\left(1-\delta\right)k_{1}-c_{1}-k_{2}-b-T_{1}\right) \\ &+\lambda_{2}\left(w_{2}l_{2}+r_{2}k_{2}+\left(1-\delta\right)k_{2}+R_{2}b-c_{1}-k_{2}-b-T_{1}\right) \end{aligned}$$

• We take partial derivatives w.r.t. c_1 , l_1 , c_2 , l_2 , λ_1 and λ_2 , make them equal to zero and solve the associated system of equations.

Solving the Household Problem III

• After going through the previous steps, we have three optimality conditions:

$$egin{array}{rll} u_c \left(c_1, 1 - l_1
ight) &=& eta \left(1 + r_2 - \delta
ight) u_c \left(c_2, 1 - l_2
ight) \ rac{u_{1-l} \left(c_1, 1 - l_1
ight)}{u_c \left(c_1, 1 - l_1
ight)} &=& w_1 \ rac{u_{1-l} \left(c_2, 1 - l_2
ight)}{u_c \left(c_2, 1 - l_2
ight)} &=& w_2 \end{array}$$

• and one arbitrage condition:

$$R_2 = 1 + r_2 - \delta$$

Problem of the Firm I

- The problem of the firm is still static.
- In the first period, wants to maximize profits given r_1 and w_1 :

$$\pi = Ak_1^{\alpha} l_1^{1-\alpha} - r_1 k_1 - w_1 l_1$$

• We take first order conditions:

$$\alpha A k_1^{\alpha - 1} l_1^{1 - \alpha} = r_1$$

(1 - \alpha) $A k_1^{\alpha} l_1^{-\alpha} = w_1$

Problem of the Firm II

• In the second period, wants to maximize profits given r_2 and w_2 :

$$\pi = Ak_2^{\alpha}l_2^{1-\alpha} - r_2k_2 - w_2l_2$$

• We take first order conditions:

$$\alpha A k_2^{\alpha - 1} l_2^{1 - \alpha} = r_2$$

(1 - \alpha) $A k_2^{\alpha} l_2^{-\alpha} = w_2$

Government and Market Clearing

- Taxes, T_1 and T_2 and expenditures, G_1 and G_2 are given.
- Then, the government budget constraint is:

$$G_1 = T_1 + b$$
$$G_2 + R_2 b = T_2$$

• Market clearing:

$$c_{1} + k_{2} + G_{1} = Ak_{1}^{\alpha}l_{1}^{1-\alpha} + (1-\delta)k_{1}$$

$$c_{2} + G_{2} = Ak_{2}^{\alpha}l_{2}^{1-\alpha} + (1-\delta)k_{2}$$

A Competitive Equilibrium

A Competitive Equilibrium is an allocation $\{c_1, l_1, c_2, l_2, k_2, b\}$, a price system $\{w_1, w_2, r_1, r_2, R_2\}$ and a government policy $\{T_1, T_2, G_1, G_2\}$ s.t:

- 1. Given the price system, the government policy and k_1 households choose $\{c_1, l_1, c_2, l_2, k_2, b\}$ to maximize their utility.
- 2. Given the price system and the policy, firms maximize profits.
- 3. Government satisfies its budget constraint.
- 4. Markets clear.

Extensions of the Model I

- More periods. In fact why not infinite?
- Problem of the household

$$\max \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1-l_t\right)$$

 $c_t + k_{t+1} + b_{t+1} + T_t = w_t l_t + r_t k_t + R_t b_t + (1 - \delta) k_t, \forall t > 0$

• Problem of the firm: maximize profits given r_t and w_t :

$$\pi = A_t k_t^{\alpha} l_t^{1-\alpha} - r_t k_t - w_t l_t$$
• Budget constraint of the government:

$$G_t + R_t b_t = T_t$$

• Market clearing:

$$c_t + k_t + G_t = Ak_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t$$

• Transversality conditions (No-Ponzi schemes):

$$\lim_{t \to \infty} \beta^t k_t = 0$$
$$\lim_{t \to \infty} \left(\prod_{j=1}^{\infty} R_j \right)^{-1} b_t = 0$$

A Competitive Equilibrium

A Competitive Equilibrium is an allocation $\{c_t, l_t, k_t, b_t\}_{t=0}^{\infty}$, a price system $\{w_t, r_t, R_t\}_{t=0}^{\infty}$ and a government policy $\{T_t, G_t\}_{t=0}^{\infty}$ s.t:

- 1. Given the price system, the government policy and k_0 households choose $\{c_t, l_t, b_t\}_{t=0}^{\infty}$ to maximize their utility.
- 2. Given the price system and the policy, firms maximize profits.
- 3. Government satisfies its budget constraint.
- 4. Markets clear:

A Competitive Equilibrium

- Proof of existence of equilibrium.
- The Welfare theorems hold.
- How do we find the equilibrium? Dynamic Programing.

Visiting Old Friends

- This model with infinite period is an old friend of ours: Solow Model with endogenous labor supply and savings.
- Nearly everything we learned in the Growth part of the class will hold here (convergence, steady states and BGPs, transitional dynamics...).
- Also, if we make A endogenous, we will have an Endogenous Growth Model with endogenous labor supply and savings.

Extensions of the Model II: Uncertainty

• Problem of the household:

$$\max \sum_{t=0}^{\infty} \sum_{s^{t} \in S^{t}} \beta^{t} \pi\left(s^{t}\right) u(c\left(s^{t}\right), l\left(s^{t}\right))$$

$$s.t. \ c\left(s^{t}\right) + k\left(s^{t}\right) + \sum_{s_{t+1} \in S} \left(q\left(s^{t}, s_{t+1}\right) a\left(s^{t}, s_{t+1}\right) + b\left(s^{t}, s_{t+1}\right)\right) = w\left(s^{t}\right) l\left(s^{t}\right) + r\left(s^{t}\right) k\left(s^{t-1}\right) + (1-\delta) k\left(s^{t-1}\right) + R\left(s^{t-1}, s_{t}\right) b\left(s^{t-1}, s_{t}\right) + a\left(s^{t-1}, s_{t}\right)$$

• Problem of the firm: maximize profits given $r\left(s^{t}
ight)$ and $w\left(s^{t}
ight)$:

$$\pi\left(s^{t}\right) = A\left(s^{t}\right)k\left(s^{t-1}\right)^{\alpha}l\left(s^{t}\right)^{1-\alpha} - r\left(s^{t}\right)k\left(s^{t-1}\right) - w\left(s^{t}\right)l\left(s^{t}\right)$$

- Role of $A\left(s^{t}\right)$.
- Government: $\left\{T\left(s^{t}\right), G\left(s^{t}\right)\right\}_{t=0}^{\infty}$

Equilibrium

A S.M. equilibrium is an allocation $\left\{c^*\left(s^t\right), l^*\left(s^t\right), k^*\left(s^t\right)\right\}_{t=0, s^t \in S^t}^{\infty}$, portfolio decisions $\left\{a^*\left(s^t, s_{t+1}\right), b^*\left(s^t, s_{t+1}\right)\right\}_{t=0, s^t \in S^t}^{\infty}$ and prices $\left\{q^*\left(s^t, s_{t+1}\right), R^*\left(s^t, s_{t+1}\right), w^*\left(s^t\right), r^*\left(s^t\right)\right\}_{t=0, s^t \in S^t}^{\infty}$ such that:

- 1. Given prices, the allocation solves the problem of the consumer and of the firm.
- 2. Government satisfies its budget constraint.
- 3. Markets clear.

Further Extensions of the Model

- No-lump taxes.
- Money.
- Life Cycle: OLG
- Different market imperfections.

Money

Jesús Fernández-Villaverde University of Pennsylvania

Why Money?

- Two questions:
 - 1. Modern economies use money. Why?
 - 2. Changes in the amount of money can affect nominal and real variables in the economy.
- It is important to answer this questions in order to implement monetary policy.

Uses of Money

- Unit of account: contracts are usually denominated in terms of money.
- Store of Value: money allows consumers to trade current goods for future goods.
- Medium of Exchange.

Other Objects

- Other objects, like stocks and bonds, can be store of value and medium of exchange.
- Moreover, as store of value, stocks and bonds are better than money, since they give a positive rate of return.
- However, stocks and bonds are not very efficient as a medium of exchange because:
 - 1. Agents are not usually well-informed about the exact value of stocks.
 - 2. It is not always easy to sell these assets.

- Hence, what distinguishes money is that it plays in a very efficient way its role as a medium of exchange.
- There is a large literature that deals with the role of money as a medium of exchange.
- In absence of regular money, other obsjects appear as mediums of exchage (cigarettes in POW's camps).

Two classes of Models

• "Deep Models"

Explicitly model the fundamental reason money is used a medium of exchange, i.e. the existence of frictions in trade.

• "Applied Models"

Simply assume that money has to be used to carry out some transactions and proceed from there.

Deep Models

- Kiyotaki and Wrigth (1989).
- The main reason for money to have value is the double-coincidence of wants problem. In a specialized economy is not easy to find someone that has what you like and, at the same time, likes what you have.
- Money reduces this problem by making exchange possible in a singlecoincidence of wants meeting.

Applied Models

- We are going to focus in the more applied perspective on money.
- We will simply assume that there are some goods that only money can buy (Cash Goods): "Cash-in-Advance Models".
- Other ways to do it: "Money in the Utility function".
- You can show both approaches are equivalent.
- Are we doing the right thing (Wallace, 2001)?

A Simple Monetary Model I

- \bullet Two assets: Money, M and Nominal Bonds, B
- Nominal Bond is an asset that sells for one unit of money in the current period and pays off 1 + R units of money in the future period.
- R = rate of return on a bond in terms of money (nominal interest rate).
- r = real interest rate.

A Simple Monetary Model II

• $\pi = inflation$

$$\pi = \frac{P_{today} - P_{yesterday}}{P_{yesterday}}$$

- Notice that now we have a level of nominal prices.
- Then:

$$1+r = \frac{1+R}{1+\pi} \simeq R - \pi$$

Household I

- Representative Household consists of a worker and a shopper.
- We assume that Cash Goods can only be acquired if an agent has money from the previous period.
- Therefore the demand for money is equivalent to the demand for future Cash Goods.
- The household has to decide the demand for Cash Goods c_t^m , Credit Goods c_t^c , Money M_t , Bonds B_t and labor supply l_t .

Household II

- Utility Function $\sum_{t=0}^{\infty} \beta^t u(c_t^m, c_t^c, 1 l_t)$
- Budget constraint

$$P_t (c_t^m + c_t^c) + B_t + M_t = M_{t-1} + (1 + R_t)B_{t-1} + P_t w_t l_t$$

• Cash In Advance Constraint:

$$P_t c_t^m \le M_{t-1}$$

• No-Ponzi Scheme condition:

$$\lim_{t \to \infty} \beta^t u_1\left(c_t^m, c_t^c, 1 - l_t\right) B_t = \mathbf{0}$$

Optimality Conditions I

• Lagrangian:

$$\sum_{t=0}^{\infty} \left\{ \beta^{t} u\left(c_{t}^{m}, c_{t}^{c}, 1-l\right) + \lambda_{t} \left(\begin{array}{c} M_{t-1} + (1+R_{t})B_{t-1} + P_{t}w_{t}l_{t} \\ -P_{t}\left(c_{t}^{m} + c_{t}^{c}\right) - B_{t} - M_{t} \end{array} \right) \right\} \\ + \mu_{t}\left(M_{t-1} - P_{t}c_{t}^{m}\right)$$

• First order conditions:

$$\beta^{t} u_{1} \left(c_{t}^{m}, c_{t}^{c}, 1 - l_{t} \right) = \left(\lambda_{t} + \mu_{t} \right) P_{t}$$
(1)

$$\beta^t u_2 \left(c_t^m, c_t^c, 1 - l_t \right) = \lambda_t P_t \tag{2}$$

$$\beta^t u_3(c_t^m, c_t^c, 1 - l_t) = \lambda_t P_t w_t \tag{3}$$

$$-\lambda_{t-1} + \lambda_t + \mu_t = 0 \tag{4}$$

$$-\lambda_{t-1} + \lambda_t (1+R_t) = 0$$
 (5)

Optimality Conditions II

• Dividing (2) by (1):

$$\frac{u_2(c_t^m, c_t^c, 1 - l_t)}{u_1(c_t^m, c_t^c, 1 - l_t)} = \frac{\lambda_t}{\lambda_t + \mu_t}$$

• But using (4) and (5):

$$\frac{u_2(c_t^m, c_t^c, 1 - l_t)}{u_1(c_t^m, c_t^c, 1 - l_t)} = \frac{1}{1 + R_t} = \frac{1}{(1 + \pi_t)(1 + r_t)}$$

• Interpretation.

Optimality Conditions III

• Labor supply:

$$\frac{u_3(c_t^m, c_t^c, 1 - l_t)}{u_2(c_t^m, c_t^c, 1 - l_t)} = w_t$$

- Then demand for money depends on r_t , π_t and w_t .
- Putting everything in equilibrium. In particular how are r_t and w_t pind down?

Using the Model

- 1. Changes in money supply (M_t changes over time): effects on output and on price level.
- 2. Changes in wages.
- 3. Changes in real interest rate.

Unemployment

Jesús Fernández-Villaverde University of Pennsylvania Why Unemployment?

- So far we have studied models where labor market clears.
- Is that a good assumption?
- Why is unemployment important?
 - 1. Reduces income
 - 2. Increases inequality.
- How can we think about unemployment in an equilibrium model?

Concepts and Facts from the Labor Market

- The labor force is the number of people, 16 or older, that are either employed or unemployed but actively looking for a job. We denote the labor force at time t by N_t.
- Note that actively looking for a job is an ambiguous term.
- Let WPt denote the total number of people in the economy that are of working age (16 - 65) at date t. The labor force participation rate ft is defined as the fraction of the population in working age that is in the labor force, i.e. ft = Nt/WPt.

The number of unemployed people are all people that don't have a job.
 We denote this number by U_t. Similarly we denote the total number of people with a job by L_t. Obviously N_t = L_t + U_t. We define the unemployment rate u_t by

$$u_t = \frac{U_t}{N_t}$$

• The job losing rate b_t is the fraction of the people with a job which is laid off during a particular time, period, say one month (it is crucial for this definition to state the time horizon). The job finding rate e_t is the fraction of unemployed people in a month that find a new job.

Basic Facts

- U.S. Labor Force in feb 2002: 142 million people
- U.S. working age population in 2002: 212 million people
- Labor force participation rate of about 67.0%.
- Between 1967 and 1993 the average job losing rate was 2.7% per month
- Average job finding rate was 43%.
- Average unemployment rate during this time period was about 6.2%

Job Creation and Destruction

- The gross job creation Crt between period t − 1 and t equals the employment gain summed over all plants that expand or start up between period t − 1 and t.
- The gross job destruction Dr_t between period t-1 and t equals the employment loss summed over all plants that contract or shut down between period t-1 and t.
- The net job creation Nc_t between period t-1 and t equals $Cr_t Dr_t$.
- The gross job reallocation Ra_t between period t 1 and t equals $Cr_t + Dr_t$.

Main Findings of Davis, Haltiwanger and Schuh (1996)

- Data from all manufacturing plants in the US with 5 or more employees from 1963 to 1987. In the years they have data available, there were between 300,000 and 400,000 plants.
- Gross job creation Crt and job destruction Drt are remarkably large.
 In a typical year 1 out of every ten jobs in manufacturing is destroyed and a comparable number of jobs is created at different plants.
- Most of the job creation and destruction reflects highly persistent plant-level employment changes. Most jobs that vanish at a particular plant fail to reopen at the same location within the next two years.

- Job creation and destruction are concentrated at plants that experience large percentage employment changes. Two-thirds of job creation and destruction takes place at plants that expand or contract by 25% or more within a twelve-month period. About one quarter of job destruction takes place at plants that shut down.
- Job destruction exhibits greater cyclical variation than job creation.
 In particular, recessions are characterized by a sharp increase in job destruction accompanied by a mild slowdown in job creation.

Unemployment and the Business Cycle

- Gross job creation is relatively stable over the business cycle, whereas gross job destruction moves strongly countercyclical: it is high in recessions and low in booms.
- In severe recessions such as the 74-75 recession or the 80-82 back to back recessions up to 25% of all manufacturing jobs are destroyed within one year, whereas in booms the number is below 5%.
- Time a worker spends being unemployed also varies over the business cycle, with unemployment spells being longer on average in recession years than in years before a recession.

• Length of unemployment spells:

Unemployment Spell	1989	1992
< 5 weeks	49%	35%
5 - 14 weeks	30%	29%
15 - 26 weeks	11%	15%
> 26 weeks	10%	21%

• Other countries: in Germany, France or the Netherlands about two thirds of all unemployed workers in 1989 were unemployed for longer than six months!!

The Evolution of the Unemployment Rate

- $U_t =$ Number of unemployed at t
- $N_t = Labor$ Force in t
- $L_t = N_t U_t =$ Number of employed in t
- $u_t = \frac{U_t}{N_t}$ = unemployment rate
- s = job losing rate

- e = job finding rate
- Assume that $N_t = (1+n)N_{t-1}$
- Then we have

$$U_t = (1-e)U_{t-1} + sL_{t-1}$$

= $(1-e)U_{t-1} + s(N_{t-1} - U_{t-1})$
• Dividing both sides by $N_t = (1+n)N_{t-1}$ yields

$$u_{t} = \frac{U_{t}}{N_{t}} = \frac{(1-e)U_{t-1}}{(1+n)N_{t-1}} + \frac{s(N_{t-1} - U_{t-1})}{(1+n)N_{t-1}}$$
$$= \frac{1-e}{1+n}u_{t-1} + \frac{s(1-u_{t-1})}{1+n}$$
$$= \frac{1-e-s}{1+n}u_{t-1} + \frac{s}{1+n}$$

Steady State Rate of Unemployment

- In theory: steady state unemployment rate, absent changes in n, s, e
- Some people call it "Natural Rate": example of theory ahead of language.
- Origin of the idea: Friedman 1969

• Solve for $u^* = u_{t-1} = u_t$

$$u^* = \frac{1-e-s}{1+n}u^* + \frac{s}{1+n}$$
$$\frac{n+e+s}{1+n}u^* = \frac{s}{1+n}$$
$$u^* = \frac{s}{n+e+s}$$

 $\bullet\,$ From data s= 2.7%, e= 43% and n= 0.09%

• $u^* = 5.9\%$

Determinants of the Rate of Unemployment

- We just presented an accounting exercise.
- There was no theory on it.
- We want to have a model to think about the different elements of the model (*b*, *e*, etc.).

Several Models to Think about Unemployment

- Search Model.
- Efficiency Wages Models.
- Sticky Wages and Prices Models.

Basic Search Model I

- 1. Matching is costly. Think about getting a date.
- 2. We can bring our intuition to the job market.
- 3. Contribution of McCall (1970).

Basic Search Model II

- 1. $V_e(w)$: utility from being employed at wage w.
- 2. V_u : utility from being unemployed. Will depend on size of unemployment benefit.
- 3. Workers get random job offers at wage w_i with some probability p.
- 4. Reservation Wage: wage such that $V_u = V_e(w^*)$.
- 5. Rule: accept the job if $w_i > w^*$. Otherwise reject.

Basic Search Model III

Finding the unemployment rate in the Search Model:

- 1. Take a separation rate s. Then flow of workers from employment into unemployment is s(1-U).
- 2. H(w): fraction of unemployees that get job offers s.t. $w_i > w$.
- 3. Then new employees are $UpH(w^*)$.

4.
$$U^*$$
 is such that $U^*pH(w^*) = s(1 - U^*)$ or
 $U^* = \frac{s}{s + pH(w^*)} = \frac{s}{s + pH(V_u)}$

Basic Search Model IV

Determinants of Unemployment Rate in the Search Model:

- Unemployment Insurance: Length and generosity of unemployment insurance vary greatly across countries. US replacement rate is 34%. Germany, France and Italy the replacement rate is about 67%, with duration well beyond the first year of unemployment.
- 2. Minimum Wages: If the minimum wage is so high that it makes certain jobs unprofitable, less jobs are offered and job finding rates decline.

Efficiency Wage Model I

- Asymmetric Information is a common factor in labor markets
- In particular monitor effort of employee is tough.
- Firms will try to induce workers to put more effort.
- How can this create unemployment?
- Shaphiro-Stiglitz (1984) model.

Efficiency Wage Model II

- The economy consist of a large number of workers L^* and of firms N.
- The worker can supply two levels of effort, e = 0 or $e = e^*$.
- Utility of the worker

$$u = \left\{ egin{array}{c} w - e & ext{if employed} \\ 0 & ext{if unemployed} \end{array}
ight.$$

• Household has a discount rate ρ (in our usual notation $\rho \simeq 1 - \beta$).

Efficiency Wage Model III

The worker can be in three different states:

- 1. E: employed and exerting effort e^* .
- 2. S: employed and shirking, e = 0.
- 3. U: unemployed.

Efficiency Wage Model IV

- If the worker exerts effort, he will lose his job with probability b.
- If the worker shirks, he will lose his job with probability q + b.
- If the worker is unemployed, it finds a job with probability a.

Efficiency Wage Model V

• Then, the value of being unemployed is

$$\rho V_E = w - e^* - b \left(V_E - V_U \right)$$

• The value of shirking is

$$\rho V_S = w - (b+q) \left(V_S - V_U \right)$$

• And the value of unemployment is:

$$\rho V_U = a \left(V_E - V_U \right)$$

Efficiency Wage Model VI

• Firm's problem: Then, the value of being unemployed is

$$\max_{w,L+S}\pi=F(e^*L)-w\left(L+S\right)$$

where F' > 0 and F'' < 0.

• We will assume

$$F'\left(\frac{e^*L^*}{N}\right) > 1$$

The No-Shirking Condition I

The firm will pick w to satisfy:

$$V_E = V_S$$

or:

$$w - e^* - b (V_E - V_U) = w - (b + q) (V_S - V_U)$$

$$e^* + b (V_E - V_U) = (b + q) (V_S - V_U)$$

$$e^* + b (V_E - V_U) = (b + q) (V_E - V_U)$$

$$V_E - V_U = \frac{e^*}{q}$$

The No-Shirking Condition II

Also:

$$\rho V_E = w - e^* - b (V_E - V_U)$$

$$\rho V_U = a (V_E - V_U)$$

Substituting

$$ho(V_E-V_U)=w-e^*-(a+b)\left(V_E-V_U
ight)$$
Using $V_E-V_U=rac{e^*}{q}$

$$\rho(\frac{e^*}{q}) = w - e^* - (a+b)\left(\frac{e^*}{q}\right)$$

or

$$w = e^* + (a + b + \rho)\left(rac{e^*}{q}
ight)$$

The No-Shirking Condition II

Now, note that in an steady state

$$NLb = (L^* - NL)a$$

or:

$$a = \frac{NLb}{L^* - NL}$$

and

$$a + b = \frac{NLb}{L^* - NL} + b = \frac{NLb + bL^* - NLb}{L^* - NL} = \frac{bL^*}{L^* - NL}$$

so we get:

$$w = e^* + \left(\frac{bL^*}{L^* - NL} + \rho\right) \left(\frac{e^*}{q}\right)$$

This equation is called the No-Shirking Condition.

Closing the Model

• If $V_E = V_S$, all the workers will exert effort and the firm's profits will be:

$$F(e^*L) - wL$$

with FOC:

$$e^*F'(e^*L) = w$$

• Since
$$w = e^* + \left(\frac{bL^*}{L^* - NL} + p\right) \left(\frac{e^*}{q}\right)$$
, we have:
 $e^*F'(e^*L) = e^* + \left(\frac{bL^*}{L^* - NL} + \rho\right) \left(\frac{e^*}{q}\right) z$



The Shapiro-Stiglitz Model

International Labor Market Comparisons I

Unemployment and Long-Term Unemployment in OECD

	Unem	ployme	nt (%)	\geq 6 Months		\geq 1 Year			
	74-9	80-9	95	79	89	95	79	89	95
Belgium	6.3	10.8	13.0	74.9	87.5	77.7	58.0	76.3	62.4
France	4.5	9.0	11.6	55.1	63.7	68.9	30.3	43.9	45.6
Germany	3.2	5.9	9.4	39.9	66.7	65.4	19.9	49.0	48.3
Netherlands	4.9	9.7	7.1	49.3	66.1	74.4	27.1	49.9	43.2
Spain	5.2	17.5	22.9	51.6	72.7	72.2	27.5	58.5	56.5
Sweden	1.9	2.5	7.7	19.6	18.4	35.2	6.8	6.5	15.7
UK	5.0	10.0	8.2	39.7	57.2	60.7	24.5	40.8	43.5
US	6.7	7.2	5.6	8.8	9.9	17.3	4.2	5.7	9.7
OECD Eur.	4.7	9.2	10.3	_	_	_	31.5	52.8	_
Tot. OECD	4.9	7.3	7.6	_	_	_	26.6	33.7	_

International Labor Market Comparisons II

Distribution of Long-Term Unemployment (longer than 1 year) by Age in 1990

	Age Group				
	15 - 24	25 - 44	\geq 45		
Belgium	17	62	20		
France	13	63	23		
Germany	8	43	48		
Netherlands	13	64	23		
Spain	34	38	28		
Sweden	9	24	67		
UK	18	43	39		
US	14	53	33		

International Labor Market Comparisons III

Net Unemployment Replacement Rate for Single-Earner Households

	Single			With Dependent Spouse			
	1. Y.	23. Y.	45. Y.	1. Y.	23. Y.	45. Y.	
Belgium	79	55	55	70	64	64	
France	79	63	61	80	62	60	
Germany	66	63	63	74	72	72	
Netherlands	79	78	73	90	88	85	
Spain	69	54	32	70	55	39	
Sweden	81	76	75	81	100	101	
UK	64	64	64	75	74	74	
US	34	9	9	38	14	14	

How can we use theory to think about these facts?

The Explanations:

- 1. Rigidities in the labor market
- 2. Wage bargaining
- 3. Welfare state
- 4. International trade
- 5. A restrictive economic policy

Rigidities in the labor market: Hopenhayn and Rogerson

- 1. Firing restrictions
- 2. Work arrangements
- 3. Minimum wages

Wage bargaining: Cole and Ohanian, Caballero and Hammour

1. Effect of "insiders" vs. "outsiders".

2. Trade Unions.

Welfare State: Ljungvist and Sargent

- 1. High welfare benefits
- 2. Rapidly changing economy

International Trade

- 1. Increase in trade during last 25 years.
- 2. Basic Trade Model Implications.
- 3. Facts do not agree with theory

Possible Policy Solutions

- Reform Labor Market Institutions.
- Reduce Trade Unions power.
- Changing Unemployment Insurance.
- Reform Educational Systems.
- Increasing Mobility.

Think about Political Economy.

Business Cycles

Jesús Fernández-Villaverde University of Pennsylvania

Business Cycles

- U.S. economy fluctuates over time.
- How can we build models to think about it?
- Do we need different models than before to do so? Traditionally the answer was yes. Nowadays the answer is no.

Business Cycles and Economic Growth

• How different are long-run growth and the business cycle?

Changes in Output per Worker	Secular Growth	Business Cycle
Due to changes in capital	1/3	0
Due to changes in labor	0	2/3
Due to changes in productivity	2/3	1/3

• We want to use the same models with a slightly different focus.

Representing the Business Cycle I

- How do we look at the data?
- What is the cyclical component of a series and what is the trend?
- Is there a unique way to decide it?

Representing the Business Cycle II

- Suppose that we have T observations of the stochastic process X, $\{x_t\}_{t=1}^{t=\infty}$.
- Hodrick-Prescott (HP) filter decomposes the observations into the sum of a trend component, x_t^t and a cyclical component x_t^c :

$$x_t = x_t^t + x_t^c$$

Representing the Business Cycle III

• How? Solve:

$$\min_{x_t^t} \sum_{t=1}^T \left(x_t - x_t^t \right)^2 + \lambda \sum_{t=2}^{T-1} \left[\left(x_{t+1}^t - x_t^t \right) - \left(x_t^t - x_{t-1}^t \right) \right]^2$$
(1)

- Intuition.
- Meaning of λ :
 - 1. $\lambda = 0 \Rightarrow$ trivial solution $(x_t^t = x_t)$.
 - 2. $\lambda = \infty \Rightarrow$ linear trend.

Representing the Business Cycle IV

• To compute the HP filter is easier to use matricial notation, and rewrite (1) as:

$$\begin{split} \min_{x^{t}} \left(x - x^{t} \right)' \left(x - x^{t} \right) + \lambda \left(A x^{t} \right)' \left(A x^{t} \right) \\ \text{where } x &= (x_{1}, ..., x_{T})', \ x^{t} = \left(x_{1}^{t}, ..., x_{T}^{t} \right)' \text{ and:} \\ \\ A &= \begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 1 & -2 & 1 & 0 \\ 0 & \cdots & \cdots & 1 & -2 & 1 \end{pmatrix}_{(T-2) \times T} \end{split}$$
Representing the Business Cycle V

• First order condition:

$$x^t - x + \lambda A' A x^t = \mathbf{0}$$

or

$$x^{t} = \left(I + \lambda A'A\right)^{-1}x$$

• $(I + \lambda A'A)^{-1}$ is a sparse matrix (with density factor $(5T - 6)/T^2$). We can exploit this property.

Figure 1: Idealized Business Cycles



Figure 2: Percentage Deviations from Trend in Real GDP from 1947 to 1999



Figure 3: Time Series Plots of x and y



Figure 4: Correlations Between Variables y and x

.



Figure 5: Leading and Lagging Variables



Figure 6: Percentage Deviations from Trend in Real GDP (colored line) and the Index of Leading Economic Indicators (black line) for the Period 1959-1999



Figure 7: Percentage Deviations from Trend in Real Consumption (black line) and Real GDP (colored line)



Figure 8: Percentage Deviations from Trend in Real Investment (black line) and Real GDP (colored line)



Figure 9: Scatter Plot for the Percentage Deviations from Trend in the Price Level (the Implicit GDP Price Deflator) and Real GDP



Figure 10: Price Level and GDP



Figure 11: Percentage Deviations from Trend in the Money Supply (black line) and Real GDP (colored line) for the Period 1947-1999



Figure 12: Percentage Deviations from Trend in Employment (black line) and Real GDP (colored line)



Table 1

Table 3.1 Correlation Coefficients and Variability of Percentage Deviations From Trend

	Correlation Coefficient (GDP)	Std. Dev. (% of S.D. of GDP)
Consumption	0.76	76.1%
Investment	0.84	464.4
Price Level	-0.29	56.2
Money Supply	0.37	77.9
Employment	0.80	58.7

Table 2

Table 3.2 Summary of Business Cycle Facts

	Cyclicality	Lead/Lag	Variability Relative to GDP
Consumption	Procyclical	Coincident	Smaller
Investment	Procyclical	Coincident	Larger
Price Level	Countercyclical	Coincident	Smaller
Money Supply	Procyclical	Leading	Smaller
Employment	Procyclical	Lagging	Smaller
Real Wage	Procyclical	?	_

Real Business Cycles

Jesús Fernández-Villaverde University of Pennsylvania

Fluctuations

- Economy fluctuates over time.
- Is there a systematic phenomenon we need to explain?
- Simple random walk:

$$egin{array}{rcl} y_t &=& y_{t-1} + arepsilon_t \ arepsilon_t &\sim& \mathcal{N}(\mathbf{0},\sigma) \end{array}$$

 Is output well described by a Random Walk? Plosser and Nelson (1992) and Unit Root testing.



Fluctuations in Equilibrium

- We want to think about business cycles using an equilibrium perspective.
- Traditionally economist did not use an equilibrium approach to address this issue.
- Big innovation: Lucas (1972).
- How can we generate fluctuations in equilibrium?

Real Business Cycles

- We learned how to map preferences (for the household), technology (for the firm) and a government policy into a Competitive Equilibrium.
- If we let preferences, technology or the government preferences change over time, the equilibrium sequence will also fluctuate.
- All these (preferences, technology, policy) are real factors (as opposed to monetary).
- This is the reason we call this approach Real Business Cycles.
- Big innovation: Kydland and Prescott (1982).

Intuition I

- Let us go back to our Robinson Crusoe's Economy.
- How will Robinson do if he wakes up and today is a sunny day?
- And if it is rainy?
- Basic idea: intertemporal substitution.

Intuition II

- We will have an initial shock: change in preferences, technology or policy.
- Then we will have a propagation mechanism: intertemporal labor substitution and capital accumulation.
- We will have fluctuations as an equilibrium outcome.

Productivity Shocks

- We will study first the effects of changes to technology.
- Remember that:

$$rac{\dot{Y}}{Y}=rac{\dot{A}}{A}+lpharac{\dot{K}}{K}+(1-lpha)rac{\dot{L}}{L}$$

and that the Solow Residual is:

$$\frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} - \alpha \frac{\dot{K}}{K} - (1 - \alpha) \frac{\dot{L}}{L}$$

• How do the Solow Residual and GDP move together?

Question

- Let us suppose that we have an economy that is hit over time by productivity shocks with the same characteristics that the ones that hit the US economy.
- How does this economy behave over time?
- In particular, how do the variances and covariances of the main variables in our economy compare with those observed in the US economy?

Household Problem

• Preferences:

$$\max E \sum_{t=0}^{\infty} \beta \left\{ \log c_t + \psi \log \left(1 - l_t\right) \right\}$$

• Budget constraint:

$$c_t + k_{t+1} = w_t l_t + r_t k_t + (1 - \delta) k_t, \ \forall \ t > 0$$

Problem of the Firm

• Neoclassical production function:

$$y_t = k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha}$$

• By profit maximization:

$$\alpha k_t^{\alpha - 1} (e^{z_t} l_t)^{1 - \alpha} = r_t$$

(1 - \alpha) $k_t^{\alpha} (e^{z_t} l_t)^{1 - \alpha} l_t^{-1} = w_t$

Evolution of the technology

- z_t changes over time.
- It follows the AR(1) process:

$$egin{array}{rcl} z_t &=&
ho z_{t-1} + arepsilon_t \ arepsilon_t &\sim& \mathcal{N}(\mathbf{0},\sigma) \end{array}$$

• Interpretation of ρ .

A Competitive Equilibrium

- We can define a competitive equilibrium in the standard way.
- The competitive equilibrium is unique.
- This economy satisfies the conditions that assure that both welfare theorems hold.
- Why is this important? We could solve instead the Social Planner's Problem associated with it.
- Advantages and disadvantages of solving the social planner's problem.

The Social Planner's Problem

• It has the form:

$$\begin{aligned} \max E \sum_{t=0}^{\infty} \beta \left\{ \log c_t + \psi \log \left(1 - l_t\right) \right\} \\ c_t + k_{t+1} &= k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} + \left(1 - \delta\right) k_t, \,\forall \, t > 0 \\ z_t &= \rho z_{t-1} + \varepsilon_t, \, \varepsilon_t \sim \mathcal{N}(0, \sigma) \end{aligned}$$

• This is a dynamic optimization problem.

Computing the RBC

- The previous problem does not have a known "paper and pencil" solution.
- We will work with an approximation: Perturbation Theory.
- We will undertake a first order perturbation of the model.
- How well will the approximation work?

Equilibrium Conditions

From the household problem+firms's problem+aggregate conditions:

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left(1 + \alpha k_t^{\alpha - 1} \left(e^{z_t} l_t \right)^{1 - \alpha} - \delta \right) \right\}$$
$$\psi \frac{c_t}{1 - l_t} = (1 - \alpha) k_t^{\alpha} \left(e^{z_t} l_t \right)^{1 - \alpha} l_t^{-1}$$
$$c_t + k_{t+1} = k_t^{\alpha} \left(e^{z_t} l_t \right)^{1 - \alpha} + (1 - \delta) k_t$$
$$z_t = \rho z_{t-1} + \varepsilon_t$$

Finding a Deterministic Solution

- We search for the first component of the solution.
- If $\sigma = 0$, the equilibrium conditions are:

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} \left(1 + \alpha k_t^{\alpha - 1} l_t^{1 - \alpha} - \delta \right)$$
$$\psi \frac{c_t}{1 - l_t} = (1 - \alpha) k_t^{\alpha} l_t^{-\alpha}$$
$$c_t + k_{t+1} = k_t^{\alpha} l_t^{1 - \alpha} + (1 - \delta) k_t$$

Deterministic Steady State

• The equilibrium conditions imply a steady state:

$$\frac{1}{c} = \beta \frac{1}{c} \left(1 + \alpha k^{\alpha - 1} l^{1 - \alpha} - \delta \right)$$
$$\psi \frac{c}{1 - l} = (1 - \alpha) k^{\alpha} l^{-\alpha}$$
$$c + \delta k = k^{\alpha} l^{1 - \alpha}$$

• Or simplifying:

$$\frac{1}{\beta} = 1 + \alpha k^{\alpha - 1} l^{1 - \alpha} - \delta$$
$$\psi \frac{c}{1 - l} = (1 - \alpha) k^{\alpha} l^{-\alpha}$$
$$c + \delta k = k^{\alpha} l^{1 - \alpha}$$

Solving the Steady State

Solution:

$$k = \frac{\mu}{\Omega + \varphi \mu}$$
$$l = \varphi k$$
$$c = \Omega k$$
$$y = k^{\alpha} l^{1-\alpha}$$

where
$$\varphi = \left(\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right)^{\frac{1}{1-\alpha}}$$
, $\Omega = \varphi^{1-\alpha} - \delta$ and $\mu = \frac{1}{\psi}\left(1 - \alpha\right)\varphi^{-\alpha}$.

Linearization I

- Loglinearization or linearization?
- Advantages and disadvantages
- We can linearize and perform later a change of variables.

Linearization II

We linearize:

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left(1 + \alpha k_t^{\alpha - 1} \left(e^{z_t} l_t \right)^{1 - \alpha} - \delta \right) \right\}$$
$$\psi \frac{c_t}{1 - l_t} = (1 - \alpha) k_t^{\alpha} \left(e^{z_t} l_t \right)^{1 - \alpha} l_t^{-1}$$
$$c_t + k_{t+1} = k_t^{\alpha} \left(e^{z_t} l_t \right)^{1 - \alpha} + (1 - \delta) k_t$$
$$z_t = \rho z_{t-1} + \varepsilon_t$$

around l, k, and c with a First-order Taylor Expansion.
Linearization III

We get:

$$\begin{aligned} &-\frac{1}{c} \left(c_{t} - c \right) = E_{t} \left\{ \begin{array}{c} -\frac{1}{c} \left(c_{t+1} - c \right) + \alpha \left(1 - \alpha \right) \beta \frac{y}{k} z_{t+1} + \\ \alpha \left(\alpha - 1 \right) \beta \frac{y}{k^{2}} \left(k_{t+1} - k \right) + \alpha \left(1 - \alpha \right) \beta \frac{y}{kl} \left(l_{t+1} - l \right) \end{array} \right\} \\ & \frac{1}{c} \left(c_{t} - c \right) + \frac{1}{\left(1 - l \right)} \left(l_{t} - l \right) = \left(1 - \alpha \right) z_{t} + \frac{\alpha}{k} \left(k_{t} - k \right) - \frac{\alpha}{l} \left(l_{t} - l \right) \\ \left(c_{t} - c \right) + \left(k_{t+1} - k \right) = \left\{ \begin{array}{c} y \left(\left(1 - \alpha \right) z_{t} + \frac{\alpha}{k} \left(k_{t} - k \right) + \frac{\left(1 - \alpha \right)}{l} \left(l_{t} - l \right) \right) \\ & + \left(1 - \delta \right) \left(k_{t} - k \right) \end{array} \right\} \\ & z_{t} = \rho z_{t-1} + \varepsilon_{t} \end{aligned}$$

Rewriting the System I

Or:

$$\begin{aligned} \alpha_1 \left(c_t - c \right) &= E_t \left\{ \alpha_1 \left(c_{t+1} - c \right) + \alpha_2 z_{t+1} + \alpha_3 \left(k_{t+1} - k \right) + \alpha_4 \left(l_{t+1} - l \right) \right\} \\ \left(c_t - c \right) &= \alpha_5 z_t + \frac{\alpha}{k} c \left(k_t - k \right) + \alpha_6 \left(l_t - l \right) \\ \left(c_t - c \right) + \left(k_{t+1} - k \right) &= \alpha_7 z_t + \alpha_8 \left(k_t - k \right) + \alpha_9 \left(l_t - l \right) \\ z_t &= \rho z_{t-1} + \varepsilon_t \end{aligned}$$

Rewriting the System II

where

$$\begin{array}{ll} \alpha_1 = -\frac{1}{c} & \alpha_2 = \alpha \left(1 - \alpha\right) \beta \frac{y}{k} \\ \alpha_3 = \alpha \left(\alpha - 1\right) \beta \frac{y}{k^2} & \alpha_4 = \alpha \left(1 - \alpha\right) \beta \frac{y}{kl} \\ \alpha_5 = \left(1 - \alpha\right) c & \alpha_6 = -\left(\frac{\alpha}{l} + \frac{1}{(1 - l)}\right) c \\ \alpha_7 = \left(1 - \alpha\right) y & \alpha_8 = y \frac{\alpha}{k} + (1 - \delta) \\ \alpha_9 = y \frac{(1 - \alpha)}{l} & y = k^{\alpha} l^{1 - \alpha} \end{array}$$

Rewriting the System III

After some algebra the system is reduced to:

$$A(k_{t+1} - k) + B(k_t - k) + C(l_t - l) + Dz_t = 0$$

$$E_t (G(k_{t+1} - k) + H(k_t - k) + J(l_{t+1} - l) + K(l_t - l) + Lz_{t+1} + Mz_t) = 0$$

$$E_t z_{t+1} = \rho z_t$$

Guess Policy Functions

We guess policy functions of the form $(k_{t+1} - k) = P(k_t - k) + Qz_t$ and $(l_t - l) = R(k_t - k) + Sz_t$, plug them in and get:

$$A (P (k_t - k) + Qz_t) + B (k_t - k) +C (R (k_t - k) + Sz_t) + Dz_t = 0$$

 $G(P(k_t - k) + Qz_t) + H(k_t - k) + J(R(P(k_t - k) + Qz_t) + SNz_t) + K(R(k_t - k) + Sz_t) + (LN + M)z_t = 0$

Solving the System I

Since these equations need to hold for any value $(k_{t+1} - k)$ or z_t we need to equate each coefficient to zero, on $(k_t - k)$:

$$AP + B + CR = 0$$
$$GP + H + JRP + KR = 0$$

and on z_t :

$$AQ + CS + D = 0$$
$$(G + JR)Q + JSN + KS + LN + M = 0$$

Solving the System II

- We have a system of four equations on four unknowns.
- To solve it note that $R = -\frac{1}{C}(AP + B) = -\frac{1}{C}AP \frac{1}{C}B$
- Then:

$$P^{2} + \left(\frac{B}{A} + \frac{K}{J} - \frac{GC}{JA}\right)P + \frac{KB - HC}{JA} = 0$$

a quadratic equation on P.

Solving the System III

• We have two solutions:

$$P = -\frac{1}{2} \left(-\frac{B}{A} - \frac{K}{J} + \frac{GC}{JA} \pm \left(\left(\frac{B}{A} + \frac{K}{J} - \frac{GC}{JA} \right)^2 - 4 \frac{KB - HC}{JA} \right)^{0.5} \right)$$

one stable and another unstable.

• If we pick the stable root and find $R = -\frac{1}{C}(AP + B)$ we have to a system of two linear equations on two unknowns with solution:

$$Q = \frac{-D(JN+K) + CLN + CM}{AJN + AK - CG - CJR}$$
$$S = \frac{-ALN - AM + DG + DJR}{AJN + AK - CG - CJR}$$

Calibration

- What does it mean to calibrate a model?
- Our choices

Calibrated Parameters

Parameter	β	ψ	α	δ	ρ	σ
Value	0.99	1.75	0.33	0.023	0.95	0.01

Computation I

- In practice you do all that in the computer.
- One of the main tools of modern macroeconomic theory is the computer.
- We build small laboratory economies inside our computers and we run experiments on them.
- If our economy behaves well in those experiments we know the answer, we are confident about its answers for questions we do not know.
- Modern macroeconomics is a *Quantitative* Science.

Computation II

- I use some Matlab code: rbc.mod for more sophisticated manipulation.
- How does it work?
- If any of you is thinking about graduate studies in economics, you should learn a programming language NOW (Matlab, Fortran 90, C++).

Solution I

Basic results are easy to understand:

- You want to work hard when productivity is high.
- You want to save more when productivity is high. Relation with consumption smoothing.
- Reverse effects happen when productivity is low.

Solution II

- We have an initial shock: productivity changes.
- We have a transmission mechanism: intertemporal substitution and capital accumulation.
- We can look at a simulation from this economy.
- Why only a simulation?

Comparison with US economy

- Simulated Economy output fluctuations are around 75% as big as observed fluctuations.
- Consumption is less volatile than output.
- Investment is much more volatile.
- Behavior of hours.

Assessment of the Basic Real Business Model

- It accounts for a substantial amount of the observed fluctuations.
- It accounts for the covariances among a number of variables.
- It has some problems accounting for the behavior of the hours worked.
- More important question: where do productivity shocks come from?

Negative Productivity Shocks I

- The model implies that half of the quarters we have negative technology shocks.
- Is this plausible? What is a negative productivity shocks?
- Role of trend.

Negative Productivity Shocks II

- s.d. of shocks is 0.01. Mean quarter productivity growth is 0.0048 (to give us a 1.9% growth per year).
- As a consequence, we would only observe negative technological shocks when $\varepsilon_t < -0.0048$.
- This happens in the model around 33% of times.
- Ways to fix it.

Some Policy Implications

- The basic model is Pareto-efficient.
- Fluctuations are the optimal response to a changing environment.
- Fluctuations are not a sufficient condition for inefficiencies or for government intervention.
- In fact in this model the government can only worsen the allocation.
- Recessions have a "cleansing" effect.

Extensions

- We can extend our model in several directions.
- Fiscal Policy shocks (McGrattan, 1994).
- Agents with Finite Lives (Ríos-Rull, 1996).
- Indivisible Labor (Rogerson, 1988, and Hansen, 1985).
- Home Production (Benhabib, Rogerson and Wright, 1991).
- Money (Cooley and Hansen, 1989).

Fiscal Policy

- We can use the model with taxes and public spending to think about fiscal policy.
- Issue non trivial: we are not in a Ricardian Equivalence world.
- Two different questions:
 - 1. Effects of the fiscal policy on the economy.
 - 2. What is the optimal fiscal policy over the cycle.

Effects of Fiscal Policy

- Basic Intuition.
- The model generates:
 - 1. a positive correlation between government spending and output.
 - 2. a positive correlation between temporarily lower taxes and output.
- How can we use our model to think about Bush's tax plan?

Optimal Fiscal Policy

- How do we want to conduct fiscal policy over the cycle?
- Remember that this is the Ramsey Problem.
- Chari, Christiano, and Kehoe (1994):
 - 1. No tax on capital (Chamley, 1986, Judd, 1985).
 - 2. Tax labor smoothly.
 - 3. Use debt as a shock absorber.
- Evaluating Bush's tax plan from a Ramsey perspective.

Time Inconsistency I

- Previous analysis imply that governments can commit themselves.
- Is this a realistic assumption?
- Time inconsistency problem.
- Original contribution by Kydland and Prescott (1977).
- Main example: tax on capital

Time Inconsistency II

- What can we say about fiscal policy in this situations?
- Use the tools of game theory.
- Main contributions:
 - 1. Chari and Kehoe (1989).
 - 2. Klein and Ríos-Rull (2001).
 - 3. Stacchetti and Phelan (2001).

Putting our Theory to Work: the Great Depression

- Great Depression is a unique event in US history.
- Timing 1929-1933.
- Major changes in the US Economic policy: New Deal.
- Can we use the theory to think about it?

Data on the Great Depression

Year	ur	Y	C	Ι	G	i	π
1929	3.2	203.6	139.6	40.4	22.0	5.9	—
1930	8.9	183.5	130.4	27.4	24.3	3.6	-2.6
1931	16.3	169.5	126.1	16.8	25.4	2.6	-10.1
1932	24.1	144.2	114.8	4.7	24.2	2.7	-9.3
1933	25.2	141.5	112.8	5.3	23.3	1.7	-2.2
1934	22.0	154.3	118.1	9.4	26.6	1.0	7.4
1935	20.3	169.5	125.5	18.0	27.0	0.8	0.9
1936	17.0	193.2	138.4	24.0	31.8	0.8	0.2
1937	14.3	203.2	143.1	29.9	30.8	0.9	4.2
1938	19.1	192.9	140.2	17.0	33.9	0.8	-1.3
1939	17.2	209.4	148.2	24.7	35.2	0.6	-1.6
1940	14.6	227.2	155.7	33.0	36.4	0.6	1.6

Output, Inputs and TFP During the Great Depression (Ohanian, 2001)

• Theory

$$\frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} - \alpha \frac{\dot{K}}{K} - (1 - \alpha) \frac{\dot{L}}{L}$$

• Data (1929=100)

Year	Y	L	K	A
1930	89.6	92.7	102.5	94.2
1931	80.7	83.7	103.2	91.2
1932	66.9	73.3	101.4	83.4
1933	65.3	73.5	98.4	81.9

• Why did TFP fall so much?

Potential Reasons

- Changes in Capacity Utilization.
- Changes in Quality of Factor Inputs.
- Changes in Composition of Production.
- Labor Hoarding.
- Increasing Returns to Scale.

Other Reasons for Great Depression (Cole and Ohanian 1999)

- Monetary Shocks: Monetary contraction, change in reserve requirements too late
- Banking Shocks: Banks that failed too small
- Fiscal Shocks: Government spending did rise (moderately)
- Sticky Nominal Wages: Probably more important for recovery

Output and Productivity after the Great Depression (Cole and Ohanian, 2003)

• Data (1929=100); data are detrended

Year	Y	A
1934	64.4	92.6
1935	67.9	96.6
1936	74.4	99.9
1937	75.7	100.5
1938	70.2	100.3
1939	73.2	103.1

• Fast Recovery of *A*, slow recovery of output. Why?

Monetary Cycle Models

Jesús Fernández-Villaverde University of Pennsylvania Money, Prices and Output

- What are the effects of changes of money supply on prices and output?
- We will think about two cases:
 - 1. The long run.
 - 2. The short run.

Some Facts about Money in the Long Run

Reported by McCandless and Weber (1995) and Rolnick and Weber (1998):

- 1. There is a high (almost unity) correlation between the rate of growth of monetary supply and the rate of inflation.
- 2. There is no correlation between the growth rates of money and real output.
- 3. There in no correlation between inflation and real output.
- 4. Inflation and money growth are higher under fiat money than under commodity standards.

Money in the Long Run

- Economist have a pretty good idea of how to think about this case.
- Take the Fisher Equation (really an accounting identity):

$$MV = PY$$

• If V and Y are roughly constant then

$$g_m = g_p$$

• "Inflation is always and everywhere a monetary phenomenon", Friedman (1956). Applying the Theory

- Earl J. Hamilton's (1934) "American Treasure and the Price Revolution in Spain, 1501-1650".
- Germany's Hyperinflation.
- Nowadays.

Money and the Cycle

- Two observations:
 - 1. Relation between Output and Money growth.
 - 2. Phillips Curve.
- "Conventional Wisdom":
 - 1. Volcker's recessions,
 - 2. Friedman and Schwartz (1963), "A Monetary History of the US".


Phillips Curve for the US between 1967-99

- It is surprisingly difficult to build models of monetary cycle.
- Two models:
 - 1. Lucas imperfect-information model.
 - 2. Sticky prices-wages model.

Imperfect-Information Model I

• Household makes labor supply decisions based on the real wage:

$$w = \frac{W}{P}$$

• Let us suppose that the substitution effect dominates in each period because of intertemporal substitution:

l'(w) > 0

• Household observes W.

Imperfect-Information Model II

- Economy is hit by shocks to productivity A (that raise w) and to the money supply (that raise P).
- With perfect information on *P*, household just finds:

$$w = \frac{W}{P}$$

and takes labor supply decisions.

• But what happens if P is not observed (or only with some noise)?

Imperfect-Information Model III

- The household observes W going up.
- Signal extraction problem: decide if W goes up because P goes up (since W = wp) or because w went up.
- Response is different. First case labor supply is constant, in the second it should increase.

Imperfect-Information Model IV

- Then money supply surprises affect labor supply and with it total output: a version of the Phillips curve.
- Policy implications:
 - 1. Government may engine an expansion with a surprise.
 - 2. Government cannot get systematic surprises.
 - 3. Time inconsistency problem: Kydland and Prescott (1977).

Imperfect-Information Model V

Criticisms:

- 1. Imperfect-Information assumption. Just get the WSJ.
- 2. Persistence.
- 3. Intertemporal Substitution.
- 4. Wages are acyclical or procyclical.
- 5. Is temporal inconsistency such a big deal? Sargent (2000)

Sticky Prices-Wages Model I

- Prices and Wages are sticky in the short run.
- Different reasons:
 - 1. Menu cost, Mankiw (1995).
 - 2. Staggered contracts, Taylor (1979).
- Empirical evidence.
- Money can have real effects.

Sticky Prices-Wages Model II

- Basic Sticky-prices model.
- A lot of firms that are monopolistic competitors: mark-ups.
- Firms set prices for several periods in advance:
 - 1. Deterministic: Taylor Pricing.
 - 2. Stochastic: Calvo Pricing.
- Firms are ready to supply any quantity of the good at the given price.

Sticky Prices-Wages Model III

- After prices are set up, money shocks occur.
- If the shock is positive, interest rate goes down and demand for goods goes up.
- Firms produce more, output goes up, unemployment goes down.
- Persistence problem: will firms change prices soon?

Sticky Prices-Wages Model IV

- Case with sticky wages is similar.
- After money shock, prices go up, real wages go down and firms produce more.
- Real wages may fail to fall in the long run: efficiency wages.

Sticky Prices-Wages Model V

- Again we have a Phillips curve.
- Policy implications:
 - 1. Government wants to keep inflation low and stable.
 - 2. Government may use monetary policy to stabilize the economy.
- Taylor's rule (Taylor, 1993, Rothemberg and Woodford, 1997).
- Are monetary rules normative or positive?

Sticky Prices-Wages Model V

Criticisms:

- 1. Why are prices and wages sticky?
- 2. Persistence problem.
- 3. Problem interpreting empirical evidence.
- 4. Wages behavior over the cycle.
- 5. Size problem: account for 15-25% of the output variability.

Coordination Failure Cycles

Jesús Fernández-Villaverde University of Pennsylvania

Externalities

- Externalities can be important.
- For instance: how much would you pay for a phone if you have the ONLY phone on earth?
- Your utility may depend on what other agents do: strategic complementarities.
- Develop by Diamond (1982) in search models.

Coordination

- Coordination issues are also important.
- Driving: UK versus US.
- We have two different Equilibria!

A Simple Model

- Cooper and John (1988).
- Utility of the agent \boldsymbol{i}

$$U_i = V\left(y_i, y\right)$$

- Best-Response $y'_i(y)$.
- Equilibrium:

$$y = y_i'(y)$$

Multiple Equilibria

- The function $y = y'_i(y)$ may have several solutions.
- Each solution is a different equilibrium.
- Which one will be selected? Opens door to sunspots and self-fulfilling prophecies.
- Equilibria may be pareto-ranked.

Example of Multiple Equilibria

- Increasing returns to scale in the production function (Farmer-Guo, 1994).
- If a lot of us work, marginal productivity is high, wages are high and as a consequence a lot of us want to work: we have an equilibrium.
- If few of us work, marginal productivity is low, wages are low and as a consequence few of us want to work: we have another, worse, equilibrium.
- Fluctuations are just jumps from one equilibrium to another.

Household Problem I

- Continuum of households $i \in [0, 1]$.
- Each household *i* has a backyard technology:

$$y_{it} = A_t k_{it}^{\alpha} l_{it}^{1-\alpha}$$

• We can rewrite it as:

$$\max E \sum_{t=1}^{\infty} eta^{t-1} u\left(c_{t}, 1-l_{t}
ight)$$
 $c_{t}+k_{t+1}=y_{it}+\left(1-\delta
ight)k_{t}, \,orall \, t>$

0

Externalities I

• The externality is given by:

$$A_t = \left(\int_i k_{it}^{\alpha} l_{it}^{1-\alpha} di\right)^{\theta}$$

- Interpretation.
- Alternatives: monopolistic competition.

Externalities II

• Since all the agents are identical:

$$y_t = k_t^{\alpha(1+\theta)} l_t^{(1-\alpha)(1+\theta)} = k_t^{\upsilon} l_t^{\mu}$$

- Increasing returns to scale production function at the aggregate level...
- but constant at the household level.

Solving the Model I

- We can solve the model as we did for the Real Business Model.
- Things only get a bit more complicated because we need to allow for the presence of shocks to beliefs.
- Instead of postulating policy functions of the form $(k_{t+1} k) = P(k_t k) + Qz_t$ and $(l_t l) = R(k_t k) + Sz_t$, we propose:

$$(k_{t+1} - k) = P(k_t - k) + Q(c_t - c)$$

(c_{t+1} - c) = R(k_t - k) + S(c_t - c) + \varepsilon_t

Solving the Model II

- Technical discussion (skip if wanted): we need as many state variables as stable roots of the system.
- $(c_t c)$ is not really a "pure" state variable.
- We will omit the details in the interest of time saving.

Dynamics of the Model I

- Interaction between the labor supply curve and the labor demand curve.
- Possibility of self-fulfilling equilibria.
- We have seen those before: money.

Dynamics of the Model II

- Note that we have recurrent shocks to beliefs.
- The dynamics of the model are stochastic.
- This means we need to simulate as in the case of the standard RBC.

Deterministic versus Stochastic Dynamics

- This is different from models that exhibit Chaos.
- It is relatively easy to build equilibrium models with permanent deterministic cycles (Benhabib-Boldrin, 1992).
- Tent mapping example.
- Why do economist use few models with chaotic dynamics?

Evaluation of the Model

- Comparison with standard RBC.
- Policy implications: government may play a role since different equilibria are pareto-ranked.
- Do we observe big increasing returns to scale in the US economy? Insufficient variation of the inputs data (Cole and Ohanian (1999)).

Introduction to Optimization

Jesús Fernández-Villaverde University of Pennsylvania

1 Unconstrained Maximization

- Consider a function $f: \Re^N \to \Re$.
- We often called this function the objective function.

Definition 1.1 A set A is an open neighborhood of x if $x \in A$ and A is an open set.

Definition 1.2 The vector $x^* \in \Re^N$ is a local maximizer of $f(\cdot)$ if \exists an open neighborhood A of x such that for $\forall x \in A : f(x^*) \ge f(x)$

Definition 1.3 The vector $x^* \in \Re^N$ is a global maximizer of $f(\cdot)$ if for $\forall x \in \Re^N : f(x^*) \ge f(x)$

- The concepts of local and global minimizer are defined analogously.
- Clearly every global maximizer is a local maximizer but the converse is not true.

Theorem 1.1 (Necessity) Suppose that $f(\cdot)$ is differentiable and that $x^* \in \Re^N$ is a local maximizer or minimizer of $f(\cdot)$. Then:

$$rac{\partial f\left(x^{*}
ight)}{\partial x_{n}}=\mathsf{0}\qquadorall n\in\left\{1,...,N
ight\}$$

or using a more concise notation:

$$\nabla f\left(x^*\right) = \mathbf{0}$$

Proof. Suppose that x^* is a local maximizer of $f(\cdot)$ but that $\frac{\partial f(x^*)}{\partial x_n} = a > 0$ (an analog argument holds if a < 0 or for local minimizers). Define the vector $e^n \in \Re^N$ as $e_n^n = 1$ and $e_h^n = 0$ for $h \neq n$ (i.e. having its nth entry equal to 1 and all the other entries equal to 0). Then, by the definition of partial derivative $\exists \varepsilon > 0$ arbitrarily small such that $[f(x^* + \varepsilon e^n) - f(x^*)]/\varepsilon > a/2 > 0$. But then $f(x^* + \varepsilon e^n) > f(x^*) + (\varepsilon a/2) f(x^*)$, a contradiction with the fact that x^* is a local maximizer.

Definition 1.4 A vector $x^* \in \Re^N$ is a critical point if $\nabla f(x^*) = 0$

Corollary 1.2 Every local maximizer or minimizer is a critical point.

Corollary 1.3 Some critical points are not local maximizers or minimizers.

Example 1.1 Consider the function $f(x_1, x_2) = x_1^2 - x_2^2$. We have that $\nabla f(0, 0) = 0$ but that point is neither a local maximizer or minimizer.

Definition 1.5 The $N \times N$ matrix M is negative semidefinite if $z'Mz \leq 0$

for $\forall z \in \Re^N$. If the inequality is strict then M is negative definite.

An analogous definition holds for positive semidefinite and positive definite matrices.

Theorem 1.4 A Matrix M is positive semidefinite (respectively, positive definite) if and only if the matrix -M is negative semidefinite (respectively, negative definite).
Theorem 1.5 Let M be an $N \times N$ matrix.

- 1. Suppose that M is symmetric. Then M is negative definite if and only if $(-1)^r |_r M_r| > 0$ for every r = 1, ..., N.
- 2. Suppose that M is symmetric. Then M is negative semidefinite if and only if $(-1)^r |_r M_r^{\pi}| > 0$ for every r = 1, ..., N and for every permutation $\pi = 1, ..., N$ of the indices of rows and columns.

Opposite results will hold for positive semidefinite and positive definite matrices.

Theorem 1.6 (Sufficiency) Suppose that $f(\cdot)$ is twice continuous differentiable and that $\nabla f(x^*) = 0$.

- 1. If $x^* \in \Re^N$ is a local maximizer then the symmetric $N \times N$ matrix $D^2 f(x^*)$ is negative semidefinite.
- 2. If $D^2f(x^*)$ is negative definite x^* is a local maximizer.

Replacing negative for positive the same is true for local minimizers.

Proof. (Outline) Take a second order expansion of $f(\cdot)$ around the local maximizers. Note that the function less the first (constant) term is negative. Since the linear term is zero by our previous result the second order term also must be negative. For that to hold $D^2f(x^*)$ must be negative semidefinite (higher order terms can be safely ignored).

Conversely if $D^2 f(x^*)$ is negative definite all points in a local neighborhood must have lower values.

• Often the condition

$$\nabla f\left(x^*\right) = \mathbf{0}$$

is called the first order condition and the fact that $D^2 f(x^*)$ is negative semidefinite a second order condition.

 Finally note that any critical point of a concave function is a global maximizer for that function and analogously any critical point of a convex function is a global minimizer. Example 1.2 Consider the function $f(x) = -ax^2 + bx$ where a and b are constant.

The first order condition is:

$$f'(x^*) = -2ax^* + b = 0 \Rightarrow x^* = b/(2a)$$

and the second order condition is:

$$f^{\prime\prime}(x^*)=-2a<\mathsf{0}$$

Then if a > 0 $x^* = b/(2a)$ is a global maximum.

Example 1.3 Consider the function $f(x_1, x_2) = -ax_1^2 - bx_2^2 + cx_1x_2 - x_1 - x_2$ where a, b > 0 and $4ab - c^2 > 0$.

The first order conditions are:

$$egin{array}{rl} \displaystyle rac{\partial f\left(x^{*}
ight)}{\partial x_{1}}&=&-2ax_{1}+cx_{2}-1=0\ \displaystyle rac{\partial f\left(x^{*}
ight)}{\partial x_{2}}&=&-2bx_{2}+cx_{1}-1=0 \end{array}$$

with solution:

$$x_1^* = \frac{c+2b}{c^2-4ba}$$
$$x_2^* = \frac{c+2a}{c^2-4ba}$$

To check the second order condition note that:

$$D^{2}f\left(x^{*}
ight)=\left(egin{array}{cc} -2a & c \ c & -2b \end{array}
ight)$$

clearly negative definite given our assumptions on the parameters.

- If the second order conditions do not hold we need to examine the logic of the problem to find whether we have a local or global maximizer.
- Also note that since we look at second derivatives for the case where the objective function is linear we cannot assure sufficiency. Again we need to study the specific conditions of the problem.

2 Constrained Optimization

We study the problem:

where
$$f: \Re^N o \Re$$
 and $g_m: \Re^N \to \Re$ for $m = 1, ..., M$.

Definition 2.1 The constraint set is

$$C = \left\{ x \in \Re^N : g_m(x) = b_m \text{ for } m = 1, ..., M \right\}$$

Definition 2.2 The vector $x^* \in C$ is a local constrained maximizer of $f(\cdot)$ if \exists an open neighborhood A of x such that for $\forall x \in A \cap C : f(x^*) \geq f(x)$

Definition 2.3 The vector $x^* \in C$ is a global constrained maximizer of $f(\cdot)$ if for $\forall x \in C : f(x^*) \ge f(x)$

The concepts of local and global constrained minimizer are defined analogously. To solve this problem we build an auxiliary function called the *Lagrangian*:

$$\mathcal{L}(x) = f(x) + \sum_{m=1}^{M} \lambda_m (b - g_m (x))$$

where the numbers λ_m are called the *Lagrange multipliers*.

Theorem 2.1 Suppose that $f : \Re^N \to \Re$ and $g_m : \Re^N \to \Re$ for m = 1, ..., M are differentiable and that x^* is a local constrained maximizer. Then \exists numbers λ_m such that:

$$abla f\left(x^{*}
ight)=\sum_{m=1}^{M}\lambda_{m}
abla g_{m}\left(x
ight)$$

The theorem shows that the constrained maximizer is a critical point of the lagrangian function for an appropriate choice of λ_m .

The Lagrangian multipliers have a nice interpretation as the shadow prices of the constraints.

Example 2.1 Consider the function $f(x_1, x_2) = -ax_1^2 - bx_2^2 - x_1 - x_2$. Suppose we want to maximize it subject to the constraint that:

$$4x_1 - x_2 = 0$$

Then we build the lagrangian:

$$-ax_1^2 - bx_2^2 - x_1 - x_2 + \lambda(x_2 - 4x_1)$$

with the first order condition:

$$\begin{array}{rcl} -2ax_1 -1 &=& 4\lambda \\ -2bx_2 -1 &=& -\lambda \end{array}$$

and the constraint:

$$4x_1 - x_2 = 0$$

i.e. a system of three equations on three unknowns with solution:

$$x_{1} = \frac{-5}{2(a+16b)}$$

$$x_{2} = \frac{-20}{2(a+16b)}$$

$$\lambda = \frac{20bx_{2}}{2(a+16b)} - 1$$

The second order theory for constrained problems is simple: we apply all previous results regarding the second derivatives matrix to the function $\mathcal{L}(x)$ instead of f(x).

The proof that the second order conditions hold for the previous example is left to you.