Continuous Time Stochastic Processes

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Filtration

- Fix a probability space $(\Omega, \mathcal{F}, P)$.
- Define $t \in [0, \infty) = \mathbb{R}_+$.
- Filtration: a family $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ of increasing $\sigma-$algebras contained in $\mathcal{F}$:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \text{ for } \forall s \leq t \text{ and } \mathcal{F}_t \subseteq \mathcal{F}$$

- Clearly, $\mathcal{F}_\infty = \mathcal{F}$ is the smallest $\sigma-$algebras containing $\forall \mathcal{F}_t$.
- $(\Omega, \mathbb{F}, P)$: filtered probability space.

Stochastic Processes

- Continuous-time stochastic process: a mapping $x : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ which is measurable with respect to $\mathcal{B}_{+} \times \mathcal{F}$ (where $\mathcal{B}_{+}$ are the Borel sets of $\mathbb{R}_{+}$).
- A stochastic process is adapted to $\mathbb{F}$ if $x(t, \omega)$ is $\mathcal{F}_t$-measurable $\forall t$.
- $x(t, \cdot)$: $\mathcal{F}_t$-measurable function of ω.
- $x(\cdot, \omega)$: realization, trajectory, or sample path of the process.
- Continuous stochastic process: $x(\cdot, \omega) \in C[0, \infty)$, a.e. $\omega \in \Omega$.

Brownian Motions: Definition

- A Wiener processes (or Brownian motion) is a stochastic process W having:
  1. continuous sample paths.
  2. independent increments.
  3. $W(t) \sim \mathcal{N}(0, t), \forall t$.

Basic Result

If a stochastic process $\{X(t), t \geq 0\}$ has continuous sample paths with stationary, independent, and i.i.d. increments, then it is a Wiener process.

- Differential:

$$dW = \lim_{dt \downarrow 0} \left(W(t + dt) - W(t) \right)$$

Properties of Differentials

- Moments:
  1. $\mathbb{E}\left[dW\right] = 0.$
  2. $\mathbb{E}\left[(dW)^2\right] = dt.$
- Also, as $dt \rightarrow 0$ (we skip the proof):
  1. $dW \sim o\left(\sqrt{dt}\right).$
  2. $(dW)^2 \rightarrow \mathbb{E}\left[(dW)^2\right] = dt.$
- Note that, while $W(t)$ has a continuous path, it is not differentiable:

$$\frac{dW}{dt} = \frac{o\left(\sqrt{dt}\right)}{dt} \rightarrow \infty \text{ as } dt \rightarrow 0$$

Diffusions I

- A Brownian motion with drift:

$$dX(t) = \mu dt + \sigma dW(t) \text{ with } X(0) = x_0$$

- More generally, we have a diffusion:

$$dX(t) = \mu(t, x)\, dt + \sigma(t, x)\, dW(t) \;\; \forall t, \forall \omega$$

- Properties:

  1. $\mathbb{E}_t[dX] = \mu(t, x)\, dt.$
  2. $var_t[dX] = \sigma^2(t, x)\, dt.$

Diffusions II

- Diffusion are important in arbitrage-free asset pricing. Aït-Sahalia (2006).

- Particularly useful cases are:

  1. Geometric Brownian motion

$$dX = \mu X dt + \sigma X dW$$

  2. Ornstein-Uhlenbeck process

$$dX = -\theta (X - \mu) dt + \sigma (t, x) X dW$$

Functions of Stochastic Processes I

- Let $F(t, x)$ be a function that is at least once differentiable in t and twice in x.
- We approximate the total differential of $F(t, X(t, \omega))$ by a Taylor expansion:

$$dF = F_t dt + F_x dX + \frac{1}{2} F_{tt} (dt)^2 + \frac{1}{2} F_{xx} (dX)^2 + F_{xt} dt (dX) + ...$$

- We substitute in:

$$\begin{aligned} dF = \; & F_t dt + F_x [\mu dt + \sigma dW] \\ & + \frac{1}{2} F_{tt} (dt)^2 \\ & + \frac{1}{2} F_{xx} \left[\mu^2 (dt)^2 + 2\mu\sigma dt dW + \sigma^2 (dW)^2 \right] \\ & + F_{xt} dt (\mu dt + \sigma dW) \\ & + H.O.T ... \end{aligned}$$

Functions of Stochastic Processes II

- Itō's lemma: we can drop the terms that have order higher than dt or $(dW)^2$ and use $(dW)^2 \rightarrow dt$:

$$\begin{aligned} dF &= F_t dt + F_x \left[\mu dt + \sigma dW\right] + \frac{1}{2}\sigma^2 F_{xx} (dW)^2 \\ &= \left(F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}\right) dt + \sigma F_x dW \end{aligned}$$

- Since $\mathbb{E}\left[dW\right] = 0$,

$$\begin{aligned} \mathbb{E}\left[dF\right] &= \left[F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}\right] dt \\ Var\left[dF\right] &= \mathbb{E}\left[dF - \mathbb{E}\left[dF\right]\right]^2 = \sigma^2 F_x^2 dt \end{aligned}$$

Functions of Stochastic Processes III

- Particular case $F(t,x) = e^{-rt} f(x)$:

$$\mathbb{E}[dF] = \left[-rf + \mu f' + \frac{1}{2}\sigma^2 f''\right] e^{-rt} dt$$

and when $r = 0$ (that is, $F(t,x) = f(x)$, an often relevant case in economics)

$$\mathbb{E}[dF] = \left[\mu f' + \frac{1}{2}\sigma^2 f''\right] dt$$