Search-Theoretic Models of Money

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Motivation

- What is money?
- Why do we use money as a society?
- Use search theory to model existence of money.
- Contrast with other two approaches:
  1. Money in DSGE models (CIA, MIU).
  2. Money in OLG models.

Essential models of money

- Hahn (1965): money is essential if it allows agents to achieve allocations they cannot achieve with other mechanisms that also respect the enforcement and information constraints in the environment.
- Why do we care about essential models of money?
- Three frictions that will make money essential:
  1. Double-coincidence of wants problem.
  2. Long-run commitment cannot be enforced.
  3. Agents are anonymous: histories are not public information.
- Money is a consequence of these frictions in trade: medium of exchange.

Three generations of models

1. 1 unit of money, 1 unit of good: Kiyotaki and Wright (1993).
2. 1 unit of money, endogenous units of good: Trejos and Wright (1995).
3. Endogenous units of money, endogenous units of good: Lagos and Wright (2005).

First generation: environment

- $[0, 1]$ continuum of anonymous agents.
- Live forever and discount future at rate r.
- $[0, 1]$ continuum of goods. Good i is produced by agent i.
- Goods are non-storable: no commodity money.
- Unit cost of production $c \geq 0$.

Double-coincidence of wants problem

- I do not produce what I like (non-restrictive: home production, specialization).
- iWj: agent i likes to consume good produced by agent j:
  1. utility $u > c$ from consuming j.
  2. utility 0 otherwise.
- Probabilities of matching:

$$p\left(iWi\right) = 0$$
$$p\left(jWi\right) = x$$
$$p\left(jWi|iWj\right) = y$$

Fixed money and fixed good

- Exogenously given quantity $M \in [0, 1]$ of an indivisible unit of storable good.
- Holding money yields zero utility γ: fiat money.
- Initial endowment: M agents are randomly endowed with one unit of money.
- Agents holding money cannot produce (for example because you need to consume before you can produce again).
- We eliminate (non-trivial) distributions.

Trades

- Pairwise random matching of agents with Poisson arrival time α.
- Bilateral trading is important, randomness is not (Corbae, Temzelides, and Wright, 2003).
- Upon meeting, agents decide whether to trade. Then, they part company and re-enter the process.
- History of previous trades is unknown.
- Exchange 1 unit of good for 1 unit of good (barter) or 1 unit of money.

Individual trading strategies

- Agents never accept a good in trade if he does not like to consume it since it is not storable.
- They will barter if they like the both agents in the pair like each other goods.
- Would they accept money for goods and vice versa?
- We will look at stationary and symmetric Nash equilibria.

Probabilities

- You meet someone with arrival rate α.
- This person can produce with probability $1 - M$.
- With probability x you like what he produces.
- With probability $\pi = \pi_0 \pi_1$ (endogenous objects to be determined) both of you want to trade.
- If $\pi > 0$, we say that money circulates.

Value functions

- Value functions with money, V_1:

$$rV_1 = \alpha x\left(1-M\right)\pi\left(u+V_0-V_1\right)$$

- Value functions without money, V_0:

$$rV_0 = \alpha xy\left(1-M\right)\left(u-c\right)+\alpha xM\pi\left(V_1-V_0-c\right)$$

- Renormalize $\alpha x = 1$ by picking time units:

$$rV_1 = \left(1-M\right)\pi\left(u+V_0-V_1\right)$$
$$rV_0 = y\left(1-M\right)\left(u-c\right)+M\pi\left(V_1-V_0-c\right)$$

Individual trading strategies

- Net gain from trading goods for money:

$$\Delta_0 = V_1 - V_0 - c = \frac{(1-M)(\pi - y)(u-c) - rc}{r+\pi}$$

- Net gain from trading money from goods:

$$\Delta_1 = u + V_0 - V_1 = \frac{(M\pi + (1-M)y)(u-c) + ru}{r+\pi}$$

- Clearly:

$$\pi_j \begin{cases} = 1 \\ \in [0,1] \\ = 0 \end{cases} \text{as } \Delta_j \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

- Plug those into the individual trading strategies, and check them.

Characterization

- Clearly $\Delta_1 > 0$. Hence $\pi_1 = 1$, i.e., the agent with money always wants to trade.
- For π_0, you have:

$$\Delta_0 = \frac{(1-M)(u-c)\pi_0}{r+\pi_0} - \frac{(1-M)y(u-c)+rc}{r+\pi_0}$$

- Then, Δ_0 has the same sign as:

$$\pi_0 - \frac{rc+(1-M)y(u-c)}{(1-M)(u-c)} = \pi_0 - \widehat{\pi}$$

Multiple equilibria

- Non-monetary equilibrium: we have an equilibrium where $\pi_0 = 0$.
- Monetary equilibrium: if

$$c < \frac{(1-M)(1-y)}{r+(1-M)(1-y)}u$$

  then $\widehat{\pi} < 1$ and $\pi_0 = 1$ is an equilibrium as well.
- Mixed-monetary equilibrium: $\pi_0 = \widehat{\pi}$. However, not robust (Schevchenko and Wrigth, 2004).

Welfare I

- Define welfare as the average utility:

$$W = MV_1 + (1 - M)\, V_0$$

- Then:

$$rW = (1 - M)\left[(1 - M)\, y + M\pi\right](u - c)$$

- Note that welfare is increasing in π.

Welfare II

- When $\pi = 1$:

$$rW = (1 - M)\left[(1 - M)\, y + M\right](u - c) = -My + M^2 y$$

- Maximize W with respect to M:

$$\begin{aligned} M^* &= \frac{1 - 2y}{2 - 2y} \text{ if } y < \frac{1}{2} \\ M^* &= 0 \text{ if } y \geq \frac{1}{2} \end{aligned}$$

- Intuition: facilitate trade versus crowding out barter.

Welfare III

- When $\pi = 0$:
$$rW = (1 - M)\left[(1 - M)\, y\right](u - c)$$
- Monotonically decreasing in $M \Rightarrow M^* = 0$.
- Result is a little bit silly: it depends on the absence of free disposal of money. Otherwise, welfare is independent of M.
- For a general π
$$rW = (1 - M)\left[(1 - M)\, y + M\pi\right](u - c)$$
monotonically increasing in M in the $[0, 1]$ interval.

Comparison with alternative arrangements

- Imagine that we have the credit arrangement: "produce for anyone you meet that wants your good."
- Value function

$$rV_c = u - c$$

- Clearly

$$rV_c > rW$$

- However, this arrangement is not self-enforceable: histories are not observed.

Second generation: endogenous prices

- We make the very strong assumption that we exchanged one good for one unit of money.
- What if we let prices to be endogenous? Shi (1995) and Trejos and Wright (1995).
- We set $y = 0$ and we let goods to be divisible.
- When agents meet, they bargain about how much q will be exchanged, or equivalently, about price $1/q$.

Utility and cost functions

- Utility is $u(q)$ and cost of production is $c(q)$.
- Assumptions:

$$u(0) = c(0) = 0$$
$$u'(0) > c(0)$$
$$u'(0) > 0, u''(0) \leq 0$$
$$c'(0) > 0, c''(0) \geq 0$$

- Also, $\widehat{q}$ and q^* are such that

$$\begin{aligned} u'(\widehat{q}) &= c'(\widehat{q}) \\ u'(q^*) &= c'(q^*) \end{aligned}$$

Value functions and bargaining

- Take $q = Q$ as given. Then:

$$rV_1 = (1 - M)\left[u(Q) + V_0 - V_1\right]$$
$$rV_0 = M\left[V_1 - V_0 - c(Q)\right]$$

- Bargaining is the generalized Nash bargaining solution:

$$q = \arg\max\left[u(q) + V_0(Q) - T_1\right]^{\theta} \times \left[V_1(Q) - c(Q) - T_0\right]^{\theta}$$
$$u(q) + V_0 \geq V_1$$
$$V_1 - c(q) \geq V_0$$

where T_j is the threat point of the agent with j units of money.

- We will set $T_j = 0$ and $\theta = 1/2$.

Equilibria

- Necessary condition taking $V_0(Q)$ and $V_1(Q)$ as given:

$$[V_1(Q) - c(q)]\, u'(q) = [u(q) + V_0(Q)]\, c'(q)$$

- The bargaining solution defines a function:

$$q = e(Q)$$

  and we look at its fixed points.

- Two fixed points:

  1. $q = 0$: nonmonetary equilibrium.
  2. $q = q^e > 0$: monetary equilibrium.

Efficiency

- Note that the efficient outcome is q^*, i.e. $u'(q^*) = c'(q^*)$.
- In the monetary equilibrium:

$$u'(q^e) = \frac{u(q^e) + V_0(q^e)}{V_1(q^e) - c(q^e)} c'(q^e) > u'(q^*)$$

since $u(q^e) + V_0(q^e) > V_1(q^e) - c(q^e)$.

- Hence $q^* > q^e$, or equivalently, the price is too high.

Third generation: endogenous prices and goods

- Relax the assumption that agents hold 0 or 1 units of money.
- Problem: endogenous distribution of money that we (and the agents!) need to keep track of.
- Computational: Molico (2006).
- Theoretical:
  1. Families: Shi (1997).
  2. Two markets: Lagos and Wright (2005).

Environment

- Discrete time.
- $[0,1]$ continuum of infinitely-lived agents with $\beta \in (0,1)$.
- Each period is divided between two subperiods: day and night.
- Utility function:

$$u(x) - c(h) + U(X) - H$$

  where x (X) is consumption during the day (night) and h (X) is labor during the day (night).

- $u(0) = 0$, $u' > 0$, $u'' < 0$, $c(0) = 0$, $c' > 0$, $c'' \geq 0$, $U' > 0$, and $U'' \leq 0$.
- Also, assume that $\exists (q^*, X^*) \in (0, \infty)^2$ s.t. $u'(q^*) = c'(q^*)$ and $U'(X^*) = 1$, $U(X^*) > X^*$.
- Key point: linearity on H.

Money

- Money holdings $m \geq 0$, perfectly divisible and storable.
- Total amount of money is M.
- $F_t(\widetilde{m})$: CDF of agents with $m \leq \widetilde{m}$ at start of day market at t.
- $G_t(\widetilde{m})$: CDF of agents with $m \leq \widetilde{m}$ at start of night market at t.
- Therefore:

$$\int m dF_t(\widetilde{m}) = \int m dG_t(\widetilde{m}) = M \text{ for all } t$$

- ϕ_t: price of money in the centralized market (inverse of the price level).

Day trading I

- Decentralized market with anonymous bilateral matching and no record-tracking technology.
- Probability of meeting: α.
- x comes in many varieties. Each agent consumes only a subset.
- Each agent can transform h one for one into one of these special goods that he himself does not consume.
- In any random meeting between to agents:
    1. Double coincidence (both agents consume what the other can produce): probability δ.
    2. Single coincidence (one agent, i.e., a "buyer" , consumes what the other can produce, i.e., a "seller" , but not the other way around): probability σ.
    3. No coincidence (neither agent consumes what the other can produce): probability $1 - 2\sigma - \delta$.

Day trading II

- m: dollars of the buyer.
- $\widetilde{m}$: dollars of the seller.
- $q_t\left(m, \widetilde{m}\right)$: quantity producer by seller (bought by buyer).
- $d_t\left(m, \widetilde{m}\right)$: dollars paid by the buyer to the seller.
- $B_t\left(m, \widetilde{m}\right)$: payoff in double coincidence meetings.

Night trading and money

- Centralized market on an homogenous good X.
- Homogeneity is irrelevant with Walrasian markets.
- We could also allow intertemporal claims in the night markets, but they will not trade in equilibrium.
- Each agent can transform H one for one into X.
- Also, x and X are perfectly divisible, but non-storable.

Value functions I

- $V_t(m)$: value function for an agent with m dollars when he enters the decentralized market:

$$
\begin{aligned}
V_t(m) \quad = \quad & \underbrace{\alpha\sigma \int u\left(q_t(m,\widetilde{m})\right) + W_t\left(m - d_t(m,\widetilde{m})\right) dF_t(\widetilde{m})}_{\text{Buying}} \\
& \underbrace{+\alpha\sigma \int -c\left(q_t(\widetilde{m},m)\right) + W_t\left(m + d_t(\widetilde{m},m)\right) dF_t(\widetilde{m})}_{\text{Selling}} \\
& \underbrace{+\alpha\delta \int B_t(m,\widetilde{m})\, dF_t(\widetilde{m})}_{\text{Bartering}} \\
& \underbrace{+(1 - 2\alpha\sigma - \alpha\delta)\, W_t(m)}_{\text{Not trading}}
\end{aligned}
$$

Value functions II

- $W_t(m)$: value function for an agent with m dollars when he enters the centralized market:

$$W_t(m) = \max_{X,H,m'} \{U(X) - H + \beta V_{t+1}(m')\}$$
$$\text{s.t. } X = H + \phi_t (m - m')$$

- Substituting the budget constraint:

$$\begin{aligned} W_t(m) &= \phi_t m + \max_{X,H} \{U(X) - X\} + \max_{m'} \{-\phi_t m' + \beta V_{t+1}(m')\} \\ &= \phi_t m + U(X^*) - X^* + \max_{m'} \{-\phi_t m' + \beta V_{t+1}(m')\} \end{aligned}$$

  where we have used the optimality condition $U'(X^*) = 1$.
- Therefore:
  1. All agents consume the same amount X^*.
  2. $W_t(m)$ is linear in m with slope ϕ_t.
  3. Some work is necessary to show that the solution to this problem is interior.

Terms of trade: double coincidence I

- Nash bargaining with symmetric power and threat points $W_t(m)$ and $W_t(\widetilde{m})$.
- Bargaining on $q_i, q_j \geq 0$ and $\Delta \leq m$:

$$\max_{q_i, q_j, \Delta} \begin{array}{c} \left[u(q_j) - c(q_i) + \left(W_t(m - \Delta) - W_t(m)\right)\right]^{1/2} * \\ \left[u(q_i) - c(q_j) + \left(W_t(\widetilde{m} + \Delta) - W_t(\widetilde{m})\right)\right]^{1/2} \end{array}$$

$$s.t.\ \Delta \leq m_i \text{ and } -\Delta \leq m_j$$

- Given the linearity of $W_t(m)$ on m, we can rewrite the problem:

$$\max_{q_i, q_j, \Delta} \log\left[u(q_j) - c(q_i) - \phi_t \Delta\right] + \log\left[u(q_i) - c(q_j) + \phi_t \Delta\right]$$

$$s.t.\ \Delta \leq m_i \text{ and } -\Delta \leq m_j$$

Terms of trade: double coincidence II

- Optimality conditions:

$$
\begin{aligned}
q_i &: \quad \frac{c'(q_i)}{u(q_j) - c(q_i) - \phi_t \Delta} = \frac{u'(q_i)}{u(q_i) - c(q_j) + \phi_t \Delta} \\
q_j &: \quad \frac{u'(q_j)}{u(q_j) - c(q_i) - \phi_t \Delta} = \frac{c'(q_j)}{u(q_j) - c(q_i) + \phi_t \Delta} \\
\Delta &: \quad u(q_j) - c(q_i) - \phi_t \Delta = u(q_j) - c(q_i) + \phi_t \Delta
\end{aligned}
$$

- Thus:

$$q^* = q_i = q_j, \text{ s.t. } u'(q^*) = c'(q^*)$$

$$\Delta = 0$$

and

$$B_t(m, \widetilde{m}) = u(q^*) - c(q^*) + W_t(m)$$

Terms of trade: single coincidence I

- Nash bargaining with power $\theta > 0$ for buyer and threat points $W_t(m)$ and $W_t(\widetilde{m})$.
- Bargaining on $q \geq 0$ and $d \leq m$:

$$\max_{q,d} \left\{ \begin{array}{c} \left[u(q) + W_t(m-d) - W_t(m)\right]^{\theta} * \\ \left[-c(q) + W_t(\widetilde{m}+d) - W_t(\widetilde{m})\right]^{1-\theta} \end{array} \right\}$$

- Given the linearity of $W_t(m)$ on m, we can rewrite the problem:

$$\max_{q,d} \frac{\theta}{1-\theta} \log\left[u(q) - \phi_t d\right] + \log\left[-c(q) + \phi_t d\right]$$

Terms of trade: single coincidence II

- Define:

$$m_t^* = \frac{1}{\phi_t}\left[(1-\theta)\,u\left(q^*\right) + \theta c\left(q^*\right)\right]$$

- Claim: solution to bargaining problem is

$$q_t\left(m, \tilde{m}\right) = \begin{cases} \hat{q}_t\left(m\right) \text{ if } m < m_t^* \\ q^* \text{ if } m \geq m_t^* \end{cases}$$

  and

$$d_t\left(m, \tilde{m}\right) = \begin{cases} m \text{ if } m < m_t^* \\ m_t^* \text{ if } m \geq m_t^* \end{cases}$$

  where $\hat{q}_t\left(m\right)$ solves:

$$m = \frac{1}{\phi_t}\frac{(1-\theta)\,u\left(q\right)c'\left(q\right) + \theta c\left(q\right)u'\left(q\right)}{(1-\theta)\,c'\left(q\right) + \theta u'\left(q\right)} = \frac{1}{\phi_t}z\left(q\right).$$

- Note that this solution does not depend on $\tilde{m}$.

Proof I

- Optimality conditions if $d \leq m$:

$$
\begin{aligned}
q &: \quad \frac{\theta}{1-\theta}\frac{u'(q)}{u(q)-\phi_t d} = \frac{c'(q)}{-c(q)+\phi_t d} \\
d &: \quad \frac{\theta}{1-\theta}\frac{1}{u(q)-\phi_t d} = \frac{1}{-c(q)+\phi_t d}
\end{aligned}
$$

- Thus:

$$\frac{\theta}{1-\theta}\frac{-c(q)+\phi_t d}{u(q)-\phi_t d} = 1 \Rightarrow u'(q^*) = c'(q^*)$$

and:

$$d^* = \frac{1}{\phi_t}\left[(1-\theta)u(q^*) + \theta c(q^*)\right] = m_t^*$$

Proof II

- Optimality conditions if $d = m$, we still have

$$q: \frac{\theta}{1-\theta}\frac{u'(q)}{u(q)-\phi_t d} = \frac{c'(q)}{-c(q)+\phi_t d}$$

but now evaluated at:

$$\frac{\theta}{1-\theta}\frac{u'(q)}{u(q)-\phi_t m} = \frac{c'(q)}{-c(q)+\phi_t m}$$

or

$$d_t(m) = m = \frac{1}{\phi_t}\underbrace{\frac{(1-\theta)u(q)c'(q)+\theta c(q)u'(q)}{(1-\theta)c'(q)+\theta u'(q)}}_{z(q)} = \frac{1}{\phi_t}z(q)$$

Characterizing strategies

- Remember $m = \frac{1}{\phi_t} z\left(q\right)$.
- Then, for any $m < m_t^*$, it follows from the *implicit function theorem* that:

$$q_t'\left(m\right) = \hat{q}_t'\left(m\right) = \frac{\phi_t}{z'\left(q_t\left(m\right)\right)}$$

- Also, remember that:

$$z\left(q\right) = \frac{\left(1-\theta\right) u\left(q\right) c'\left(q\right) + \theta c\left(q\right) u'\left(q\right)}{\theta u'\left(q\right) + \left(1-\theta\right) c'\left(q\right)}$$

  and then (where all functions are evaluated at q):

$$z' = \frac{u'c'\left[\theta u' + \left(1-\theta\right) c'\right] + \theta\left(1-\theta\right)\left(u-c\right)\left(u'c'' - c'u''\right)}{\left[\theta u' + \left(1-\theta\right) c'\right]^2} > 0 > 0$$

- $\hat{q}_t\left(m\right) \rightarrow q^*$ as $m \rightarrow m^*$.
- $\hat{q}_t\left(m\right)$ is increasing for any $m < m_t^*$ and continuous.

Equilibrium value function I

- Then:

$$
\begin{aligned}
V_t(m) \quad = \quad & \underbrace{\alpha\sigma\left[u\left(q_t(m)\right) - \phi_t d_t(m)\right] + \alpha\sigma W_t(m)}_{\text{Buying}} \\
& + \underbrace{\alpha\sigma \int \left[-c\left(q_t(\tilde{m})\right) + \phi_t d_t(\tilde{m})\right] dF_t(\tilde{m}) + \alpha\sigma W_t(m)}_{\text{Selling}} \\
& + \underbrace{\alpha\delta\left(\left[u(q^*) - c(q^*)\right] + W_t(m)\right)}_{\text{Bartering}} \\
& + \underbrace{(1 - 2\alpha\sigma - \alpha\delta)\, W_t(m)}_{\text{Not trading}}
\end{aligned}
$$

Equilibrium value function II

- Grouping $W_t(m)$ and substituting its value:

$$\begin{aligned} V_t(m) &= \alpha\sigma\left[u\left(q_t(m)\right) - \phi_t d_t(m)\right] + \\ &\quad + \alpha\sigma \int \left[-c\left(q_t(\tilde{m})\right) + \phi_t d_t(\tilde{m})\right] dF_t(\tilde{m}) \\ &\quad + \alpha\delta\left[u(q^*) - c(q^*)\right] \\ &\quad + \phi_t m + U(X^*) - X^* + \max_{m'}\left\{-\phi_t m' + \beta V_{t+1}(m')\right\} \end{aligned}$$

or

$$V_t(m) = \nu_t(m) + \phi_t m + \max_{m'}\left\{-\phi_t m' + \beta V_{t+1}(m')\right\}$$

where

$$\begin{aligned} v_t(m) &\equiv \alpha\sigma\left[u\left(q_t(m)\right) - \phi_t d_t(m)\right] \\ &\quad + \alpha\sigma \int \left[-c\left(q_t(\tilde{m})\right) + \phi_t d_t(\tilde{m})\right] dF_t(\tilde{m}) \\ &\quad + \alpha\delta\left[u(q^*) - c(q^*)\right] + U(X^*) - X^* \end{aligned}$$

Equilibrium value function III

- By repeated substitution:

$$\begin{aligned} V_t(m) &= v_t(m_t) + \phi_t m_t + \max_{m_{t+1}} \left[-\phi_t m_{t+1} + \beta V_{t+1}(m_{t+1}) \right] \\ &= v_t(m_t) + \phi_t m_t \\ &\quad + \sum_{j=t}^{\infty} \beta^{j-t} \max_{m_{j+1}} \left\{ -\phi_j m_{j+1} + \beta \left[v_{j+1}(m_{j+1}) + \phi_{j+1} m_{j+1} \right] \right\} \end{aligned}$$

- Let us focus on the maximization problem:

$$\max_{m_{j+1}} \left\{ -\phi_j m_{j+1} + \beta \left[v_{j+1}(m_{j+1}) + \phi_{j+1} m_{j+1} \right] \right\}$$

- The optimality condition is:

$$-\phi_t + \beta \phi_{t+1} + \beta v'_{t+1}(m_{t+1}) = 0$$

Monetary Equilibrium

A monetary equilibrium is a list $\{V_t, W_t, X_t, H_t, m'_t, q_t, d_t, \phi_t, F_t, G_t\}_{t=0}^{\infty}$ such that:

1. the value functions V_t and W_t and the decision rules (X_t, H_t, m'_t) satisfy the equations above for every t given $\{q_t, d_t, \phi_t, F_t, G_t\}_{t=0}^{\infty}$;
2. the terms of trade (q_t, d_t) in the decentralized market solve the problems above for every t given $\{V_t, W_t\}_{t=0}^{\infty}$;
3. $\phi_t > 0$ for every t;
4. $\int m'_t(m)\, dG_t(m) = M$ for all t;
5. $\{F_t, G_t\}_{t=0}^{\infty}$ is consistent with initial conditions and the evolution of money holdings implied by trades in the centralized and decentralized markets.

Some properties of the equilibrium I

- Note that, for any $m \geq m_t^*$

$$v_t(m) \equiv \alpha(\sigma\theta+\delta)\left[u(q^*)-c(q^*)\right] +\alpha\sigma\int\left[-c(q_t(\tilde{m}))+\phi_t d_t(\tilde{m})\right]dF_t(\tilde{m}) +U(X^*)-X^*$$

  and:

$$v_t'(m)=0$$

- In equilibrium, $\phi_t \geq \beta\phi_{t+1}$. Otherwise, the agent can pick $m_{t+1} \geq m_t^*$ and the optimality condition

$$-\phi_t+\beta\phi_{t+1}>0$$

  implies that the maximization problem does not have a solution.
- This restriction allows the Friedman rule

$$\frac{\phi_t}{\phi_{t+1}}=\beta.$$

Some properties of the equilibrium II

- Note that, for any $m < m_t^*$,

$$v_t'(m) = \alpha\sigma\left[u'(q_t(m))\,q_t'(m) - \phi_t d_t'(m)\right] = \alpha\sigma\left[u'(q_t(m))\,q_t'(m) - \phi_t\right]$$

- Then, coming back to the optimality condition:

$$\begin{aligned}
\phi_t - \beta\phi_{t+1} &= \beta v_{t+1}'(m_{t+1}) \\
&= \beta\alpha\sigma\left[u'(q_{t+1}(m_{t+1}))\,q_{t+1}'(m_{t+1}) - \phi_{t+1}\right] \\
&= \beta\alpha\sigma\phi_{t+1}\left[\frac{u'(q_{t+1}(m_{t+1}))}{z'(_{t+1}(m_{t+1}))} - 1\right] \\
&= \beta\alpha\sigma\phi_{t+1}\left[\frac{u'\left[\theta u' + (1-\theta)c'\right]^2}{\begin{pmatrix} u'c'\left[\theta u' + (1-\theta)c'\right] \\ +\theta(1-\theta)(u-c)(u'c'' - c'u'') \end{pmatrix}} - 1\right]
\end{aligned}$$

where in the last line we evaluate the u and c functions at $q_{t+1}(m_{t+1})$.

Some properties of the equilibrium III

- As $m_{t+1} \to m^*_{t+1}$ from below, the slope of the objective function is:

$$\phi_t - \beta\phi_{t+1} =$$

$$\beta\alpha\sigma\phi_{t+1}\underbrace{\left[\frac{\left[u'\left(q^*\right)\right]^2}{\left[u'\left(q^*\right)\right]^2 + \theta\left(1-\theta\right)\left[u\left(q^*\right) - c\left(q^*\right)\right]\left[c''\left(q^*\right) - u''\left(q^*\right)\right]} - 1\right]}_{\Sigma}$$

- $\Sigma \leq 0$.
- $\Sigma = 0$ if and only if $\phi_t = \beta\phi_{t+1}$ and $\theta = 1$.
- Since $\phi_t \geq \beta\phi_{t+1}$,

$$-\phi_t + \beta\phi_{t+1} + \beta\alpha\sigma\phi_{t+1}\Sigma < 0$$

  which means that any solution must satisfy $m_{t+1} < m^*_{t+1}$ (i.e., close to m^*_{t+1} by the left, the return function is already falling).

Some properties of the equilibrium IV

- By imposing conditions either on the bargaining power θ or on preferences, one can also prove that $v''_{t+1} < 0$.
- This ensures that the choice of $m_{t+1} = M$ is unique in equilibrium.
- Therefore, we have:

$$\phi_t = \beta \left(v'_{t+1} (M) + \phi_{t+1} \right)$$

or

$$\phi_t = \beta \left[\alpha\sigma u' \left(q_{t+1} (M) \right) q'_{t+1} (M) + (1 - \alpha\sigma) \phi_{t+1} \right]$$

Some properties of the equilibrium V

- In any monetary equilibrium, we have $q_t \in (0, q^*)$:

$$M = \frac{z(q_t)}{\phi_t} < \frac{z(q^*)}{\phi_t} \equiv m_t^*$$

  for every $t = 0, 1, 2, ...$

- Then:

$$q_t'(M) = \frac{\phi_t}{z'(q_t)}$$

  and:

$$\begin{aligned} \phi_t &= \beta \left(v_{t+1}'(m_{t+1}) + \phi_{t+1} \right) \\ &= \beta \phi_{t+1} \left[\alpha\sigma \frac{u'(q_{t+1}(m_{t+1}))}{z'(q_{t+1}(m_{t+1}))} + 1 - \alpha\sigma \right] \end{aligned}$$

  or

$$z(q_t) = \beta z(q_{t+1}) \left[\alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} + 1 - \alpha\sigma \right]$$

Stationary equilibria

- A necessary and sufficient condition for a stationary equilibrium is:

$$1 = \beta \left[\alpha\sigma \frac{u'(q)}{z'(q)} + 1 - \alpha\sigma \right]$$

or

$$\frac{u'(q)}{z'(q)} = 1 + \frac{1-\beta}{\alpha\sigma\beta}$$

- When $\theta < 1$, a stationary solution exists, but we cannot guarantee uniqueness.

- In this case, we always have $q < q^*$ and the steady state is not efficient.

Monetary policy I

- New money injected as lump-sum transfers in the centralized market:

$$M_{t+1} = (1+\tau)\, M_t$$

  with a constant $\tau \geq 0$.
- New equilibrium condition:

$$(1+\tau)\, z(q_t) = \beta z(q_{t+1}) \left[\alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} + 1 - \alpha\sigma \right]$$

- Consider only stationary equilibria in which aggregate real balances are constant over time. Then:

$$\frac{\phi_t}{\phi_{t+1}} = 1 + \tau$$

  and

$$\frac{u'(q)}{z'(q)} = 1 + \frac{1+\tau-\beta}{\alpha\sigma\beta}$$

  with $\partial q / \partial \tau < 0$.

Monetary policy II

- Thus, a higher steady state inflation rate implies a lower level of output in the decentralized market: a higher inflation rate raises the opportunity cost of holding money for transaction purposes, which reduces the demand for real money balances.

- The Friedman rule is the optimal monetary policy. However, it delivers the efficient outcome q^* only in the case $\theta = 1$.

- The quantity traded in the decentralized market is smaller than the socially efficient level owing to a hold-up problem. An additional unit of money carried into the decentralized market will not reap its full return when the seller retains some of the bargaining power, creating an additional inefficiency in the model.