Random Matching Models

Jesús Fernández-Villaverde[1]

December 4, 2021

[1]University of Pennsylvania

Introduction

Motivation

- Trade in the labor market is a decentralized economic activity:
  1. It takes time and effort.
  2. It is uncoordinated.
- Central points:
  1. Matching arrangements.
  2. Productivity opportunities constantly arise and disappear.

Empirical observations

- Huge amount of labor turnover.
- Pioneers in this research: Davis and Haltiwanger.
- Micro data:
  1. Current population survey (CPS)
  2. Job opening and labor turnover survey (JOLTS): 16.000 establishments, monthly.
  3. Business employment dynamics (BED): entry and exit of establishments.
  4. Longitudinal employer household dynamics (LEHD): matched data.

Basic accounting identity

- For each period t and level of aggregation i:

$$\begin{aligned}\text{Net Employment Change}_{ti} &= \underbrace{\text{Hires}_{ti} - \text{Separations}_{ti}}_{\text{Workers Flows}} \\ &= \underbrace{\text{Creation}_{ti} - \text{Destruction}_{ti}}_{\text{Jobs Flows}}\end{aligned}$$

- Difficult to distinguish between voluntary and involuntary separations.

Four models of random matching

- Pissarides (1985).
- Mortensen and Pissarides (1994).
- Burdett and Mortensen (1998).
- Moen (1997).

Model I: Pissarides

Setup

- Pissarides (1985).
- Continuous time.
- Constant and exogenous interest rate r: stationary world.
- No capital (we will change this later).

Workers

- Continuum of measure L of worker. A law of large numbers hold in the economy.
- Workers are identical.
- Linear preferences (risk neutrality).
- Thus, worker maximizes total discounted income:

$$\int_0^\infty e^{-rt} y(t)\, dt$$

where r is the interest rate and $y(t)$ is income per period.

Firms

- Endogenous number of small firms:
  1. One firm=one job.
  2. Competitive producers of the final output at price p.
- Free entry into production:
  1. Perfectly elastic supply of firm operators.
  2. Zero-profit condition.
- Vacancy cost $c > 0$ per unit of time.

Matching function, I

- L workers, u unemployment rate, and v vacancy rate.
- How do we determine how many matches do we have?
- Define matching function:

$$fL = m(uL, vL)$$

  where f is the rate of jobs created.
- Increasing in both argument, concave, and constant returns to scale.
- Why CRS?
  1. Argument against decreasing returns to scale: submarkets.
  2. But possibly increasing returns to scale (we will come back to this).
- Then, $f = m(u, v)$.

Matching function, II

- All matches are random.
- Microfoundation of the matching function? Butters (1977).
- Empirical evidence:

$$f_t = e^{\varepsilon_t} u_t^{0.72} v_t^{0.28}$$

- ε_t is the sum of:

  1. High frequency noise.
  2. Very low frequency movement (for example, demographics).

What if increasing returns to scale?

- Multiple equilibria:
  1. High activity equilibrium.
  2. Low activity equilibrium.
- Diamond (1982), Howitt and McAfee (1987).
- In any case, a matching function implies externalities and opens door to inefficiencies.

Properties of matching function, I

- Define vacancy unemployment ratio (or market tightness) as $\theta = \frac{v}{u}$.
- Then:

$$q(\theta) = m\left(\frac{u}{v}, 1\right) = m\left(\frac{1}{\theta}, 1\right)$$

- We can show:

1. $q'(\theta) \leq 0$.
2. $\frac{q'(\theta)}{q(\theta)}\theta \in [-1, 0]$.

Properties of matching function, II

- Since $\frac{f}{v} = \frac{m(u,v)}{v} = q(\theta)$, we have:
  1. $q(\theta)$ is the (Poisson) rate at which vacant jobs become filled.
  2. Mean duration of a vacancy is $\frac{1}{q(\theta)}$.
- Since $\frac{f}{u} = \frac{m(u,v)}{u} = \theta q(\theta)$, we have:
  1. $\theta q(\theta)$ is the (Poisson) rate at which unemployed workers find a job.
  2. Mean duration of unemployment is $\frac{1}{\theta q(\theta)}$.

Externalities

- Note that $q(\theta)$ and $\theta q(\theta)$ depend on market tightness.
- This is called a search or congestion externality.
- Think about a party where you take 5 friends.
- Prices and wages do not play a direct role for the rates.
- Competitive vs. search equilibria.

Job creation and destruction

- Job creation: a firm and a worker match and they agree on a wage.
- Job creation in a period: $fL = u\theta q(\theta) L$.
- Job creation rate: $\frac{u\theta q(\theta)}{1-u}$.
- Job destruction: exogenous at (Poisson) rate λ.
- Job destruction in a period: $\lambda (1-u) L$.
- Job destruction rate: $\frac{\lambda(1-u)}{1-u}$.

Evolution of unemployment

- Evolution of unemployment:

$$\dot{u} = \lambda\left(1 - u\right) - u\theta q\left(\theta\right)$$

- In steady state:

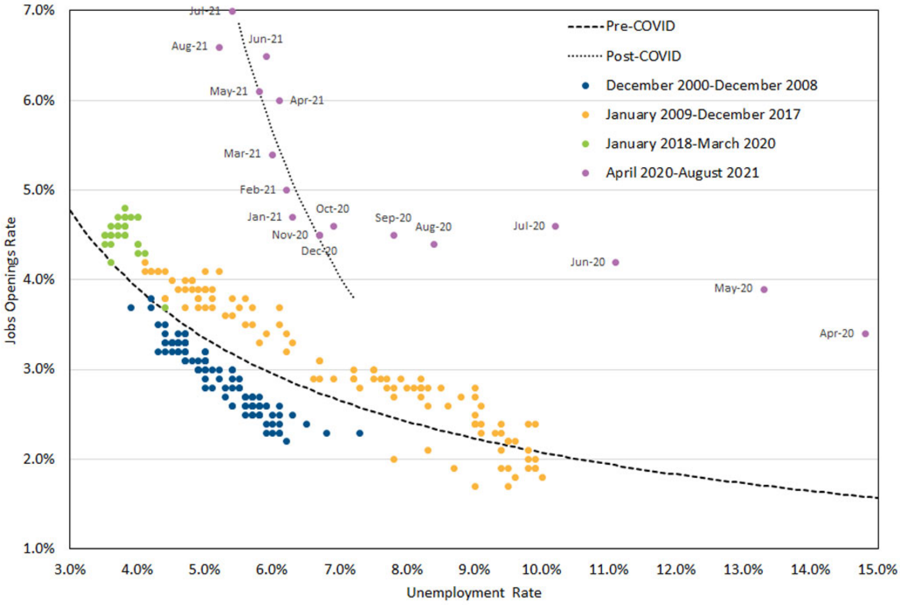
$$\lambda\left(1 - u\right) = u\theta q\left(\theta\right)$$

or

$$u = \frac{\lambda}{\lambda + \theta q\left(\theta\right)}$$

- This relation is a downward-slopping and convex to the origin curve: the Beveridge Curve.

Figure 1: Beveridge Curve



Sources: Bureau of Labor Statistics' Job Openings and Labor Turnover Survey and author's calculations.

Labor contracts and firm's value functions

- Wage w.
- Hours fixed and normalized to 1.
- Either part can break the contract at any time without cost.
- J is the value function of an occupied job.
- V is the value function of a vacant job.
- Then, in a stationary equilibrium:

$$rV = -c + q(\theta)(J - V)$$
$$rJ = p - w - \lambda J$$

- Note $J = \frac{p-w}{r+\lambda}$ and $J' = -\frac{1}{r+\lambda}$.

Job creation condition

- Because of free entry

$$V = 0$$
$$J = \frac{c}{q(\theta)}$$

- Then:

$$p - w - (r + \lambda) J = 0 \Rightarrow$$
$$p - w - (r + \lambda) \frac{c}{q(\theta)} = 0$$

- This equation is know as the job creation condition.
- Interpretation.

Workers, I

- Value of not working: z.
- Includes leisure, UI, home production.
- Because of linearity of preferences, we can ignore extra income.
- U is the value function of unemployed worker.
- W is the value function of employed worker.
- Then:

$$rU = z + \theta q(\theta)(W - U)$$
$$rW = w + \lambda(U - W)$$

- Notice $W = \frac{w}{r+\lambda} + \frac{\lambda}{r+\lambda}U$ and $W' = \frac{1}{r+\lambda}$.

Workers, II

- With some algebra:

$$(r + \theta q(\theta))\, U - \theta q(\theta)\, W = z$$
$$-\lambda U + (r + \lambda)\, W = w$$

and

$$U = \frac{(r+\lambda)\, z + \theta q(\theta)\, w}{(r + \theta q(\theta))(r+\lambda) - \lambda \theta q(\theta)} = \frac{\lambda z + \theta q(\theta)\, w + rz}{r^2 + r\theta q(\theta) + \lambda r}$$
$$W = \frac{(r + \theta q(\theta))\, w + \lambda z}{(r + \theta q(\theta))(r+\lambda) - \lambda \theta q(\theta)} = \frac{\lambda z + \theta q(\theta)\, w + rw}{r^2 + r\theta q(\theta) + \lambda r}$$

- Clearly, for $r > 0$, $W > U$ if and only if $w > z$.
- If $r = 0$, $W = U$.

Wage determination, I

- We can solve Nash Bargaining solution:

$$w = \arg\max \left(W - U\right)^{\beta} \left(J - V\right)^{1-\beta}$$

- First order conditions:

$$\beta \frac{W'}{W - U} = -\left(1 - \beta\right) \frac{J'}{J - V}$$

- Since $W' = -J' = \frac{1}{r+\lambda}$ and $V = 0$:

$$W = U + \beta \left(\underbrace{W - U + J}_{\text{surplus of the relation}} \right) = U + \beta S$$

- Also:

$$W - U = \frac{\beta}{1-\beta} J = \frac{\beta}{1-\beta} \frac{c}{q\left(\theta\right)}$$

Wage determination, II

- Since $J = \frac{p-w}{r+\lambda}$ and $W = \frac{w}{r+\lambda} + \frac{\lambda}{r+\lambda} U$

$$\frac{w}{r+\lambda} - \frac{r}{r+\lambda} U = \beta \left(\frac{w}{r+\lambda} - \frac{r}{r+\lambda} U + \frac{p-w}{r+\lambda} \right) \Rightarrow w = rU + \beta \left(p - rU \right)$$

- Interpretation.

- Now, notice:

$$w = rU + \beta \left(p - rU \right) \Rightarrow$$
$$w = \left(1-\beta\right) rU + \beta p \Rightarrow$$
$$w = \left(1-\beta\right) \left(z + \theta q\left(\theta\right) \left(W - U\right)\right) + \beta p \Rightarrow$$
$$w = \left(1-\beta\right) \left(z + \theta q\left(\theta\right) \frac{\beta}{1-\beta} \frac{c}{q\left(\theta\right)}\right) + \beta p \Rightarrow$$
$$w = \left(1-\beta\right) z + \beta \left(p + \theta c\right)$$

- The last condition is known as the wage equation.

Steady state

- Three equations:

$$w = (1-\beta) z + \beta\theta c + \beta p$$

$$p - w - (r+\lambda)\frac{c}{q(\theta)} = 0$$

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

- Combine the first two conditions:

$$(1-\beta)(p-z) - \frac{r+\lambda+\beta\theta q(\theta)}{q(\theta)} c = 0$$

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

that we can plot in the Beveridge Diagram.

Comparative statics

- Raise z: higher unemployment because less surplus to firms. Relation with unemployment insurance.
- Changes in matching function.
- Changes in Nash parameter.
- Dynamics?

Efficiency, I

- Can the equilibrium achieve social efficiency despite search externalities?
- Social planner:

$$\max_{u,\theta} \int_0^\infty e^{-rt} \left(p\left(1-u\right) + zu - c\theta u\right) dt$$

$$s.t.\ u = \frac{\lambda}{\lambda + \theta q\left(\theta\right)}$$

- The social planner faces the same matching frictions as the agents.
- First-order conditions of the Hamiltonian:

$$-e^{-rt}\left(p - z + c\theta\right) + \mu\left(\lambda + \theta q\left(\theta\right)\right) - \dot{\mu} = 0$$

$$-e^{-rt} cu + \mu u q\left(\theta\right)\left(1 - \eta\left(\theta\right)\right) = 0$$

where μ is the multiplier and $\eta\left(\theta\right)$ is (minus) the elasticity of $q\left(\theta\right)$.

Efficiency, II

- From the second equation:

$$\mu = e^{-rt} \frac{cu}{uq(\theta)(1-\eta(\theta))}$$

- Now:

$$e^{-rt} cu = \mu uq(\theta)(1-\eta(\theta))$$
$$-rt + \log cu = \log \mu + \log uq(\theta)(1-\eta(\theta))$$

and taking time derivatives:

$$-r = \frac{\dot{\mu}}{\mu} \Rightarrow -\dot{\mu} = r\mu$$

and

$$\begin{aligned} -e^{-rt}(p - z + c\theta) + \mu(\lambda + \theta q(\theta)) - \dot{\mu} &= 0 \Rightarrow \\ -e^{-rt}(p - z + c\theta) + \mu(r + \lambda + \theta q(\theta)) &= 0 \end{aligned}$$

Efficiency, III

- Thus, we get:

$$
\begin{aligned}
-e^{-rt}\left(p-z+c\theta\right)+e^{-rt}\frac{cu\left(r+\lambda+\theta q\left(\theta\right)\right)}{uq\left(\theta\right)\left(1-\eta\left(\theta\right)\right)} &= 0 \Rightarrow \\
\left(1-\eta\left(\theta\right)\right)\left(p-z\right)-\frac{r+\lambda+\eta\left(\theta\right)\theta q\left(\theta\right)}{q\left(\theta\right)}c &= 0
\end{aligned}
$$

- Remember that the market job creation condition:

$$\left(1-\beta\right)\left(p-z\right)-\frac{r+\lambda+\beta\theta q\left(\theta\right)}{q\left(\theta\right)}c=0$$

- Both conditions are equal if, and only if, $\eta\left(\theta\right)=\beta$.

Hosios' rule

- Imagine that matching function is $m = Au^{\eta}v^{1-\eta}$.
- Then $\eta(\theta) = \eta$.
- We have that efficiency is satisfied if $\eta = \beta$.
- This result is know as the Hosios Rule (Hosios, 1990):
  1. If $\eta > \beta$ equilibrium unemployment is below its social optimum.
  2. If $\eta < \beta$ equilibrium unemployment is above its social optimum.
- Intuition: externalities equal to share of surplus.

Introducing capital

- Production function $f(k)$ per worker with depreciation rate δ.
- Arbitrage condition in capital market $f'(k) = (r+\delta)$.
- We have four equations:

$$f'(k) = (r+\delta)$$
$$w = (1-\beta)z + \beta\theta c + \beta p(f(k) - (r+\delta)k)$$
$$p(f(k) - (r+\delta)k) - w - (r+\lambda)\frac{c}{q(\theta)} = 0$$
$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

Model II: Mortensen and Pissarides

Setup

- Mortensen and Pissarides (1994).
- Similar to previous model but we endogeneize job destruction.
- Why? Empirical Evidence from Davis, Haltiwanger, and Schuh (1996).
- Productivity of a job px where x is the idiosyncratic component.
- New x's arrive with Poisson rate λ.
- Distribution is $G(\cdot)$.
- Distribution is memoriless and with bounded support $[0, 1]$.
- Initial draw is $x = 1$. Why?

Policy function of the firm

- Value function for a job is $J(x)$.
- Then:
  1. If $J(x) \geq 0$, the job is kept.
  2. If $J(x) < 0$, the job is destroyed.
- There is an R such that $J(R) = 0$.
- This R is the reservation productivity.

Flows into unemployment

- A law of large numbers hold for the economy.
- Job destruction: $\lambda G(R)(1-u)$.
- Unemployment evolves:

$$\dot{u} = \lambda G(R)(1-u) - u\theta q(\theta)$$

- In steady state:

$$u = \frac{\lambda G(R)}{\lambda G(R) + \theta q(\theta)}$$

Value functions

- Value functions for the firm:

$$rV = -c + q(\theta)(J(1) - V)$$

$$rJ(x) = px - w(x) + \lambda \int_R^1 J(s)\,dG(s) - \lambda J(x)$$

- Value functions for the worker:

$$rU = z + \theta q(\theta)(W(1) - U)$$

$$rW(x) = w(x) + \lambda \int_R^1 W(s)\,dG(s) + \lambda G(R)\,U - \lambda W(x)$$

- Because of free entry, $V = 0$ and $J(1) = \frac{c}{q(\theta)}$.
- Also, by Nash bargaining:

$$W(x) - U = \beta(W(x) - U + J(x))$$

Equilibrium equations

$$u = \frac{\lambda G\left(R\right)}{\lambda G\left(R\right) + \theta q\left(\theta\right)}$$

$$J\left(R\right) = 0$$

$$J\left(1\right) = \frac{c}{q\left(\theta\right)}$$

$$W\left(x\right) - U = \beta\left(W\left(x\right) - U + J\left(x\right)\right)$$

Solving the model, I

- First, repeating the same steps than in the Pissarides model:

$$w(x) = (1-\beta)z + \beta(px + \theta c)$$

- Second:

$$W(R) - U = \beta(W(R) - U + J(R)) = \beta(W(R) - U) \Rightarrow$$
$$W(R) = U$$

- Third:

$$rJ(x) = px - (1-\beta)z - \beta(px + \theta c) + \lambda\int_R^1 J(s)\,dG(s) - \lambda J(x) \Rightarrow$$
$$(r+\lambda)J(x) = (1-\beta)px - (1-\beta)z - \beta\theta c + \lambda\int_R^1 J(s)\,dG(s)$$

Solving the model, II

- At $x = R$

$$(r+\lambda) J(R) = (1-\beta) pR - (1-\beta) z - \beta\theta c + \lambda \int_R^1 J(s)\, dG(s) = 0$$

- Thus:

$$(r+\lambda) J(x) = (1-\beta) p (x - R) \Rightarrow$$
$$(r+\lambda) J(1) = (1-\beta) p (1 - R) \Rightarrow$$
$$(r+\lambda) \frac{c}{q(\theta)} = (1-\beta) p (1 - R) \Rightarrow$$
$$(1-\beta) p \frac{1-R}{r+\lambda} = \frac{c}{q(\theta)}$$

Solving the model, III

- Notice that:

$$(r+\lambda)\,J\,(x) = (1-\beta)\,p\,(x-R) \Rightarrow J\,(x) = \frac{(1-\beta)}{r+\lambda}p\,(x-R)$$

- Then:

$$(r+\lambda)\,J\,(x) = (1-\beta)\,(px-z) - \beta\theta c + \lambda\int_R^1 J\,(s)\,dG\,(s) \Rightarrow$$

$$(r+\lambda)\,J\,(x) = (1-\beta)\,(px-z) - \beta\theta c + \frac{\lambda\,(1-\beta)\,p}{r+\lambda}\int_R^1 (s-R)\,dG\,(s)$$

- Evaluate the previous expression at $x = R$ and using the fact that $J\,(R) = 0$:

$$(r+\lambda)\,J\,(R) = 0 = (1-\beta)\,(pR-z) - \beta\theta c + \frac{\lambda\,(1-\beta)\,p}{r+\lambda}\int_R^1 (s-R)\,dG\,(s) \Rightarrow$$

$$R - \frac{z}{p} - \frac{\beta}{1-\beta}\theta c + \frac{\lambda}{r+\lambda}\int_R^1 (s-R)\,dG\,(s) = 0$$

Solving the model, IV

- We have two equations on two unknowns, R and θ:

$$(1-\beta)\, p\frac{1-R}{r+\lambda} = \frac{c}{q(\theta)}$$

$$R - \frac{z}{p} - \frac{\beta}{1-\beta}\theta c + \frac{\lambda}{r+\lambda}\int_R^1 (s-R)\, dG(s) = 0$$

- The first expression is known as the job creation condition.
- The second expression is known as the job destruction condition.
- Together with $u = \frac{\lambda G(R)}{\lambda G(R)+\theta q(\theta)}$ and $w(x) = (1-\beta) z + \beta (px + \theta c)$, we complete the characterization of the equilibrium.

Efficiency

- Social welfare:

$$\max_{u,\theta} \int_0^\infty e^{-rt} \left(y + zu - c\theta u\right) dt$$

$$s.t.\ u = \frac{\lambda G(R)}{\lambda G(R) + \theta q(\theta)}$$

where y is the average product per person in the labor market.

- The evolution of y is given by:

$$\dot{y} = p\theta q(\theta)\, u + \lambda (1-u) \int_R^1 psdG(s) - \lambda y$$

- Again, Hosios' rule.

Model III: Burdett and Mortensen

Motivation

- Burdett and Mortensen (1998).
- Wage dispersion: different wages for the same work.
- Violates the law of one price.
- What is same work? Observable and unobservable heterogeneity.
- Evidence of wage dispersion: Mincerian regression

$$w_i = X_i' \beta + \varepsilon_i$$

- Typical Mincerian regression accounts for 25-30% of variation in the data.

Theoretical challenge

- Remember Diamond's paradox: elasticity of labor supply was zero for the firm.
- Not all the deviations from a competitive setting deliver wage dispersion.
- Wage dispersion you get from Mortensen-Pissarides is very small (Krusell, Hornstein, Violante, 2007).
- Main mechanism to generate wage dispersion: on-the-job search.

Environment

- Unit measure of identical workers.
- Unit measure of identical firms.
- Each worker is unemployed (state 0) or employed (state 1).
- Poisson arrival rate of new offers λ. Same for workers and unemployed agents.
- Offers come from an equilibrium distribution F.

Previous assumptions that we keep

- No recall of offers.
- Job-worker matches are destroyed at rate δ.
- Value of not working: z.
- Discount rate r.
- Vacancy cost c.

Value functions for workers

- Utility of unemployed agent:

$$rV_0 = z + \lambda \left[\int \max\left\{ V_0, V_1\left(w'\right) \right\} dF\left(w'\right) - V_0 \right]$$

- Utility of worker employed at wage w:

$$\begin{aligned} rV_1\left(w\right) &= w + \lambda \int \left[\max\left\{ V_1\left(w\right), V_1\left(w'\right) \right\} - V_1\left(w\right) \right] dF\left(w'\right) \\ &\quad + \delta \left[V_0 - V_1\left(w\right) \right] \end{aligned}$$

- As before, there is a reservation wage w_R such that $V_0 = V_1\left(w_R\right)$.

- Clearly, $w_R = z$.

Firms' problem

- $G(w)$: distribution of workers.
- Wage posting: Butters (1977), Burdett and Judd (1983), and Mortensen (1990).
- The profit for a firm:

$$\pi(p, w) = \frac{[u + (1-u) G(w)]}{r + \delta + \lambda (1 - F(w))} (p - w)$$

- Firm sets wages w to maximize $\pi(p, w)$. No symmetric pure strategy equilibrium.
- Firms will never post w lower than z.

Unemployment

- Steady state unemployment:

$$\lambda (1 - F(z)) u = \delta (1 - u)$$

- Then:

$$u = \frac{\delta}{\delta + \lambda [1 - F(z)]} = \frac{\delta}{\delta + \lambda}$$

  where we have used the fact that no firm will post wage lower than z and that F will not have mass points (equilibrium property that we have not shown yet).

Distribution of workers

- Workers gaining less than w:

$$E(w) = (1-u)\,G(w)$$

- Then:

$$\dot{E}(w) = \lambda F(w)\,u - (\delta + \lambda[1 - F(w)])\,E(w)$$

- In steady state:

$$E(w) = \frac{\lambda F(w)}{\delta + \lambda[1 - F(w)]}u \Rightarrow$$
$$G(w) = \frac{E(w)}{1-u} = \frac{\delta F(w)}{\delta + \lambda[1 - F(w)]}$$

Solving for an equilibrium, I

- Equilibrium objects: u, $F(w)$, λ, $G(w)$.
- Simple yet boring arguments show that $F(w)$ does not have mass points and has connected support.
- First, by free entry:

$$\pi(p, z) = \frac{\delta}{\delta + \lambda} \frac{p - z}{r + \delta + \lambda} = c$$

  which we solve for λ.
- Hence, we also know $u = \frac{\delta}{\delta+\lambda}$.

Solving for an equilibrium, II

- Second, by the equality of profits and with some substitutions:

$$
\begin{aligned}
\pi(p,w) &= \frac{\left[\frac{\delta}{\delta+\lambda} + \left(\frac{\lambda}{\delta+\lambda}\right)\frac{\delta F(w)}{\delta+\lambda[1-F(w)]}\right]}{r+\delta+\lambda(1-F(w))}(p-w) \\
&= \frac{\delta}{\delta+\lambda[1-F(w)]}\frac{p-w}{r+\delta+\lambda(1-F(w))} \\
&= \frac{\delta}{\delta+\lambda}\frac{p-z}{r+\delta+\lambda}
\end{aligned}
$$

- Previous equality is a quadratic equation on $F(w)$.

Solving for an equilibrium, III

- To simplify the solution, set $r = 0$. Then:

$$F(w) = \frac{\delta + \lambda}{\delta} \left[1 - \left(\frac{p - w}{p - z} \right)^{0.5} \right]$$

- Now, we get:

$$G(w) = \frac{\delta}{\lambda} \left[\left(\frac{p - w}{p - z} \right)^{0.5} - 1 \right]$$

- Highest wage is $F(w^{\max}) = 1$

$$w^{\max} = \left(1 - \frac{\delta}{\delta + \lambda} \right)^2 p + \left(\frac{\delta}{\delta + \lambda} \right)^2 z$$

- Empirical content.
- Modifications to fit the data.

Model IV: Moen

Competitive search

- Moen (1997).
- A market maker chooses a number of markets m and determines the wage w_i in each submarket.
- Workers and firms are free to move between markets.
- Two alternative interpretations:
    1. Clubs charging an entry fee. Competition drives fees to zero.
    2. Wage posting by firms.

Workers, I

- Value functions:

$$
\begin{aligned}
rU_i &= z + \theta_i q(\theta_i)(W_i - U_i) \\
rW_i &= w_i + \lambda (U_i - W_i)
\end{aligned}
$$

- Then:

$$
\begin{aligned}
W_i &= \frac{1}{r+\lambda} w_i + \frac{\lambda}{r+\lambda} U_i \\
rU_i &= z + \theta_i q(\theta_i) \left(\frac{w_i - rU_i}{r+\lambda} \right)
\end{aligned}
$$

- Workers will pick the highest U_i.

Workers, II

- In equilibrium, all submarkets should deliver the same U_i. Hence:

$$\theta_i q(\theta_i) = \frac{rU - z}{w_i - rU}(r + \lambda)$$

- Negative relation between wage and labor market tightness.

- If $w_i < rU$, the market will not attract workers and it will close.

Firms

- Value Functions:

$$\begin{aligned} rV_i &= -c + q(\theta_i)(J_i - V_i) \\ rJ_i &= p - w_i - \lambda J_i \end{aligned}$$

- Thus:

$$rV_i = -c + q(\theta_i)\left(\frac{p - w_i}{r + \lambda} - V_i\right)$$

- Each firm solves

$$rV_i = \max_{w_i, \theta_i}\left(-c + q(\theta_i)\left(\frac{p - w_i}{r + \lambda} - V_i\right)\right)$$

$$s.t.\ rU_i = z + \theta_i q(\theta_i)\left(\frac{w_i - rU}{r + \lambda}\right)$$

Equilibrium

- Impose equilibrium condition $V_i = 0$ and solve the dual:

$$rU_i = \max_{w_i,\theta_i} \left(z + \theta_i q(\theta_i) \frac{w_i - rU}{r+\lambda} \right)$$
$$s.t.\ c = q(\theta_i) \frac{p - w_i}{r+\lambda}$$

- Plugging the value of w_i from the constraint into the objective function:

$$rU_i = \max_{\theta_i} \left(z - c\theta_i + \theta_i q(\theta_i) \frac{p - rU}{r+\lambda} \right)$$

- Solution:

$$c = q(\theta_i) \frac{p - rU}{r+\lambda} + \theta_i q'(\theta_i) \frac{p - rU}{r+\lambda},$$

which is unique if $\theta_i q(\theta_i)$ is concave.