Production on OLG models

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Production

Introducing production in an OLG model

- Are the properties of OLG models consequence of the absence of production?
- First explored by Diamond (1965).
- We want to have models with production for policy purposes.
- Tradition of Auerbach and Kotlikoff (1987).

Setup

Demographics

- Individuals live for two periods.
- N_t^t: number of young people in period t.
- N_t^{t-1}: number of old people at period t.
- Normalize the size of the initial old generation to 1: $N_0^0 = 1$.
- People do not die early, $N_t^t = N_{t+1}^t$.
- Population grows at constant rate n:

$$N_t^t = (1+n)^t N_0^0 = (1+n)^t$$

- The total population at period t:

$$N_t^{t-1} + N_t^t = (1+n)^t \left(1 + \frac{1}{1+n}\right)$$

Preferences

- Preferences over consumption streams given by:

$$u(c_t^t, c_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t)$$

- U is strictly increasing, strictly concave, twice continuously differentiable and satisfies the Inada conditions.
- All individuals are assumed to be purely selfish and have no bequest motives whatsoever.
- The initial old generation has preferences

$$u(c_1^0) = U(c_1^0)$$

Endowments

- Each individual of generation $t \geq 1$ has as endowments one unit of time to work when young and no endowment when old.
- How we can generalize it?
    1. Life cycle profile of productivity.
    2. Leisure in the utility function.
- Hence the labor force in period t is of size N_t^t with maximal labor supply of $1 * N_t^t$.
- Each member of the initial old generation is endowed with capital stock $(1+n)\bar{k}_1 > 0$.

Firms

- Constant returns to scale technology:
$$Y_t = F(K_t, L_t)$$
- Profits are zero in equilibrium and we do not have to specify ownership of firms.
- Single, representative firm that behaves competitively in that it takes as given the rental prices of factor inputs (r_t, w_t) and the price for its output.
- Defining the capital-labor ratio $k_t = \frac{K_t}{L_t}$, we have:
$$y_t = \frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t)$$
- We assume that f is twice continuously differentiable, strictly concave, and satisfies the Inada conditions.

Timing

1. At the beginning of period t, production takes place with labor of generation t and capital saved by the now old generation $t-1$ from the previous period. The young generation earns a wage w_t.
2. At the end of period t, the young generation decides how much of the wage income to consume, c_t^t, and how much to save for tomorrow, s_t^t. The saving occurs in form of physical capital, which is the only asset in this economy.
3. At the beginning of period $t+1$, production takes place with labor of generation $t+1$ and the saved capital of the now old generation t. The return on savings equals $r_{t+1}-\delta$, the real interest rate from period t to $t+1$.
4. At the end of period $t+1$, generation t consumes its savings plus interest rate, i.e. $c_{t+1}^t = (1 + r_{t+1} - \delta)s_t^t$ and then dies.

Sequential markets equilibrium, I

Given $\bar{k}_1$, a sequential markets equilibrium is allocations for households $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$, allocations for the firm $\{(\hat{K}_t, \hat{L}_t)\}_{t=1}^{\infty}$ and prices $\{(\hat{r}_t, \hat{w}_t)\}_{t=1}^{\infty}$ such that:

1. For all $t \geq 1$, given $(\hat{w}_t, \hat{r}_{t+1})$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solves

$$\max_{c_t^t, c_{t+1}^t \geq 0, s_t^t} U(c_t^t) + \beta U(c_{t+1}^t)$$

$$\text{s.t. } c_t^t + s_t^t \leq \hat{w}_t$$

$$c_{t+1}^t \leq (1 + \hat{r}_{t+1} - \delta) s_t^t$$

2. Given $\bar{k}_1$ and $\hat{r}_1$, $\hat{c}_1^0$ solves

$$\max_{c_1^0 \geq 0} U(c_1^0)$$

$$\text{s.t. } c_1^0 \leq (1 + \hat{r}_1 - \delta) \bar{k}_1$$

Sequential markets equilibrium, II

3. For all $t \geq 1$, given $(\hat{r}_t, \hat{w}_t)$, $(\hat{K}_t, \hat{L}_t)$ solves

$$\max_{K_t, L_t \geq 0} F(K_t, L_t) - \hat{r}_t K_t - \hat{w}_t L_t$$

4. For all $t \geq 1$:

   4.1 (Goods Market) $N_t^t \hat{c}_t^t + N_t^{t-1} \hat{c}_t^{t-1} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t)$.

   4.2 (Asset Market) $N_t^t \hat{s}_t^t = \hat{K}_{t+1}$.

   4.3 (Labor Market) $N_t^t = \hat{L}_t$.

Stationary equilibrium

A steady state (or stationary equilibrium) is $(\bar{k}, \bar{s}, \bar{c}_1, \bar{c}_2, \bar{r}, \bar{w})$ such that the sequences $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$, $\{(\hat{K}_t, \hat{L}_t)\}_{t=1}^{\infty}$ and $\{(\hat{r}_t, \hat{w}_t)\}_{t=1}^{\infty}$, defined by

$$
\begin{aligned}
\hat{c}_t^t &= \bar{c}_1 \\
\hat{c}_t^{t-1} &= \bar{c}_2 \\
\hat{s}_t^t &= \bar{s} \\
\hat{r}_t &= \bar{r} \\
\hat{w}_t &= \bar{w} \\
\hat{K}_t &= \bar{k} * N_t^t \\
\hat{L}_t &= N_t^t
\end{aligned}
$$

are an equilibrium, for given initial condition $\bar{k}_1 = \bar{k}$.

Saving equals investment

- Investment:

$$\hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t) - \left(N_t^t \hat{c}_t^t + N_t^{t-1} \hat{c}_t^{t-1}\right)$$

- Saving:

$$\underbrace{N_t^t \hat{s}_t^t}_{\text{savings of the young}} - \underbrace{N_{t-1}^{t-1} \hat{s}_{t-1}^{t-1}}_{\text{dissavings of the old}}$$

- Also:

$$N_{t-1}^{t-1} \hat{s}_{t-1}^{t-1} = (1-\delta)\hat{K}_t$$

- Hence,

$$\hat{K}_{t+1} - (1-\delta)\hat{K}_t = N_t^t \hat{s}_t^t - (1-\delta)\hat{K}_t$$

  or our asset market equilibrium condition

$$N_t^t \hat{s}_t^t = \hat{K}_{t+1}$$

Characterizating equilibrium

- Characterizing the equilibrium in an OLG model with production is difficult.
- However, we can prove in general existence of equilibrium.
- We can have multiplicity of equilibria, even without money.
- Moreover, we may even have chaotic dynamics.
- Welfare theorems break down.

Optimality of allocations, I

- Consider first steady state equilibria.
- Let c_1^*, c_2^* be the steady state consumption levels when young and old, respectively, and k^* be the steady state capital labor ratio.
- Consider the goods market clearing (or resource constraint):

$$N_t^t \hat{c}_t^t + N_t^{t-1} \hat{c}_t^{t-1} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t)$$

- Divide by $N_t^t = \hat{L}_t$ to obtain:

$$\hat{c}_t^t + \frac{\hat{c}_t^{t-1}}{1+n} + (1+n)\hat{k}_{t+1} - (1-\delta)\hat{k}_t = f(k_t)$$

- Use the steady state allocations to obtain:

$$c_1^* + \frac{c_2^*}{1+n} + (1+n)k^* - (1-\delta)k^* = f(k^*)$$

Optimality of allocations, II

- Define $c^* = c_1^* + \frac{c_2^*}{1+n}$ to be total (per worker) consumption in the steady state. We have that:

$$c^* = f(k^*) - (n + \delta)k^*$$

- Now suppose that the steady state equilibrium satisfies:

$$f'(k^*) - \delta < n$$

something that may or may not hold, depending on functional forms and parameter values.

- This steady state is not Pareto optimal: the equilibrium is dynamically inefficient.

Intuition

- If $f'(k^*) - \delta < n$, it is possible to decrease the capital stock per worker marginally, and the effect on per capita consumption is
$$\frac{dc^*}{dk^*} = f'(k^*) - (n + \delta) < 0$$
so that a marginal decrease of the capital stock leads to higher available overall consumption.
- An allocation is inefficient if the interest rate (in the steady state) is smaller than the population growth rate, that is, if we are in the Samuelson case.

General result

Theorem

Cass (1972), Balasko and Shell (1980). A feasible allocation is Pareto optimal if and only if

$$\sum_{t=1}^{\infty} \prod_{\tau=1}^{t} \frac{(1 + r_{\tau+1} - \delta)}{(1 + n_{\tau+1})} = +\infty$$

As an obvious corollary, alluded to before we have that a steady state equilibrium is Pareto optimal (or dynamically efficient) if and only if

$$f'(k^*) - \delta \geq n$$

With technological progress:

$$f'(k^*) - \delta \geq n + g$$

Empirical relevance

- Dynamic inefficiency is not purely an academic matter.
- Some reasonable numbers: U.S. population growth $n \approx 1\%$, $g \approx 2\%$.
- Is rate of return higher or lower than 3%?
- Abel, Mankiw, Summers, and Zeckhauser (1989) extend result to an economy with uncertainty.
- Sufficient condition for dynamic efficiency: net capital income exceeds investment.
- U.S., net capital income investment$\approx$ 17% of GDP, net capital income $\approx$ 19% of GDP.

Policy implications

- If the competitive equilibrium of the economy features dynamic inefficiency, its citizens save more than is socially optimal.
- Hence, we need government programs that reduce national saving:
  - Tax on capital.
  - An unfunded, or pay-as-you-go social security system.
  - Having government debt.