

OLG models

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Overlapping generations

The overlapping generations model

- Besides the neoclassical growth model, the OLG model is the second major workhorse of modern macroeconomics.
- Pioneered by **Allais (1947)**, **Samuelson (1958)**, and **Diamond (1965)**.
- Important features of the model:
 1. Competitive equilibria may be Pareto suboptimal.
 2. Outside money may have positive value.
 3. There may exist a continuum of equilibria.

Motivation I

- Shortcoming of the infinitely lived agents model: individuals, apparently, do not live forever.
- But:
 1. Role of assumptions in economic theory. Friedman's *Essays on Positive Economics*.
 2. An altruistic bequest motive makes individuals that live for a finite number of periods to maximize the utility of the entire dynasty.
- More relevant motivation: we want models where agents undergo an interesting life cycle with low-income youth, high-income middle ages, and retirement where labor income drops to zero.

Motivation II

- Why?
 1. Integrate micro and macro data.
 2. Analyze issues like social security, the effect of taxes on retirement decisions, the distributive effects of taxes versus government deficits, the effects of life-cycle saving on capital accumulation, educational policies, etc.
- Final motivation: because of its interesting (some say, pathological) theoretical properties, it is also an area of intense study among economic theorists.
- How much should we believe those theoretical properties?
- Role of quantitative OLG models with a large number of generations.

Setup

Basic setup of the model

- Time is discrete, $t = 1, 2, 3, \dots$ and the economy (but not its people) lives forever.
- In each period, there is a single, nonstorable consumption good.
- In each period, a new generation (of measure 1) is born, which we index by its date of birth.
- People live for two periods and then die.
- Alternative: stochastic aging (Blanchard, 1985). We do not need to keep track of age distributions.

Endowments and consumption

- (e_t^t, e_{t+1}^t) : generation t 's endowment of the consumption good in the first and second period of their live.
- (c_t^t, c_{t+1}^t) : consumption allocation of generation t .
- At time t , there are two generations alive:
 1. One old generation $t - 1$ that has endowment e_t^{t-1} and consumption c_t^{t-1} .
 2. One young generation t that has endowment e_t^t and consumption c_t^t .
- In period 1, there is an initial old generation 0 that has endowment e_1^0 and consumes c_1^0 .

Timing

generation \ time	1	2	...	t	$t + 1$
0	(c_1^0, e_1^0)				
1	(c_1^1, e_1^1)	(c_2^1, e_2^1)			
\vdots			\ddots		
$t - 1$				(c_t^{t-1}, e_t^{t-1})	
t				(c_t^t, e_t^t)	(c_{t+1}^t, e_{t+1}^t)
$t + 1$					$(c_{t+1}^{t+1}, e_{t+1}^{t+1})$

Double infinities

- There are both an infinite number of periods as well as an infinite number of agents.
- This “double infinity” has been cited to be the major source of the theoretical peculiarities of the OLG model (prominently by **Karl Shell**).
- For example, double infinity will be key for the failure of the first fundamental welfare theorem to hold in the model.
- Mechanism: value of aggregate endowment at equilibrium prices may not be finite.

Outside and inside money

- In some of our applications, we will endow the initial generation with an amount of outside money m .
- **Outside money**: money that is, on the net, an asset of the private economy. This includes fiat currency issued by the government.
- **Inside money** (such as bank deposits) is both an asset as well as a liability of the private sector (in the case of deposits: an asset of the deposit holder, a liability to the bank).
- If $m \geq 0$, then m can be interpreted as fiat money.
- If $m < 0$, one should envision the initial old people having borrowed from some institution (outside the model) and m is the amount to be repaid.

- Preferences of individuals are representable by:

$$u_t(c) = U(c_t^t) + \beta U(c_{t+1}^t)$$

- Preferences of the initial old generation is representable by:

$$u_0(c) = U(c_1^0)$$

- We will assume that U is strictly increasing, strictly concave, and twice continuously differentiable.

Allocations I

Definition

An **allocation** is a sequence $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^\infty$.

Definition

An allocation is **feasible** if $c_t^{t-1}, c_t^t \geq 0$ for all $t \geq 1$ and

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1$$

Definition

An allocation is **stationary** if $c_{t-1}^t, c_t^t \geq 0$ for all $t \geq 1$ and

$$c_t^{t-1} = c_t^t = c \text{ for all } t \geq 1$$

Definition

An allocation $c_1^0, \{(c_t^t, c_{t+1}^t)\}_{t=1}^\infty$ is **Pareto optimal** if it is feasible and if there is no other feasible allocation $\hat{c}_0^1, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^\infty$ such that:

$$\begin{aligned}u_t(\hat{c}_t^t, \hat{c}_{t+1}^t) &\geq u_t(c_t^t, c_{t+1}^t) \text{ for all } t \geq 1 \\u_0(\hat{c}_1^0) &\geq u_0(c_1^0)\end{aligned}$$

with strict inequality for at least one $t \geq 0$.

- In the presence of money ($m \neq 0$), we will take money to be the numéraire.
- This is important since we can only normalize the price of one commodity to **1**.
- With money, no further normalizations are admissible.
- Let p_t be the price of one unit of the consumption good at period t .

Markets structure

- As in the infinite horizon model, we have two frameworks: **Arrow-Debreu** and **sequential trading**.
- Arrow-Debreu framework, trading takes place in a hypothetical centralized marketplace at period 0 (even though the generations are not born yet).
- Plausibility?
- Alternative interpretation: standard GE framework except agents care about consumption only in two periods.

Sequential trading

- Trade takes place sequentially in spot markets for consumption goods that open in each period.
- In addition, there is an asset market through which individuals carry their savings.
- Let r_{t+1} be the interest rate from period t to period $t + 1$ and s_t^t be the savings of generation t from period t to period $t + 1$.
- We will consider assets that cost one unit of consumption in period t and deliver $1 + r_{t+1}$ units tomorrow. Those assets are easier to handle than zero-coupon bonds if the asset at hand is fiat money. However, both assets have identical implications.
- We do not need a Ponzi condition.

Arrow-Debreu equilibrium

Given m , an Arrow-Debreu equilibrium is an allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^\infty$ and prices $\{p_t\}_{t=1}^\infty$ such that

1. Given $\{p_t\}_{t=1}^\infty$, for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ solves:

$$\begin{aligned} & \max_{(c_t^t, c_{t+1}^t) \geq 0} u_t(c_t^t, c_{t+1}^t) \\ \text{s.t. } & p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t e_t^t + p_{t+1} e_{t+1}^t \end{aligned}$$

2. Given p_1, \hat{c}_1^0 solves:

$$\begin{aligned} & \max_{c_1^0} u_0(c_1^0) \\ \text{s.t. } & p_1 c_1^0 \leq p_1 e_1^0 + m \end{aligned}$$

3. For all $t \geq 1$ (resource balance or goods market clearing):

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1$$

Sequential markets equilibrium

Given m , a sequential markets equilibrium is an allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^\infty$ and interest rates $\{r_t\}_{t=1}^\infty$ such that:

1. Given $\{r_t\}_{t=1}^\infty$ for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solves:

$$\begin{aligned} & \max_{(c_t^t, c_{t+1}^t) \geq 0, s_t^t} u_t(c_t^t, c_{t+1}^t) \\ & \text{s.t. } c_t^t + s_t^t \leq e_t^t \\ & c_{t+1}^t \leq e_{t+1}^t + (1 + r_{t+1})s_t^t \end{aligned}$$

2. Given r_1, \hat{c}_1^0 solves:

$$\begin{aligned} & \max_{c_1^0} u_0(c_1^0) \\ & \text{s.t. } c_1^0 \leq e_1^0 + (1 + r_1)m \end{aligned}$$

3. For all $t \geq 1$ (resource balance or goods market clearing):

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1$$

Market clearing condition for the asset market I

- Given that the period utility function U is strictly increasing, the budget constraints hold with equality.
- Summing the budget constraints of agents:

$$c_{t+1}^t + c_{t+1}^{t+1} + s_{t+1}^{t+1} = e_{t+1}^t + e_{t+1}^{t+1} + (1 + r_{t+1})s_t^t$$

- By resource balance:

$$s_{t+1}^{t+1} = (1 + r_{t+1})s_t^t$$

- Doing the same manipulations for generation 0 and 1:

$$s_1^1 = (1 + r_1)m$$

Market clearing condition for the asset market II

- By repeated substitution:

$$s_t^t = \prod_{\tau=1}^t (1 + r_\tau) m$$

- The amount of saving (in terms of the period t consumption good) has to equal the value of the outside supply of assets, $\prod_{\tau=1}^t (1 + r_\tau) m$.
- Interpretation.
- This condition should appear in the definition of equilibrium. By Walras' law, however, either the asset market or the good market equilibrium condition is redundant.

Equivalence between equilibria

- For $r_{t+1} > -1$, we combine both budget constraints into:

$$c_t^t + \frac{1}{1+r_{t+1}} c_{t+1}^t = e_t^t + \frac{1}{1+r_{t+1}} e_{t+1}^t$$

- Divide by $p_t > 0$:

$$c_t^t + \frac{p_{t+1}}{p_t} c_{t+1}^t = e_t^t + \frac{p_{t+1}}{p_t} e_{t+1}^t$$

- Divide initial old generation by $p_1 > 0$ to obtain:

$$c_1^0 \leq e_1^0 + \frac{m}{p_1}$$

- Hence, it looks that $1 + r_{t+1} = \frac{p_t}{p_{t+1}}$ must play a key role.

Equivalence proposition

- Given equilibrium Arrow-Debreu prices $\{p_t\}_{t=1}^{\infty}$, define interest rates:

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}}$$
$$1 + r_1 = \frac{1}{p_1}$$

- These interest rates induce a sequential markets equilibrium with the same allocation as the Arrow-Debreu equilibrium.
- Conversely, given equilibrium sequential markets interest rates $\{r_t\}_{t=1}^{\infty}$, define Arrow-Debreu prices by:

$$p_1 = \frac{1}{1 + r_1}$$
$$p_{t+1} = \frac{p_t}{1 + r_{t+1}}$$

- These prices induce allocations that are equivalent to the sequential markets equilibrium.

- From the equivalence, the return on the asset equals:

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}} = \frac{1}{1 + \pi_{t+1}}$$

$$(1 + r_{t+1})(1 + \pi_{t+1}) = 1$$

$$r_{t+1} \approx -\pi_{t+1}$$

where π_{t+1} is the inflation rate from period t to $t + 1$.

- The real return on money equals the negative of the inflation rate.

More on the equivalence I

- Using:

$$p_1 = \frac{1}{1 + r_1}$$
$$p_{t+1} = \frac{p_t}{1 + r_{t+1}}$$

with repeated substitution delivers:

$$p_t = \frac{1}{\prod_{\tau=1}^t (1 + r_\tau)} \Rightarrow \prod_{\tau=1}^t (1 + r_\tau) = \frac{1}{p_t}$$

- Interpretation.

More on the equivalence II

- Now, note that we argued before that

$$s_t^t = \prod_{\tau=1}^t (1 + r_\tau) m$$

Hence:

$$s_t^t = \frac{m}{p_t}$$

- You can think about this last condition both as:
 1. An equilibrium condition.
 2. A money demand function.

Offer curves

Offer curves

- Gale (1973) developed a nice way of analyzing the equilibria of a two-period OLG economy graphically: using offer curves.
- First, assume that the economy is stationary in that $e_t^t = w_1$ and $e_{t+1}^t = w_2$, that is, the endowments are time invariant. This is to simplify derivations and avoid carrying (e_t^t, e_{t+1}^t) as arguments of functions.
- For given $p_t, p_{t+1} > 0$, let by $c_t^t(p_t, p_{t+1})$ and $c_{t+1}^t(p_t, p_{t+1})$ denote the solution to maximization problem of agent for all $t \geq 1$.
- Given our assumptions, this solution is unique.

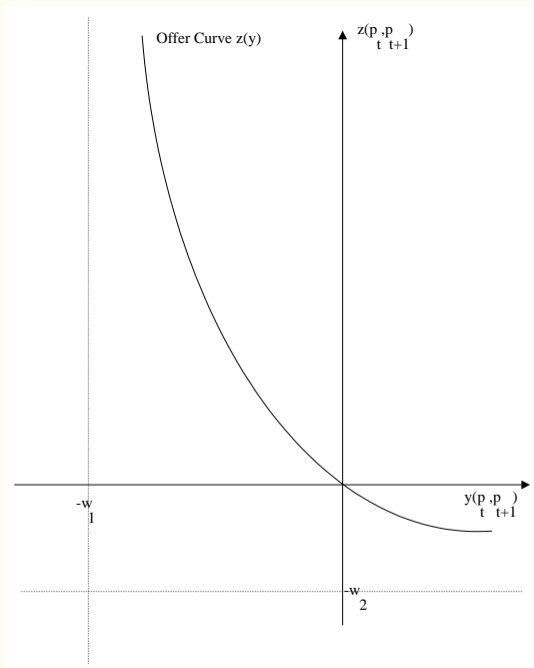
Excess demand functions

- Define the excess demand functions:

$$y(p_t, p_{t+1}) = c_t^t(p_t, p_{t+1}) - w_1$$

$$z(p_t, p_{t+1}) = c_{t+1}^t(p_t, p_{t+1}) - w_2$$

- These functions summarize, for given prices, consumer optimization.
- y and z only depend on $\frac{p_{t+1}}{p_t}$, but not on p_t and p_{t+1} separately (the excess demand functions are homogeneous of degree zero in prices).
- Varying $\frac{p_{t+1}}{p_t}$ between 0 and ∞ (not inclusive), we obtain the offer curve: a locus of optimal excess demands in (y, z) space $(y, f(y))$.
- f can be a multi-valued correspondence.
- A point on the offer curve is an optimal excess demand function for some $\frac{p_{t+1}}{p_t} \in (0, \infty)$.



More on offer curves I

- Since $c_t^t(p_t, p_{t+1}) \geq 0$ and $c_{t+1}^t(p_t, p_{t+1}) \geq 0$, we have $y(p_t, p_{t+1}) \geq -w_1$ and $z(p_t, p_{t+1}) \geq -w_2$.
- Since the optimal choices obviously satisfy the budget constraint:

$$p_t y(p_t, p_{t+1}) + p_{t+1} z(p_t, p_{t+1}) = 0 \Rightarrow$$
$$\frac{z(p_t, p_{t+1})}{y(p_t, p_{t+1})} = -\frac{p_t}{p_{t+1}},$$

one equation in two unknowns (p_t, p_{t+1}) for a given $t \geq 1$.

- $(y, z) = (0, 0)$ is on the offer curve, as for appropriate prices, no trade is the optimal trading strategy.
- For a given point on the offer curve $(y(p_t, p_{t+1}), z(p_t, p_{t+1}))$ with $y(p_t, p_{t+1}) \neq 0$, the slope of the straight line through the point (y, z) and the origin is $-\frac{p_t}{p_{t+1}}$.

More on offer curves II

- We can express goods market clearing in terms of excess demand functions as

$$y(p_t, p_{t+1}) + z(p_{t-1}, p_t) = 0$$

- Also, for the initial old generation the excess demand function is given by

$$z_0(p_1, m) = \frac{m}{p_1}$$

so that the goods market equilibrium condition for the first period reads as

$$y(p_1, p_2) + z_0(p_1, m) = 0$$

- Finally, notice that we have:

$$s_t^t = -y(p_t, p_{t+1}) = \frac{m}{p_t}$$

and

$$z(p_t, p_{t+1}) = \frac{m}{p_{t+1}}$$

- These conditions highlight the role of money as a mechanism for intertemporal trade.

More on offer curves IV

- Using homogeneity, an alternative way to express them is as:

$$s_t^t = f(r_{t+1}) = \frac{m}{p_t}$$

and

$$g(r_{t+1}) = \frac{m}{p_{t+1}}$$

- Also, notice that

$$f(r_{t+1}) = \frac{m}{p_t} = g(r_t)$$

is an aggregate resource constraint that implies a difference equation on r_t .

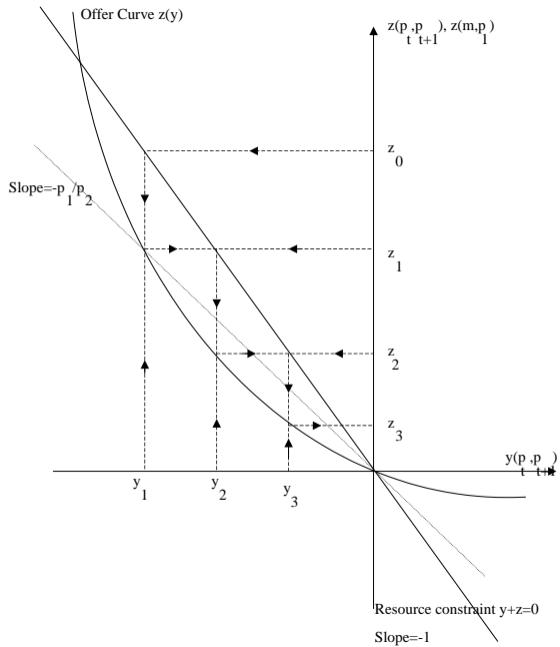
- This motivates us to propose a simple algorithm to find equilibria.

Algorithm to find equilibria

1. Pick an initial price p_1 (this is NOT a normalization since p_1 determines the real value of money $\frac{m}{p_1}$ the initial old generation is endowed with; we have already normalized the price of money). Hence, we know $z_0(p_1, m)$. This determines $y(p_1, p_2)$.
2. From the offer curve, we determine $z(p_1, p_2) \in f(y(p_1, p_2))$. Note that if f is a correspondence then there are multiple choices for z .
3. Once we know $z(p_1, p_2)$, we can find $y(p_2, p_3)$ and so forth. In this way we determine the entire equilibrium consumption allocation:

$$\begin{aligned}c_1^0 &= z_0(p_1, m) + w_2 \\c_t^t &= y(p_t, p_{t+1}) + w_1 \\c_{t+1}^t &= z(p_t, p_{t+1}) + w_2\end{aligned}$$

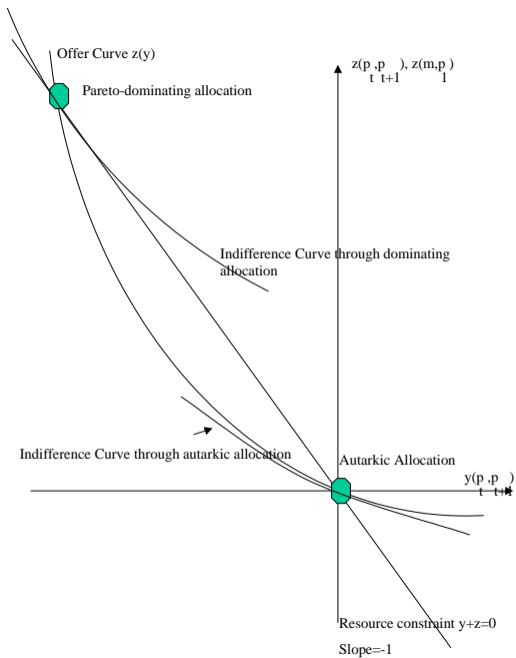
4. Equilibrium prices can then be found, given p_1 .



- Any initial p_1 that induces sequences $c_1^0, \{(c_t^t, c_{t+1}^t), p_t\}_{t=1}^\infty$ such that the consumption sequence satisfies $c_t^{t-1}, c_t^t \geq 0$ is an equilibrium for given money stock.
- This already indicates the possibility of a lot of equilibria for this model.
- In general, the price ratio supporting the autarkic equilibrium satisfies:

$$\frac{p_t}{p_{t+1}} = \frac{U'(e_t^t)}{\beta U'(e_{t+1}^t)} = \frac{U'(w_1)}{\beta U'(w_2)}$$

and this ratio represents the slope of the offer curve at the origin.



Inefficiencies

- Define the autarkic interest rate as:

$$1 + \bar{r} = \frac{U'(w_1)}{\beta U'(w_2)}$$

- Gale (1973):
 - Samuelson case: $\bar{r} < 0$.
 - Classical case: $\bar{r} \geq 0$.

Inefficient equilibria I

- Competitive equilibria in OLG models may not be Pareto optimal.

Sufficient Condition

If $\sum_{t=1}^{\infty} p_t < \infty$, then the competitive equilibrium allocation for any pure exchange OLG economy is Pareto-efficient.

- If, however, the value of the aggregate endowment is infinite (at the equilibrium prices), then the competitive equilibrium MAY not be Pareto optimal.
- Inefficiency is therefore associated with low (negative) interest rates.

Inefficient equilibria II

- **Balasko and Shell (1980)** show that, under certain technical conditions, the autarkic equilibrium is Pareto optimal if and only if:

$$\sum_{t=1}^{\infty} \prod_{\tau=1}^t (1 + r_{\tau}) = +\infty$$

where $\{r_{t+1}\}$ is the sequence of autarkic equilibrium interest rates.

- Remember that:

$$p_t = \frac{1}{\prod_{\tau=1}^t (1 + r_{\tau})}$$

- Hence, an autarkic equilibrium is Pareto optimal if and only if:

$$\sum_{t=1}^{\infty} \frac{1}{p_t} = +\infty$$

that is, if prices do not explode.

Intuition I

- Take the autarkic allocation and try to construct a Pareto improvement.
- In particular, give additional $\delta_0 > 0$ units of consumption to the initial old generation. This obviously improves this generation's life.
- From resource feasibility, this requires taking away δ_0 from generation 1 in their first period of life.
- To make them not worse off, they have to receive δ_1 in additional consumption in their second period of life, with δ_1 satisfying:

$$\delta_0 U'(e_1^1) = \delta_1 \beta U'(e_2^1)$$

or

$$\delta_1 = \delta_0 \frac{U'(e_1^1)}{\beta U'(e_2^1)} = \delta_0(1 + r_2) > 0$$

Intuition II

- In general, the required transfers in the second period of generation t 's life to compensate for the reduction of first period consumption:

$$\delta_t = \delta_0 \prod_{\tau=1}^t (1 + r_{\tau+1})$$

- Such a scheme does not work if the economy ends at fine time T since the last generation (that lives only through youth) is worse off.
- But as our economy extends forever, such an intergenerational transfer scheme is feasible provided that the δ_t do not grow too fast, that is, if interest rates are sufficiently low.
- But if such a transfer scheme is feasible, then we found a Pareto improvement over the original autarkic allocation, and hence the autarkic equilibrium allocation is not Pareto efficient.

More on money

Positive valuation of outside money

- Second main result of OLG models: outside money may have positive value.
- Money in this equilibrium is a **bubble**:
 1. The fundamental value of an asset is the value of its dividends, evaluated at the equilibrium Arrow-Debreu prices.
 2. An asset has a bubble if its price does not equal its fundamental value.
 3. Since money does not pay dividends, its fundamental value is zero and the fact that it is valued positively in equilibrium makes it a bubble.

Intuition

- The current young generation transfer some of their endowment to the old people for pieces of paper because they expect (correctly so, in equilibrium) to exchange these pieces of paper against consumption goods when they are old.
- Hence, we achieve an intertemporal allocation of consumption goods that dominates the autarkic allocation.
- Without the outside asset, again, this economy can do nothing else but remain in the possibly dismal state of autarky.
- This is why the social contrivance of money is so useful in this economy.
- As we will see later, other institutions (for example, a pay-as-you-go social security system or a gift-giving mechanism) may achieve the same as money.
- Relation with search models of money.
- More general point: money is memory ([Kocherlakota, 1998](#)).

Comparison with exchange economies

Theorem

In pure exchange economies with a finite number of infinitely lived agents, there cannot be an equilibrium in which outside money is valued.

Necessary conditions

Suppose, that there is an equilibrium $\{(\hat{c}_t^i)_{i \in I}\}_{t=1}^{\infty}, \{\hat{p}_t\}_{t=1}^{\infty}$ for initial endowments of outside money $(m^i)_{i \in I}$ such that $\sum_{i \in I} m^i \neq 0$. By local nonsatiation:

$$\sum_{t=1}^{\infty} \hat{p}_t \hat{c}_t^i = \sum_{t=1}^{\infty} \hat{p}_t e_t^i + m^i < \infty$$

Summing over all individuals $i \in I$ yields $\sum_{t=1}^{\infty} \hat{p}_t \sum_{i \in I} (\hat{c}_t^i - e_t^i) = \sum_{i \in I} m^i$.

But resource feasibility requires $\sum_{i \in I} (\hat{c}_t^i - e_t^i) = 0$ for all $t \geq 1$ and hence $\sum_{i \in I} m^i = 0$, a contradiction.

Deficit finance I

- Presence of money allows thinking about government financing: issuing or retiring currency. Hence, we will index m_t .
- Imagine government consumption g .
- Thrown into the sea (or enters separably in the utility function).
- Lump sum taxes on each generation τ_1 and τ_2 .
- Constant endowment (as in the offer curves section).
- Then:

$$m_t - m_{t-1} = p_t (g - \tau_1 - \tau_2) = p_t d$$

Deficit finance II

- Now, remember that $f(r_{t+1}) = \frac{m_t}{p_t}$.
- Hence:

$$\begin{aligned} \underbrace{f(r_{t+1})}_{\text{Young Saving}} &= \underbrace{\frac{m_{t-1}}{p_t}}_{\text{Old Dissaving}} + \underbrace{\frac{m_t - m_{t-1}}{p_t}}_{\text{Government Dissaving}} \\ &= \frac{m_{t-1}}{p_t} + d \\ &= \frac{m_{t-1}}{p_{t-1}} \frac{p_{t-1}}{p_t} + d \\ &= f(r_t)(1 + r_t) + d \end{aligned}$$

with initial equation

$$f(r_1) = \frac{m_0}{p_1} + d$$

Deficit finance III

- Since endowments are constant, we can solve the difference equation by “guess-and-verify” a constant interest rate:

$$\begin{aligned}f(r) &= f(r)(1+r) + d \Rightarrow \\rf(r) &= d\end{aligned}$$

and

$$f(r) = \frac{m_0}{p_1} + d$$

- Since r is a tax on real balances, $rf(r)$ is a Laffer curve.
- Multiple equilibria:
 1. Stationary and non-stationary (continuum).
 2. Pareto-ranked.
- More general property: the existence of interesting equilibria.

Interesting equilibria

Continuum of equilibria I

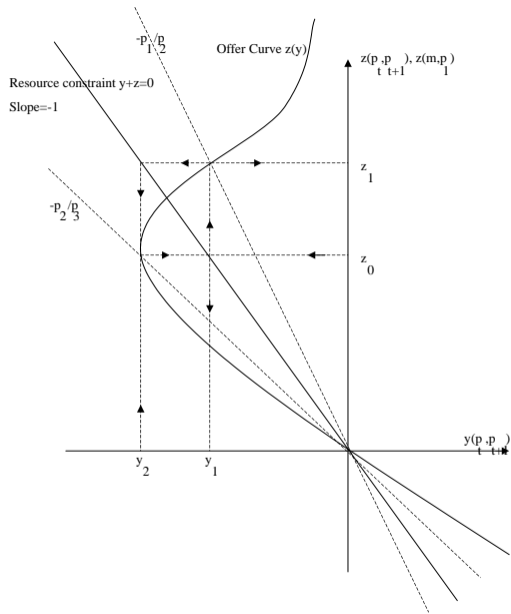
- Third major difference: the possibility of a whole continuum of equilibria in OLG models.
- General proof is complicated.
- We can build non-stationary equilibria that in the limit converge to the same allocation (autarky), they differ in the sense that at any finite t , the consumption allocations and price ratios (and levels) differ across equilibria. These equilibria are arbitrarily close to each other.
- This is again in stark contrast to standard Arrow-Debreu economies where, generically, the set of equilibria is finite and all equilibria are locally unique.
- Generically: for almost all endowments, that is, the set of possible values for the endowments for which this statement does not hold is of measure zero.

Continuum of equilibria II

- Local uniqueness: for every equilibrium price vector, there exists ε such that any ε -neighborhood of the price vector does not contain another equilibrium price vector, apart from the trivial ones involving a different normalization (Debreu, 1970).
- If we are in the *Samuelson case* $\bar{r} < 0$, then (and only then) all these equilibria are Pareto-ranked.
- If we introduce a productive asset with positive dividends and no money, there exists a unique equilibrium, which is Pareto optimal.
- It is not the existence of a long-lived outside asset that is responsible for the existence of a continuum of equilibria.
- If we introduce a Lucas tree with negative dividends (the initial old generation is an eternal slave, say, of the government and has to come up with d in every period to be used for government consumption), then the existence of the whole continuum of equilibria is restored.

Endogenous cycles

- The equilibria in OLG economies need not be monotonic.
- Instead, equilibria with cycles are possible.
- Take an offer curve that is backward bending.
- After period $t = 2$, the economy repeats the cycle from the first two periods.
- In addition, we will have sunspots.



Equilibrium

- The equilibrium allocation is of the form:

$$c_t^{t-1} = \begin{cases} c^{ol} = z_0 - w_2 & \text{for } t \text{ odd} \\ c^{oh} = z_1 - w_2 & \text{for } t \text{ even} \end{cases}$$
$$c_t^t = \begin{cases} c^{yl} = y_1 - w_1 & \text{for } t \text{ odd} \\ c^{yh} = y_2 - w_1 & \text{for } t \text{ even} \end{cases}$$

with $c^{ol} < c^{oh}$, $c^{yl} < c^{yh}$.

- Prices satisfy:

$$\frac{p_t}{p_{t+1}} = \begin{cases} \alpha^h & \text{for } t \text{ odd} \\ \alpha^l & \text{for } t \text{ even} \end{cases}$$
$$\pi_{t+1} = -r_{t+1} = \begin{cases} \pi^l < 0 & \text{for } t \text{ odd} \\ \pi^h > 0 & \text{for } t \text{ even} \end{cases}$$

- These cycles are purely endogenous in the sense that the environment is completely stationary: nothing distinguishes odd and even periods.
- Also, notice that it is not particularly difficult to construct cycles of length bigger than 2 periods.
- We can also build chaotic economies.
- Some economists have taken this feature of OLG models to be the basis of a theory of endogenous business cycles (see, for example, [Grandmont, 1985](#)).