Equilibrium with Complete Markets

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Arrow-Debreu versus Sequential Markets

- In previous lecture, we discussed the preferences of agents in a situation with uncertainty.

- Now, we will discuss how markets operate in a simple endowment economy.

- Two approaches: Arrow-Debreu set-up and sequential markets.

- Under some technical conditions both approaches are equivalent.

Environment

- We have I agents, $i = 1, ..., I$.

- Endowment:

$$(e^1, ..., e^I) = \{e_t^1(s^t), ..., e_t^I(s^t)\}_{t=0, s^t \in S^t}^{\infty}$$

- Tradition in macro of looking at endowment economies. Why? Consumption, risk-sharing, asset pricing.

- Advantages and shortcomings.

Allocation

Definition

An allocation is a sequence of consumption in each period and event for each individual:

$$(c^1, ..., c^I) = \{c_t^1(s^t), ..., c_t^I(s^t)\}_{t=0, s^t \in S^t}^{\infty}$$

Definition

Feasible allocation: an allocation such that:

$$\begin{aligned} c_t^i(s^t) &\geq 0 \text{ for all } t, \text{ all } s^t \in S^t, \text{ for } i = 1, 2 \\ \sum_{i=1}^{I} c_t^i(s^t) &\leq \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, \text{ all } s^t \in S^t \end{aligned}$$

Pareto Efficiency

Definition

An allocation $\left\{\left(c_t^1(s^t), ..., c_t^I(s^t)\right)\right\}_{t=0, s^t \in S^t}^{\infty}$ is Pareto efficient if it is feasible and if there is no other feasible allocation

$$\left\{\left(\tilde{c}_t^1(s^t), ..., \tilde{c}_t^I(s^t)\right)\right\}_{t=0, s^t \in S^t}^{\infty}$$

such that

$$\begin{aligned} u(\tilde{c}^i) &\geq u(c^i) \text{ for all } i \\ u(\tilde{c}^i) &> u(c^i) \text{ for at least one } i \end{aligned}$$

- Ex ante versus ex post efficiency.

Arrow-Debreu Market Structure

- Trade takes place at period 0, *before* any uncertainty has been realized (in particular, before s_0 has been realized).

- As for allocation and endowment, Arrow-Debreu prices have to be indexed by event histories in addition to time.

- Let $p_t(s^t)$ denote the price of one unit of consumption, quoted at period 0, delivered at period t if (and only if) event history s^t has been realized.

- We need to normalize one price to 1 and use it as numeraire.

Arrow-Debreu Equilibrium

Definition

An Arrow-Debreu equilibrium are prices $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ and allocations $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ such that:

1. Given $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$, for $i = 1, .., I$, $\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ solves:

$$\max_{\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t \pi(s^t) u(c_t^i(s^t))$$

$$\text{s.t. } \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \widehat{p}_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \widehat{p}_t(s^t) e_t^i(s^t)$$

$$c_t^i(s^t) \geq 0 \text{ for all } t$$

2. Markets clear:

$$\sum^{I} \hat{c}_t^i(s^t) = \sum^{I} e_t^i(s^t) \text{ for all } t, \text{ all } s^t \in S^t$$

Welfare Theorems

Theorem

Let $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ *be a competitive equilibrium allocation. Then,* $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ *is Pareto efficient.*

Theorem

Let $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ *be Pareto efficient. Then, there is a an A-D equilibrium with price* $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ *that decentralizes the allocation* $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$.

Sequential Markets Market Structure

- Now we will let trade take place sequentially in spot markets in each period, event-history pair.
- One period contingent IOU's: financial contracts bought in period t that pay out one unit of the consumption good in $t+1$ only for a particular realization of $s_{t+1} = j$ tomorrow.
- $Q_t(s^t, s_{t+1})$: price at period t of a contract that pays out one unit of consumption in period $t+1$ if and only if tomorrow's event is s_{t+1} (zero-coupon bonds).
- $a^i_{t+1}(s^t, s_{t+1})$: quantities of these Arrow securities bought (or sold) at period t by agent i.
- These contracts are often called Arrow securities, contingent claims or one-period insurance contracts.

Period-by-period Budget Constraint

- The period t, event history s^t budget constraint of agent i is given by

$$c_t^i(s^t) + \sum_{s_{t+1}|s^t} Q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \leq e_t^i(s^t) + a_t^i(s^t)$$

- Note: we only have prices and quantities.

- Many economists use expectations in the budget constraint. We will later see why. However, this is *bad* practice.

Natural Debt Limit

- We need to rule out Ponzi schemes.

- Tail of endowment distribution:

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{p_\tau(s^\tau)}{p_t(s^t)} e_\tau^i(s^\tau)$$

- $A_t^i(s^t)$ is known as the *natural debt limit*.

- Then:

$$-a_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1})$$

Sequential Markets Equilibrium

Definition

A SME is prices for Arrow securities $\left\{\widehat{Q}_t(s^t, s_{t+1})\right\}_{t=0,s^t\in S^t,s_{t+1}\in S}^{\infty}$ and allocations $\left\{\left(\hat{c}_t^i(s^t), \left\{\hat{a}_{t+1}^i(s^t, s_{t+1})\right\}_{s_{t+1}\in S}\right)_{i=1,..,I}\right\}_{t=0,s^t\in S^t}^{\infty}$ such that:

1. For $i = 1, .., I$, given $\left\{\widehat{Q}_t(s^t, s_{t+1})\right\}_{t=0,s^t\in S^t,s_{t+1}\in S}^{\infty}$, for all i, $\left\{\hat{c}_t^i(s^t), \left\{\hat{a}_{t+1}^i(s^t, s_{t+1})\right\}_{s_{t+1}\in S}\right\}_{t=0,s^t\in S^t}^{\infty}$ solves:

$$\max_{\left\{c_t^i(s^t),\left\{a_{t+1}^i(s^t,s_{t+1})\right\}_{s_{t+1}\in S}\right\}_{t=0,s^t\in S^t}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t \pi(s^t) u(c_t^i(s^t))$$

$$\text{s.t. } c_t^i(s^t) + \sum_{s_{t+1}|s^t} \widehat{Q}_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \leq e_t^i(s^t) + a_t^i(s^t)$$

$$c_t^i(s^t) \geq 0 \text{ for all } t, s^t \in S^t$$

$$a_{t+1}^i(s^t, s_{t+1}) \geq -A_{t+1}^i(s^{t+1}) \text{ for all } t, s^t \in S^t$$

Definition (cont.)

2. For all $t \geq 0$

$$\begin{aligned} \sum_{i=1}^{I} \hat{c}_t^i(s^t) &= \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, s^t \in S^t \\ \sum_{i=1}^{I} \hat{a}_{t+1}^i(s^t, s_{t+1}) &= 0 \text{ for all } t, s^t \in S^t \text{ and all } s_{t+1} \in S \end{aligned}$$

Equivalence of Arrow-Debreu and Sequential Markets Equilibria

- A full set of one-period Arrow securities is sufficient to make markets "sequentially complete."

- Any (nonnegative) consumption allocation is attainable with an appropriate sequence of Arrow security holdings $\{a_{t+1}(s^t, s_{t+1})\}$ satisfying all sequential markets budget constraints.

- Later, when we talk about asset pricing, we will discuss how to use $Q_t(s^t, s_{t+1} = j)$ to price any other security.

Pareto Problem

- We will extensively exploit the two welfare theorems.
- Negishi's (1960) method to compute competitive equilibria:
  1. We fix some Pareto weights.
  2. We solve the Pareto problem associated to those weights.
  3. We decentralize the resulting allocation using the second welfare theorem.
- All competitive equilibria correspond to some Pareto weights.

Social Planner's Problem

- We solve the social planners problem:

$$\max_{\left(\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}\right)_{i=1,..,I}} \sum_{i=1}^{I}\alpha_i \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi(s^t)u(c_t^i(s^t))$$

$$\text{s.t. } \sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, s^t \in S^t$$

$$c_t^i(s^t) \geq 0 \text{ for all } t, s^t \in S^t$$

where α_i are the Pareto weights.

Definition

An allocation $\left(\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}\right)_{i=1,..,I}$ is Pareto efficient if and only if it solves the social planners problem for some $(\alpha_i)_{i=1,...,I} \in [0,1]$.

Perfect Insurance

- We write the lagrangian for the problem:

$$\max_{\left(\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}\right)_{i=1,..,I}} \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\left\{\begin{array}{c}\sum_{i=1}^{I}\alpha_i\beta^t\pi(s^t)u(c_t^i(s^t)) \\ +\lambda_t\left(s^t\right)\left(\sum_{i=1}^{I}\left[e_t^i(s^t)-c_t^i(s^t)\right]\right)\end{array}\right\}$$

  where $\lambda_t\left(s^t\right)$ is the state-dependent lagrangian multiplier.
- We forget about the non-negativity constraints and take FOCs:

$$\alpha_i\beta^t\pi(s^t)u'(c_t^i(s^t))=\lambda_t\left(s^t\right)\ \text{for all } i,t,s^t\in S^t$$

- Then, by dividing the condition for two different agents:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))}=\frac{\alpha_j}{\alpha_i}$$

Definition

An allocation $\left(\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}\right)_{i=1,..,I}$ has perfect consumption insurance if the ratio of marginal utilities between two agents is constant across time and states

Irrelevance of History

- From previous equation, and making $j = 1$:

$$c_t^i(s^t) = u'^{-1}\left(\frac{\alpha_1}{\alpha_i} u'(c_t^1(s^t))\right)$$

- Summing over individuals and using aggregate resource constraint:

$$\sum_{i=1}^{I} e_t^i(s^t) = \sum_{i=1}^{I} u'^{-1}\left(\frac{\alpha_1}{\alpha_i} u'(c_t^1(s^t))\right)$$

which is one equation on one unknown, $c_t^1(s^t)$.

- Then, the pareto-efficient allocation $\left(\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}\right)_{i=1,..,I}$ only depends on aggregate endowment and not on s^t.

Theory Confronts Data

- Perfect insurance implies proportional changes in marginal utilities as a response to aggregate shocks.
- Do we see perfect risk-sharing in the data?
- Surprasingly more difficult to answer than you would think.
- Let us suppose we have CRRA utility function. Then, perfect insurance implies:

$$\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\alpha_i}{\alpha_j}\right)^{\frac{1}{\gamma}}$$

Individual Consumption

- Now, $c_t^i(s^t) = \frac{c_t^j(s^t)}{\alpha_j^{\frac{1}{\gamma}}} \alpha_i^{\frac{1}{\gamma}}$ and we sum over i

$$\sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} e_t^i(s^t) = \frac{c_t^j(s^t)}{\alpha_j^{\frac{1}{\gamma}}} \sum_{i=1}^{I} \alpha_i^{\frac{1}{\gamma}}$$

- Then:

$$c_t^j(s^t) = \frac{\alpha_j^{\frac{1}{\gamma}}}{\sum_{i=1}^{I} \alpha_i^{\frac{1}{\gamma}}} \sum_{i=1}^{I} e_t^i(s^t) = \theta_j y_t(s^t)$$

i.e., each agent consumes a constant fraction of the aggregate endowment.

Individual Level Regressions

- Take logs:

$$\log c_t^j(s^t) = \log \theta_j + \log y_t(s^t)$$

- If we take first differences,

$$\Delta \log c_t^j(s^t) = \Delta \log y_t(s^t)$$

- Equation we can estimate:

$$\Delta \log c_t^j(s^t) = \alpha_1 \Delta \log y_t(s^t) + \alpha_2 \Delta \log e_t^i(s^t) + \varepsilon_t^i$$

Estimating the Equation

- How do we estimate?

$$\Delta \log c_t^j(s^t) = \alpha_1 \Delta \log y_t(s^t) + \alpha_2 \Delta \log e_t^i(s^t) + \varepsilon_t^i$$

- CEX data.
- We get α_2 is different from zero (despite measurement error).
- Excess sensitivity of consumption by another name!
- Possible explanation?

Permanent Income Hypothesis

- Build the Lagrangian of the problem of the household i:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t)) - \mu_i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) \left(e_t^i(s^t) - c_t^i(s^t) \right)$$

Note: we have only one multiplier μ_i.

- Then, first order conditions are

$$\beta^t \pi(s^t) u'(c_t^i(s^t)) = \mu_i p_t(s^t) \text{for all } t, s^t \in S^t$$

- Substituting into the budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \frac{1}{\mu_i} \beta^t \pi(s^t) u'(c_t^i(s^t)) \left(e_t^i(s^t) - c_t^i(s^t) \right) = 0$$

No Aggregate Shocks

- Assume that $\sum_{i=1}^{I} e_t^i(s^t)$ is constant over time. From perfect insurance, we know then that $c_t^i(s^t)$ is also constant. Let's call it $\widehat{c}^i$.

- Then (cancelling constants)

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) \left(e_t^i(s^t) - \widehat{c}^i \right) = 0 \Rightarrow$$

$$\widehat{c}^i = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) e_t^i(s^t)$$

- Later, we will see that with no aggregate shocks, $\beta^{-1} = 1 + r$.

- Then,

$$\widehat{c}^i = \frac{r}{1+r} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \left(\frac{1}{1+r} \right)^t \pi(s^t) e_t^i(s^t)$$