Endogenous growth models

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Introduction

Endogenous technological change

- First generation endogenous growth models: human capital, AK, learning-by-doing, externalities.
- Second generation endogenous growth models (Romer, 1987 and 1990): new varieties of products (or processes) that increase the division of labor.
- Alternative by Grossman and Helpman (1991): product innovation.
- Schumpeterian growth models by Aghion and Howitt (1992) and Grossman and Helpman (1991).

Ideas as engine of growth

- Technology: the way inputs to the production process are transformed into output.
- Technological progress due to new ideas:
    1. Products.
    2. Managerial practices.
    3. Business models.
- Why (and under what circumstances) are resources are spent on the development of new ideas?

Historians of science vs. economists

- Many historians of science focus on the autonomous role of science in developing inventions and progress (the "Newton paradigm").
- However, economists emphasize the role of profit.
- Classical study of Schmookler (*Invention and Economic Growth*, 1963): innovation is determined by the size of the market.
- Examples:
  1. Horseshoe, many innovations in the late 19^{th} century and early 20^{th} century, stop afterwards.
  2. Air conditioners sold at Sears, between 1960 and 1980 and between 1980 and 1990.
  3. Drugs for Malaria versus drugs for acne.

Ideas

- What is an idea?
- What are the basic characteristics of an idea?
  1. Ideas are *nonrivalrous* goods.
  2. Ideas are, at least partially, *excludable*.
- Nonrivalrousness: implies that cost of providing the good to one more consumer, the *marginal cost* of this good, is constant at zero. Production process for ideas is usually characterized by substantial fixed costs and low marginal costs. Think about software.
- Excludability: required so that firm can recover fixed costs of development. Existence of intellectual property rights like patent or copyright laws are crucial for the private development of new ideas.

Intellectual property rights and the industrial revolution

- Ideas engine of growth.
- Intellectual property rights needed for development of ideas.
- Sustained growth recent phenomenon.
- Coincides with establishment of intellectual property rights.

Data on ideas

- Measure technological progress directly through ideas.
- Measure ideas via measuring patents.
- Measure ideas indirectly by measuring resources devoted to development of ideas.

Environment

Household

- Representative household with a utility function:

$$U(0) = \int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

- $L = 1$, no population growth ($c(t) = C(t)$).
- Asset evolution:

$$\dot{a}(t) = r(t)\, a(t) + w(t) - c(t)$$

- Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}(r(t) - \rho)$$

- I will skip being explicit about initial and transversality conditions.

Final good producer, I

- Competitive producer with technology:

$$Y(t) = \frac{1}{\alpha}\left(\int_0^{N(t)} x(\upsilon, t)^{\alpha} \, d\upsilon\right) L^{1-\alpha}$$

- The elasticity of substitution τ among different intermediate goods is just

$$\alpha = \frac{\tau - 1}{\tau} \Rightarrow \tau = \frac{1}{1-\alpha}$$

- Price of final good normalized to 1.
- $x(\upsilon, t)$: input, fully depreciated in production.
- $p(\upsilon, t)$: price of inputs.

Final good producer, II

- Problem of final good producer:

$$\max_{[x(v,t)]_0^{N(t)},L} \left\{ \begin{array}{c} \frac{1}{\alpha}\left(\int_0^{N(t)} x(v,t)^{\alpha}\,dv\right) L^{1-\alpha} \\ -\int_0^{N(t)} p(v,t)\,x(v,t)\,dv - w(t)\,L \end{array} \right\}$$

- Hence, necessary conditions are:

$$x(v,t)^{\alpha-1} L^{1-\alpha} - p(v,t) = 0 \Rightarrow$$
$$x(v,t) = p(v,t)^{\frac{1}{\alpha-1}} L$$

and:

$$(1-\alpha)\frac{Y(t)}{L} = w(t)$$

Varieties producers, I

- Each varieties producer is a monopolist in its own type.
- Production of inputs at (constant) marginal cost α (we can always define the units of the final good and the units of each variety to get this result).
- Then, per unit profit is $p(\upsilon, t) - \alpha$.
- Given demand function $x(\upsilon, t) = p(\upsilon, t)^{\frac{1}{\alpha-1}} L$, we have:

$$\max_{p(\upsilon,t)} \left(p(\upsilon, t) - \alpha\right) x(\upsilon, t) =$$

$$\max_{p(\upsilon,t)} \left(p(\upsilon, t) - \alpha\right) p(\upsilon, t)^{\frac{1}{\alpha-1}} L$$

$$\propto \max_{p(\upsilon,t)} p(\upsilon, t)^{\frac{\alpha}{\alpha-1}} - \alpha p(\upsilon, t)^{\frac{1}{\alpha-1}}$$

- This problem is static!

Varieties producers, II

- Optimality condition:

$$\frac{\alpha}{\alpha - 1} p(v,t)^{\frac{1}{\alpha-1}} - \frac{\alpha}{\alpha - 1} p(v,t)^{\frac{1}{\alpha-1}-1} = 0 \Rightarrow$$
$$p(v,t) = p = 1$$

  Classical condition (mark-up over marginal cost). Same for all producers.

- Demand is then:

$$x(v,t) = x = L$$

  Same for all producers.

- Hence, total profit

$$\pi(v,t) = \pi = (p(v,t) - \alpha)\, x(v,t) = (1 - \alpha)\, L$$

Innovation, I

- Innovation:

$$\dot{N}(t) = \eta Z(t)$$

  where $Z(t)$ is the amount of final good used in innovation, given some initial $N(0)$.

- Recall that unit price of $Z(t)$ is 1.
- Then, cost of developing one new idea (i.e., $\dot{N}(t) = 1$) is $\frac{1}{\eta}$.
- Innovation gives you a perpetual patent.
- Value of a patent:

$$V(\upsilon, t) = \int_t^\infty e^{-\int_t^s r(s')ds'} \pi(\upsilon, s)\, ds$$

Innovation, II

- Optimality condition:

$$r(t)\,V(v,t) - \dot{V}(v,t) = \pi(v,t) \Rightarrow$$
$$r(t)\,V(v,t) - \dot{V}(v,t) = (1-\alpha)\,L$$

- Free entry into inputs market determines $Z(t)$. Thus,

$$V(v,t) = \frac{1}{\eta}$$

- Clearly:

$$\dot{V}(v,t) = 0$$

  and then:

$$r(t)\,V(v,t) - \dot{V}(v,t) = (1-\alpha)\,L \Rightarrow r(t) = r = \eta\,(1-\alpha)\,L$$

Aggregate constraints

- Assets:

$$a = \int_0^{N(t)} V(v, t)\, dv$$

- Aggregate resource constraint:

$$Y(t) = C(t) + X(t) + Z(t)$$

Equilibrium

Equilibrium

A equilibrium is a sequence of allocations $\{Y(t), C(t), X(t), Z(t)\}_{t=0}^{\infty}$, available varieties $\{N(t)\}_{t=0}^{\infty}$, quantities and prices for varieties $\left\{[p(\upsilon,t), x(\upsilon,t)]_0^{N(t)}\right\}_{t=0}^{\infty}$, and input prices $\{r(t), w(t)\}_{t=0}^{\infty}$ such that:

- Given input prices, $\{r(t), w(t)\}_{t=0}^{\infty}$, the representative household maximizes its utility.
- Given prices for varieties $\left\{[p(\upsilon,t)]_0^{N(t)}\right\}_{t=0}^{\infty}$, and wages $\{w(t)\}_{t=0}^{\infty}$, the final good producer maximizes.
- Given demand function, the varieties producers set up prices of varieties to maximize profits.
- Free entry determines $Z(t)$.
- Markets clear:

$$a = \int_0^{N(t)} V(\upsilon,t)\, d\upsilon$$

$$Y(t) = C(t) + X(t) + Z(t)$$

Solving

Solving for equilibrium, I

- First, since:

$$x(\upsilon, t) = x = L$$

we have:

$$Y(t) = \frac{1}{\alpha}\left(\int_0^{N(t)} x(\upsilon, t)^{\alpha}\, d\upsilon\right) L^{1-\alpha} = \frac{1}{\alpha} N(t)\, L$$

Increases in varieties increase productivity of labor. Hence:

$$g_Y = g_N$$

Also,

$$X(t) = \alpha \int_0^{N(t)} x(\upsilon, t)\, d\upsilon = \alpha \int_0^{N(t)} L d\upsilon = \alpha N(t)\, L = \alpha^2 Y(t)$$

- Second, we have:

$$w(t) = (1-\alpha)\frac{Y(t)}{L} = \frac{1-\alpha}{\alpha} N(t)$$

Solving for equilibrium, II

- Third, recall that:

$$r(t) = r = \eta(1-\alpha)L$$

- Fourth:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}(r(t)-\rho) \Rightarrow g_C = \frac{1}{\sigma}(\eta(1-\alpha)L-\rho)$$

- Now, this is a model without transitional dynamics, and hence:

$$g_Y = g_N = g_C = \frac{1}{\sigma}(\eta(1-\alpha)L-\rho)$$

Solving for equilibrium, III

- Finally,

$$g_N = \frac{\dot{N}(t)}{N(t)} = \eta \frac{Z(t)}{N(t)} = \frac{1}{\sigma}(\eta(1-\alpha)L - \rho)$$

and we get:

$$\begin{aligned} Z(t) &= \frac{1}{\sigma}\left((1-\alpha)L - \frac{\rho}{\eta}\right)N(t) \\ &= \frac{1}{\sigma}\left((1-\alpha) - \frac{\rho}{\eta L}\right)\alpha Y(t) \end{aligned}$$

- Since:

$$X(t) = \alpha^2 Y(t)$$

we find, to close the model

$$C(t) = \left[1 - \alpha^2 - \frac{1}{\sigma}\left((1-\alpha) - \frac{\rho}{\eta L}\right)\right] Y(t)$$

Optimal growth

The social planner's problem, I

- The Social planner's problem can be written as:

$$\max_{\{c(t),N(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma} dt$$
$$\text{s.t. } Y(t) = C(t) + X(t) + Z(t)$$

- We can rewrite the resource constraint in terms of net output:

$$\begin{aligned} \widetilde{Y}(t) &= \frac{1}{\alpha}\left(\int_0^{N(t)} x(\upsilon,t)^{\alpha} d\upsilon\right) L^{1-\alpha} - \int_0^{N(t)} \alpha x(\upsilon,t)\, d\upsilon \\ &= C(t) + Z(t) \end{aligned}$$

The social planner's problem, II

- Maximizing net output is a static problem with optimality conditions:

$$x(\upsilon, t)^{\alpha-1} L^{1-\alpha} - \alpha = 0 \Rightarrow x(\upsilon, t) = x = \alpha^{\frac{1}{\alpha-1}} L$$

  and then:

$$\widetilde{Y}^*(t) = (1-\alpha)\,\alpha^{\frac{1}{\alpha-1}} N^*(t)\, L$$

  and:

$$Y^*(t) = \frac{1}{\alpha^{\frac{1}{1-\alpha}}} N^*(t)\, L$$

- Compare with equilibrium output:

$$Y^*(t) = \frac{1}{\alpha^{\frac{1}{1-\alpha}}} N^*(t)\, L > Y(t) = \frac{1}{\alpha} N(t)\, L$$

The social planner's problem, III

- Now, we can rewrite the whole problem as:

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma} dt$$

$$\text{s.t. } \dot{N}(t) = \eta \left[(1-\alpha) \alpha^{\frac{1}{\alpha-1}} N(t) L - C(t) \right]$$

- The Hamiltonian:

$$\mathcal{H}(N, C, \mu) = \frac{C(t)^{1-\sigma}}{1-\sigma} + \mu(t) \eta \left[(1-\alpha) \alpha^{\frac{1}{\alpha-1}} N(t) L - C(t) \right]$$

with optimality conditions:

$$\mathcal{H}_C(N, C, \mu) = C^*(t)^{-\sigma} - \mu(t) \eta = 0$$

$$\mathcal{H}_N(N, C, \mu) = \mu(t) \eta (1-\alpha) \alpha^{\frac{1}{\alpha-1}} L = \rho \mu(t) - \dot{\mu}(t)$$

The social planner's problem, IV

- Then:

$$\mu(t) = \frac{C^*(t)^{-\sigma}}{\eta} \Rightarrow \dot{\mu}(t) = -\sigma \frac{C^*(t)^{-\sigma-1}}{\eta} \dot{C}^*(t) \Rightarrow -\frac{\dot{\mu}(t)}{\mu(t)} = \sigma \frac{\dot{C}^*(t)}{C^*(t)}$$

and we get:

$$\frac{\dot{C}^*(t)}{C^*(t)} = \frac{1}{\sigma}\left[\eta(1-\alpha)\alpha^{\frac{1}{\alpha-1}}L - \rho\right]$$

- In comparison, in equilibrium we get:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}\left[\eta(1-\alpha)L - \rho\right]$$

Since $\alpha^{\frac{1}{\alpha-1}} > 1$, the growth rate under a social planner is higher than in the equilibrium.

- Intuition.
- Policy remedies.

The social planner's problem, V

- Also:

$$g_N^* = \frac{\dot{N}^*(t)}{N^*(t)} = \eta \frac{Z^*(t)}{N^*(t)} = \frac{1}{\sigma}\left(\eta (1-\alpha)\alpha^{\frac{1}{\alpha-1}} L - \rho\right)$$

  and we get:

$$Z^*(t) = \frac{1}{\sigma}\left((1-\alpha)\alpha^{\frac{1}{\alpha-1}} - \frac{\rho}{\eta L}\right)\alpha Y^*(t)$$

- Two sources of higher growth:

  1. Higher output, $Y^*(t)$.
  2. Higher fraction of output on R&D.

Extensions

Growth with knowledge spillovers

- We substitute:

$$\dot{N}(t) = \eta Z(t)$$

  by:

$$\dot{N}(t) = \eta N(t) L_R(t)$$

- Labor used in production:

$$L_Y(t) = L - L_R(t)$$

- Rest of the analysis is pretty much the same as before.

Scale effects

- Notice that:

$$\frac{Y(t)}{L} = \frac{1}{\alpha} N(t)$$

and

$$\frac{\dot{N}(t)}{N(t)} = \frac{1}{\sigma}\left(\eta(1-\alpha)L - \rho\right)$$

- Scale effect on L.
- Problems:
  1. Size of markets do not seem to matter that much.
  2. With population growth, the economy will explode.
  3. Increases in observed R&D without increases in long-run growth rates.

Growth without scale effects

- Modification by Jones (1995).
- Instead of:

$$\dot{N}(t) = \eta N(t) L_R(t)$$

  we have:

$$\dot{N}(t) = \eta N^{\phi}(t) L_R(t)$$

  where $\phi < 1$.

- Growth rate:

$$g = \frac{n}{1 - \phi}$$

  where n is population growth.