The neoclassical growth model

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Introduction

Neoclassical growth model

- Original contribution of Ramsey (1928). That is why sometimes it is known as the Ramsey model.
- Completed by David Cass (1965) and Tjalling Koopmans (1965). That is why some times it is known as the Cass-Koopmans model.
- William Brock and Leonard Mirman (1972) introduced uncertainty.
- Finn Kydland and Edward Prescott (1982) used it to create the real business cycle research agenda.

Environment

Utility function

- Representative household with a utility function:

$$u(c(t))$$

Definition

$u(c)$ is strictly increasing, concave, twice continuously differentiable with derivatives u' and u'', and satisfies Inada conditions:

$$\lim_{c \to 0} u'(c) = \infty$$
$$\lim_{c \to \infty} u'(c) = 0$$

Dynastic structure

- Population evolves:

$$L(t) = \exp(nt)$$

  with $L_0 = 1$.

- Intergenerational altruism.
- Intertemporal utility function:

$$U(0) = \int_0^{\infty} e^{-(\rho - n)t} u(c(t))\, dt$$

- ρ: subjective discount rate, such that $\rho > n$.
- $\rho - n$: “effective” discount rate.

Budget constraint

- Asset evolution:

$$\dot{a} = (r - \delta - n)\, a + w - c$$

- Who owns the capital in the economy? Role of complete markets.
- Modigliani-Miller theorems.
- Arrow securities.
- No-Ponzi game condition:

$$\lim_{t \to \infty} a(t) \exp\left(- \int_0^t (r(s) - \delta - n)\, ds \right) = 0$$

- Historical examples.

Production side

- Cobb-Douglas aggregate production function:

$$Y = K^{\alpha} L^{1-\alpha}$$

- Per capita terms:

$$y = k^{\alpha}$$

- From the first order condition of firm with respect to capital k:

$$r = \alpha k^{\alpha-1}$$

$$w = k^{\alpha} - k\alpha k^{\alpha-1} = (1-\alpha)\, k^{\alpha}$$

- Interest rate:

$$r - \delta$$

Aggregate consistency conditions

- Asset market clearing:

$$a = k$$

- Implicitly, labor market clearing.
- Resource constraint:

$$\dot{k} = k^{\alpha} - c - (n + \delta) k$$

Competitive equilibrium

Competitive equilibrium I

A competitive equilibrium is a sequence of per capita allocations $\{c(t), k(t)\}_{t=0}^{\infty}$ and input prices $\{r(t), w(t)\}_{t=0}^{\infty}$ such that:

- Given input prices, $\{r(t), w(t)\}_{t=0}^{\infty}$, the representative household maximizes its utility:

$$\max_{\{c(t),a(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho-n)t} u(c(t))\, dt$$

$$\text{s.t. } \dot{a} = (r - \delta - n)\, a + w - c$$

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s) - \delta - n)\, ds\right) = 0$$

$$a_0 = k_0$$

Competitive equilibrium II

- Input prices, $\{r(t), w(t)\}_{t=0}^{\infty}$, are equal to the marginal productivities:

$$r(t) = \alpha k(t)^{\alpha-1}$$
$$w(t) = (1-\alpha) k(t)^{\alpha}$$

- Markets clear:

$$a(t) = k(t)$$
$$\dot{k} = k(t)^{\alpha} - c(t) - (n+\delta) k(t)$$

Solving the model

Household maximization

- We can come back now to the problem of the household.

- We build the Hamiltonian:

$$\mathcal{H}(a, c, \mu) = u(c(t)) + \mu(t)((r(t) - n - \delta)a(t) - w(t) - c(t))$$

where:

1. $a(t)$ is the state variable.
2. $c(t)$ is the control variable.
3. $\mu(t)$ is the current-value co-state variable.

Necessary conditions

1. Partial derivative of the Hamiltonian with respect to controls is equal to zero:

$$\mathcal{H}_c(a, c, \mu) = u'(c(t)) - \mu(t) = 0$$

2. Partial derivative of the Hamiltonian with respect to states is:

$$\mathcal{H}_a(a, c, \mu) = \mu(t)(r(t) - n - \delta) = (\rho - n)\mu(t) - \dot{\mu}(t)$$

3. Partial derivative of the Hamiltonian with respect to co-states is:

$$\mathcal{H}_\mu(a, c, \mu) = (r(t) - n - \delta)a(t) - c(t) = \dot{a}(t)$$

4. Transversality condition:

$$\lim_{t \to \infty} e^{-\rho t}\mu(t)a(t) = 0$$

Working with the necessary conditions I

- From the second condition:

$$\mu\left(r-n-\delta\right)=\left(\rho-n\right)\mu-\dot{\mu}\Rightarrow$$

$$\left(r-n-\delta\right)=\left(\rho-n\right)-\frac{\dot{\mu}}{\mu}\Rightarrow$$

$$\frac{\dot{\mu}}{\mu}=-\left(r-\delta-\rho\right)$$

- From the first condition:

$$u'\left(c\right)=\mu$$

and taking derivatives with respect to time:

$$u''\left(c\right)\dot{c}=\dot{\mu}\Rightarrow$$

$$\frac{u''\left(c\right)}{u'\left(c\right)}\dot{c}=\frac{\dot{\mu}}{\mu}=-\left(r-\delta-\rho\right)$$

Working with the necessary conditions II

- Now, we can combine both expression:

$$-\sigma \frac{\dot{c}}{c} = -\left(r - \delta - \rho\right)$$

where

$$\sigma = -\frac{u''\left(c\right)}{u'\left(c\right)} c = \frac{d \log\left(c\left(s\right)/c\left(t\right)\right)}{d \log\left(u'\left(c\left(s\right)\right)/u'\left(c\left(t\right)\right)\right)}$$

is the (inverse of) elasticity of intertemporal substitution (EIS).

- Thus:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(r - \delta - \rho\right)$$

This expression is known as the consumer Euler equation.

CRRA

- In the previous equation, we have implicitly assumed that σ is a constant.
- This will be only true of a class of utility functions.
- Constant Relative Risk Aversion (CRRA):

$$\begin{aligned} \frac{c^{1-\sigma}-1}{1-\sigma} \text{ for } \sigma &\neq 1 \\ \log c \text{ for } \sigma &= 1 \end{aligned}$$

  (you need to take limits and apply L'Hôpital's rule).

- Why is it called CRRA?

Applying equilibrium conditions

- First, note that $r = \alpha k^{\alpha-1}$. Then:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(\alpha k^{\alpha-1} - \delta - \rho\right)$$

- Second, $k = a$. Then:

$$\dot{a} = (r - \delta - n)\, a + w - c \Rightarrow$$
$$\dot{k} = \left(\alpha k^{\alpha-1} - \delta - n\right) k + w - c \Rightarrow$$
$$\dot{k} = k^{\alpha} - c - (n + \delta)\, k$$

where in the last step we use the fact that $k^{\alpha} = \alpha k^{\alpha-1} k + w$.

System of differential equations

System of differential equations

- We have two differential equations:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(\alpha k^{\alpha-1} - \delta - \rho\right)$$
$$\dot{k} = k^{\alpha} - c - (n+\delta)\,k$$

on two variables, k and c, plus the transversality condition:

$$\lim_{t\to\infty} e^{-\rho t}\mu a = \lim_{t\to\infty} e^{-\rho t}\mu k = 0$$

- How do we solve it?

Steady state

- We search for a steady state where $\dot{c} = \dot{k} = 0$.
- Then:

$$\begin{aligned} \frac{1}{\sigma}\left(\alpha\left(k^*\right)^{\alpha-1}-\delta-\rho\right) &= 0 \\ \left(k^*\right)^{\alpha}-c^*-(n+\delta)\,k^* &= 0 \end{aligned}$$

- System of two equations on two unknowns k^* and c^* with solution:

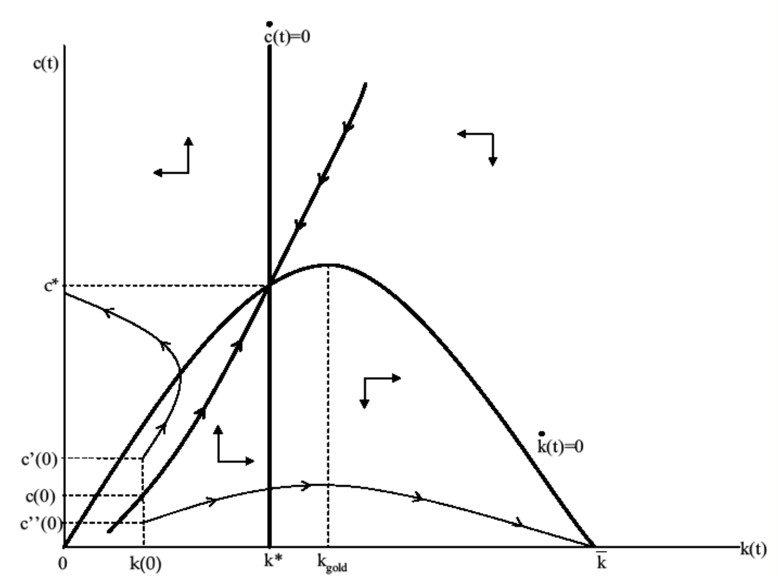
$$\begin{aligned} k^* &= \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \\ c^* &= \left(k^*\right)^{\alpha}-(n+\delta)\,k^* \end{aligned}$$

- Note that EIS does not enter into the steady state. In fact, the form of the utility function is irrelevant!

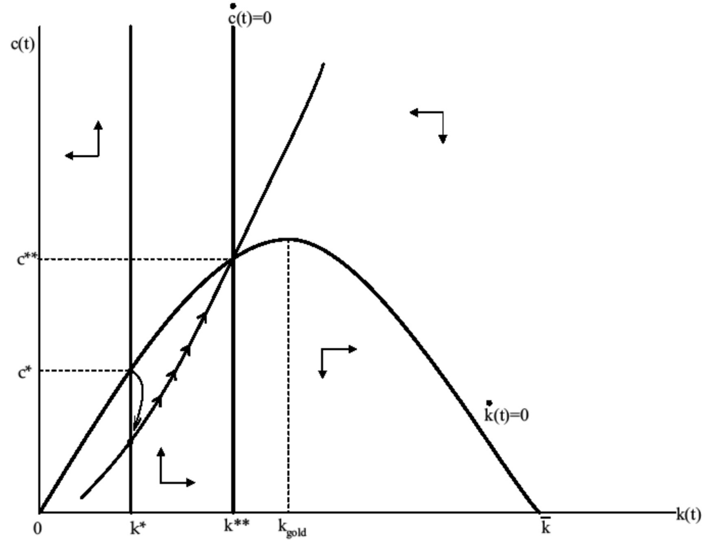
Transitional dynamics

- The neoclassical growth model does not have a closed-form solution.
- We can do three things:
  1. Use a phase diagram.
  2. Solve an approximated version of the model where we linearize the equations.
  3. Use the computer to approximate the solution numerically.

Phase diagram



Comparative statics: lower discount rate



Linearization I

- We can linearize the system:

$$\begin{aligned}\frac{\dot{c}}{c} &= \frac{1}{\sigma}\left(\alpha k^{\alpha-1} - \delta - \rho\right) \\ \dot{k} &= k^{\alpha} - c - (n+\delta)\,k\end{aligned}$$

- We get:

$$\begin{aligned}\dot{c} &\simeq \frac{c^{*}\alpha\,(\alpha-1)\,(k^{*})^{\alpha-2}}{\sigma}\,(k-k^{*}) + \frac{\alpha\,(k^{*})^{\alpha-1} - \delta - \rho}{\sigma}\,(c-c^{*}) \\ &= \frac{c^{*}}{\sigma}\left(\alpha\,(\alpha-1)\,(k^{*})^{\alpha-2}\right)(k-k^{*})\end{aligned}$$

and:

$$\begin{aligned}\dot{k} &\simeq \left(\alpha\,(k^{*})^{\alpha-1} - n - \delta\right)(k-k^{*}) - (c-c^{*}) \\ &= (\rho-n)\,(k-k^{*}) - (c-c^{*})\end{aligned}$$

Linearization II

- The behavior of the linearized system is given by the roots (eigenvalues) ξ of:

$$\det\begin{pmatrix} \rho - n - \xi & -1 \\ \frac{c^*}{\sigma}\left(\alpha\left(\alpha-1\right)\left(k^*\right)^{\alpha-2}\right) & -\xi \end{pmatrix}$$

- Solving:

$$\begin{aligned} -\xi\left(\rho - n - \xi\right) + \frac{c^*}{\sigma}\left(\alpha\left(\alpha-1\right)\left(k^*\right)^{\alpha-2}\right) &= 0 \Rightarrow \\ \xi^2 - \xi\left(\rho - n\right) + \frac{c^*}{\sigma}\left(\alpha\left(\alpha-1\right)\left(k^*\right)^{\alpha-2}\right) &= 0 \end{aligned}$$

- Thus:

$$\xi = \frac{\left(\rho - n\right) \pm \sqrt{1 - 4\left(\alpha\left(\alpha-1\right)\left(k^*\right)^{\alpha-2}\right)}}{2}$$

  and since $\alpha\left(\alpha-1\right) < 1$, we have one positive and one negative eigenvalue$\Rightarrow$one stable manifold.

Linearization III

- We will call ξ_1 the positive eigenvalue and ξ_2 the negative one.
- With some results in differential equations, we can show:

$$k = k^* + \eta_1 e^{\xi_1 t} + \eta_2 e^{\xi_2 t} \Rightarrow$$
$$k - k^* = \eta_1 e^{\xi_1 t} + \eta_2 e^{\xi_2 t}$$

  where η_1 and η_2 are arbitrary constants of integration.
- It must be that $\eta_1 = 0$. If $\eta_1 > 0$, we will violate the transversality condition and $\eta_1 < 0$ will take k_t to 0.
- Then, η_2 is determined by:

$$\eta_2 = k_0 - k^*$$

- Hence:

$$k = \left(1 - e^{\xi_2 t}\right) k^* + e^{\xi_2 t} k_0 \Rightarrow$$
$$k - k^* = \eta_2 e^{\xi_2 t} = \left(k_0 - k^*\right) e^{\xi_2 t}$$

Linearization IV

- Also:

$$\dot{c} = \frac{c^*}{\sigma}\left(\alpha\left(\alpha-1\right)\left(k^*\right)^{\alpha-2}\right)\left(k-k^*\right)$$

  or

$$c = \frac{c^*}{\sigma}\left(\alpha\left(\alpha-1\right)\left(k^*\right)^{\alpha-2}\right)\frac{\eta_2}{\xi_2}e^{\xi_2 t} + c^*$$

  where the constant c^* ensures that we converge to the steady state.

- Since $y = k^\alpha$, we get:

$$\log y = \alpha \log\left(k^* + \left(k_0 - k^*\right)e^{\xi_2 t}\right)$$

Linearization V

- Taking time derivatives and making $y = y_0$:

$$\frac{\dot{y}}{y_0} = \frac{\alpha}{k^* + (k_0 - k^*)\, e^{\xi_2 t}} \left((k_0 - k^*)\, \xi_2 e^{\xi_2 t}\right)$$

$$= \alpha\xi_2 - \alpha\xi_2 \frac{k^*}{k_0}$$

$$= \alpha\xi_2 - \alpha\xi_2 \left(\frac{y^*}{y_0}\right)^{\frac{1}{\alpha}}$$

- This suggest to go to the data and run convergence regressions of the form:

$$g_{i,t,t-1} = b^0 + b^1 \log y_{i,t-1} + \varepsilon_{i,t}$$

- We need to be careful about interpreting the coefficient $\hat{b}^1$.
- Where does the error come from?

Selecting parameter values

- In general, computers cannot approximate the solution for arbitrary parameter values.
- How do we determine the parameter values?
- Two main approaches:
    1. Calibration.
    2. Statistical methods: Methods of Moments, ML, Bayesian.
- Advantages and disadvantages.

Calibration as an empirical methodology

- Emphasized by Lucas (1980) and Kydland and Prescott (1982).
- Two sources of information:
  1. Well accepted microeconomic estimates.
  2. Matching long-run properties of the economy.
- Problems of 1 and 2.
- References:
  1. Browning, Hansen, and Heckman (1999) chapter in *Handbook of Macroeconomics*.
  2. Debate in *Journal of Economic Perspectives*, Winter 1996: Kydland and Prescott, Hansen and Heckman, Sims.

Calibration of the standard model

- Parameters: n, α, δ, ρ, and σ.
- n: population growth in the data.
- α: capital income. Proprietor's income?
- δ: in steady state

$$\delta k^* = x^* \Rightarrow \delta = \frac{x^*}{k^*}$$

- ρ: in steady state

$$r^* = \alpha \left(\frac{\alpha}{\rho + \delta} \right)^{\frac{\alpha}{1-\alpha} - 1} - \delta$$

  Then, we take r^* from the data and given α and δ, we find ρ.
- σ: from microeconomic evidence.

Running the model in the computer

- We have the system:

$$
\begin{aligned}
\frac{\dot{c}}{c} &= \frac{1}{\sigma}\left(\alpha k^{\alpha-1} - \delta - \rho\right) \\
\dot{k} &= k^{\alpha} - c - (n+\delta)\,k
\end{aligned}
$$

- Many methods to solve it.
- A simple one is a shooting algorithm.
- A popular alternative: Runge-Kutta methods.

A shooting algorithm

- Approximate the system by:

$$\frac{\frac{c(t+\Delta t)-c(t)}{\Delta t}}{c(t)} = \frac{1}{\sigma}\left(\alpha k(t)^{\alpha-1} - \delta - \rho\right)$$
$$\frac{k(t+\Delta t)-k(t)}{\Delta t} = k(t)^{\alpha} - c(t) - (n+\delta)k(t)$$

  for a small Δt.

- Steps:

  1. Given $k(0)$, guess $c(0)$.
  2. Trace dynamic system for a long t.
  3. Is $k(t) \to k^*$? If yes, we got the right $c(0)$. If $k(t) \to \infty$, raise $c(0)$, if $k(t) \to 0$, lower $c(0)$.

- Intuition: phase diagram.

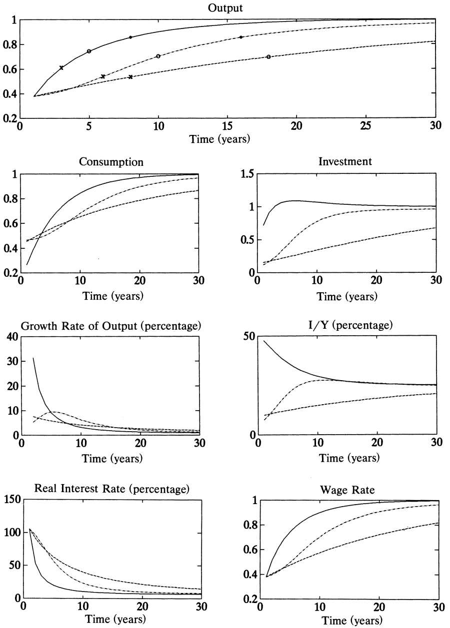


FIGURE 4. IMPLICATIONS OF INTERTEMPORAL PREFERENCES FOR TRANSITIONAL DYNAMICS

Savings rate

Savings rate I

- We can actually work on our system of differential equations a bit more to show a more intimate relation between the Solow and the neoclassical growth model.
- The savings rate is defined as:

$$s(t) = 1 - \frac{c(t)}{y(t)}$$

- Now

$$\frac{d\left(c(t)/y(t)\right)}{dt}\frac{1}{c(t)/y(t)} = \frac{\dot{c}}{c} - \frac{\dot{y}}{y} = \frac{\dot{c}}{c} - \alpha\frac{\dot{k}}{k}$$

Savings rate II

- If we substitute in the differential equations for $\frac{\dot{c}}{c}$ and $\dot{k}$:

$$
\begin{aligned}
&\frac{d\left(c\left(t\right)/y\left(t\right)\right)}{dt}\frac{1}{c\left(t\right)/y\left(t\right)} \\
&= \frac{1}{\sigma}\left(\alpha k^{\alpha-1}-\delta-\rho\right)-\alpha\left(k^{\alpha-1}-\frac{c}{k}-n-\delta\right) \\
&= \frac{1}{\sigma}\left(\alpha k^{\alpha-1}-\delta-\rho\right)-\alpha\left(k^{\alpha-1}-\frac{c}{y}k^{\alpha-1}-n-\delta\right) \\
&= -\frac{1}{\sigma}\left(\delta+\rho\right)+\alpha\left(n+\delta\right)+\left(\frac{1}{\sigma}-1+\frac{c}{y}\right)\alpha k^{\alpha-1}
\end{aligned}
$$

- Then:

$$
\frac{d\left(c\left(t\right)/y\left(t\right)\right)}{dt}\frac{1}{c\left(t\right)/y\left(t\right)} = -\frac{1}{\sigma}\left(\delta+\rho\right)+\alpha\left(n+\delta\right)+\left(\frac{1}{\sigma}-1+\frac{c}{y}\right)\alpha k^{\alpha-1}
$$

$$
\dot{k} = k^{\alpha}-c-\left(n+\delta\right)k
$$

is another system of differential equations.

Savings rate III

- This system implies that the saving rate is monotone (always increasing, always decreasing, or constant).

- We find the locus $\frac{d(c(t)/y(t))}{dt} = 0$:

$$\left(\frac{1}{\sigma} - 1 + \frac{c}{y}\right) \alpha k^{\alpha-1} = \frac{1}{\sigma}(\delta+\rho) - \alpha(n+\delta) \Rightarrow$$
$$\frac{c}{y} = 1 - \frac{1}{\sigma} + \left(\frac{1}{\sigma}(\delta+\rho) - \alpha(n+\delta)\right)\frac{1}{\alpha}k^{1-\alpha}$$

- Hence, if:

$$\frac{1}{\sigma}(\delta+\rho) = \alpha(n+\delta)$$

the savings rate is constant, and we are back into the basic Solow model!

Optimal growth

The social planner's problem

- The Social planner's problem can be written as:

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho-n)t} u(c(t))\, dt$$

$$\text{s.t. } \dot{k} = k(t)^{\alpha} - c(t) - (n+\delta) k(t)$$

$$\lim_{t\to\infty} k(t) \exp\left(-\int_0^t (r(s)-\delta-n)\, ds\right) = 0$$

$$k_0 \text{ given}$$

- Interpretation of r here.

- This problem is very similar to the household's problem.

- We can also apply the optimality principle to the Hamiltonian:

$$\mathcal{H}(a,c,\mu) = u(c(t)) + \mu(t)\left(k(t)^{\alpha} - c(t) - (n+\delta) k(t)\right)$$

Necessary conditions

1. Partial derivative of the Hamiltonian with respect to controls is equal to zero:

$$\mathcal{H}_c(a, c, \mu) = u'(c(t)) - \mu(t) = 0$$

2. Partial derivative of the Hamiltonian with respect to states is:

$$\mathcal{H}_a(a, c, \mu) = \mu(t)\left(\alpha k(t)^{\alpha-1} - n - \delta\right) = (\rho - n)\mu(t) - \dot{\mu}(t)$$

3. Partial derivative of the Hamiltonian with respect to co-states is:

$$\mathcal{H}_\mu(a, c, \mu) = k(t)^\alpha - c(t) - (n+\delta)k(t) = \dot{k}(t)$$

4. Transversality condition:

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)k(t) = 0$$

Comparing the necessary conditions

- Following very similar steps than in the problem of the consumer we find:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(\alpha k^{\alpha-1} - \delta - \rho\right)$$
$$\dot{k} = k^{\alpha} - c - (n+\delta)\,k$$
$$\lim_{t\to\infty} e^{-\rho t}\mu(t)\,k(t) = 0$$

- From the household problem:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(\alpha k^{\alpha-1} - \delta - \rho\right)$$
$$\dot{k} = k^{\alpha} - c - (n+\delta)\,k$$
$$\lim_{t\to\infty} e^{-\rho t}\mu(t)\,k(t) = 0$$

- Both problems have the same necessary conditions!