On the Solution of the Growth Model with Investment-Specific Technological Change*

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Abstract

Recent work by Greenwood, Hercowitz, and Krusell (1997 and 2000) and Fisher (2003) has emphasized the importance of investment-specific technological change as a main driving force behind long-run growth and the business cycle. This paper shows how the growth model with investment-specific technological change has a closed-form solution if capital fully depreciates. This solution furthers our understanding of the model and it constitutes a useful benchmark to check the accuracy of numerical procedures to solve dynamic macroeconomic models in cases with several state variables.

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1. Introduction

The recent work of Greenwood, Hercowitz, and Krusell (1997 and 2000) has focused the attention of economists on the role of investment-specific technological change as a main driving force behind economic growth and business cycle fluctuations. Fisher (1999) documents two key empirical observations that support these conclusions. First, the relative price of business equipment in terms of consumption goods has fallen in nearly every year since the 1950s. Second, the fall in the relative price of capital is faster during expansions than during recessions.

Models of investment-specific technological change have also being successfully used to account for the evolution of the skill premium in the U.S. since the Second World War (Krusell *et al.*, 2000) or the cyclical behavior of hours and productivity (Fisher, 2003), among several other applications.

Unfortunately, the standard growth model with investment-specific technological change, as presented in Fisher (2003), does not have a known analytic solution. Therefore, researchers have employed computational methods to solve the model.

In this paper, we show how this standard model has a closed-form solution when there is full depreciation of capital. We derive the exact solution in the case where there is a labor/leisure choice and long-run growth in the economy. The solution has a simple backward representation that allows to gauge the importance of each parameter on the behavior of the model.

There are, at least, two reasons that make our result important. First, the closed-form solution improves our understanding of the dynamics of the model beyond the findings provided by numerical computations. The law of motions for variables uncover the main driving forces in the model and develop intuition that is difficult to obtain from the computer output. In particular, we illustrate how shocks propagate over time and which factors determine the persistence of the model. This exercise highlights the importance of the capital participation share as a determinant of propagation.

Second, the closed-form solution is an excellent test case to check the behavior of numerical procedures like solution methods for a dynamic macroeconomic models. The approximated solutions generated by those algorithms in the case of full depreciation can be compared against the closed-form solution. In that way, we can evaluate the accuracy of the solution method. The model with investment-specific technological change is a more interesting test case than the neoclassical growth model because the presence of two shocks increases the dimensionality of the problem and, consequently, makes it more representative of interesting macro applications.

2. A Growth Model with Two Shocks

We present a simple growth model with two shocks, one to the general technology and one to investment as described in Fisher (2003).

There is a representative household in the economy, whose preferences over stochastic sequences of consumption c_t and leisure l_t can be represented by the utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t + \psi \log(1 - L_t) \right) \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor, ψ controls labor supply, and E_0 is the conditional expectation operator.

There is one final good produced according to the production function $A_t K_t^{\alpha} L_t^{1-\alpha}$, where K_t is the aggregate capital stock, L_t is the aggregate labor input, and A_t is a stochastic process representing random general-purpose technological progress. The final good can be used for consumption, C_t, or for investment, X_t:

$$C_t + X_t = A_t K_t^{\alpha} L_t^{1-\alpha}. \tag{2}$$

One unit of investment is transformed into V_t units of capital, where V_t is a stochastic process representing random investment-specific technological progress. Consequently, and given a depreciation factor δ, the law of motion for capital is given by $K_{t+1} = (1-\delta)K_t + V_t X_t$. In equilibrium, $1/V_t$ will be equal to the relative price of capital in terms of consumption.

The laws of motion for the two stochastic processes are given by:

$$A_t = e^{\gamma + \varepsilon_{at}} A_{t-1}, \ \gamma \geq 0 \tag{3}$$

$$V_t = e^{\upsilon + \varepsilon_{\upsilon t}} V_{t-1}, \ \upsilon \geq 0 \tag{4}$$

where $[\varepsilon_{at}, \varepsilon_{\upsilon t}]' \sim \mathcal{N}(0, D)$, and D is a diagonal matrix. This stochastic process implies that the logs of A_t and V_t follow a random walk with drifts γ and υ. This specification generates long run growth in the economy and the possibility of changes in the long run relative price of capital.

We could rewrite the model to accommodate deterministic trends and transitory shocks on the stochastic processes for technology. The main thrust of the results would be the same. We omit details because of space considerations.

A competitive equilibrium can be defined in a standard way as a sequence of allocations and prices such that both the representative household and the firm maximize and markets clear. Also, since both welfare theorems hold in this economy, we can instead solve the equivalent and simpler social planner's problem that maximizes the utility of the representative household subject to the economy resource constraint, the law of motion for capital, the stochastic process, and some initial conditions K_0, A_0, and V_0.

An alternative interpretation of this model is to think of an environment with two sectors, one that produces the consumption good and one that produces the investment good. Each sector uses the same production function except that the total factor productivity is different. In an equilibrium where factor are mobile, this environment aggregates to the same economy that the one presented here with V_t capturing the differences in total factor productivity.

3. Transforming the Model

The previous model is nonstationary because of the presence of two unit roots, one in each technological process. Since standard solution methods do not apply to nonstationary models, we need to transform the model into an stationary problem. The key requirement for any transformation of this short is to use a scaling variable that is fully known before the current period shocks are realized. In the simultaneous equations language, we require that the variable is predetermined.

To transform in that way our model, we begin by plugging the law of motion for capital in the resource constraint:

$$C_t + \frac{K_{t+1}}{V_t} = A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)\frac{K_t}{V_t}.$$

Then, if we divide by $Z_t = A_{t-1}^{\frac{1}{1-\alpha}} V_{t-1}^{\frac{\alpha}{1-\alpha}} = \left(A_{t-1}V_{t-1}^{\alpha}\right)^{\frac{1}{1-\alpha}}$, we find:

$$\frac{C_t}{Z_t} + \frac{K_{t+1}}{Z_{t+1}V_t}\frac{Z_{t+1}}{Z_t} = \frac{A_t V_{t-1}^{\alpha}}{Z_t^{1-\alpha}}\left(\frac{K_t}{Z_t V_{t-1}}\right)^{\alpha} L_t^{1-\alpha} + (1-\delta)\frac{K_t}{Z_t V_{t-1}}\frac{V_{t-1}}{V_t}.$$

First, note that since:

$$Z_{t+1} = A_t^{\frac{1}{1-\alpha}} V_t^{\frac{\alpha}{1-\alpha}} = A_{t-1}^{\frac{1}{1-\alpha}} V_{t-1}^{\frac{\alpha}{1-\alpha}} e^{\frac{\gamma+\alpha\upsilon+C_a(L)\varepsilon_{at}+\alpha C_\upsilon(L)\varepsilon_{\upsilon t}}{1-\alpha}},$$

we have that:

$$\frac{Z_{t+1}}{Z_t} = \frac{A_{t-1}^{\frac{1}{1-\alpha}} V_{t-1}^{\frac{\alpha}{1-\alpha}} e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}}}{A_{t-1}^{\frac{1}{1-\alpha}} V_{t-1}^{\frac{\alpha}{1-\alpha}}} = e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}}.$$

Also $\frac{A_t V_{t-1}^{\alpha}}{Z_t^{1-\alpha}} = e^{\gamma+\varepsilon_{at}}$, $\frac{V_{t-1}}{V_t} = e^{-\upsilon-\varepsilon_{\upsilon t}}$, and $Z_t V_{t-1} = A_{t-1}^{\frac{1}{1-\alpha}} V_{t-1}^{\frac{1}{1-\alpha}}$.

As a consequence, if we define $\widetilde{C}_t = \frac{C_t}{Z_t}$ and $\widetilde{K}_t = \frac{K_t}{Z_t V_{t-1}}$, we can rewrite the resource constraint as:

$$\widetilde{C}_t + e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}}\widetilde{K}_{t+1} = e^{\gamma+\varepsilon_{at}}\widetilde{K}_t^{\alpha}L_t^{1-\alpha} + (1-\delta)\,e^{-\upsilon-\varepsilon_{\upsilon t}}\widetilde{K}_t \tag{5}$$

and the utility function as:

$$E_0\sum_{t=0}^{\infty}\beta^t\left(\log\widetilde{C}_t + \psi\log(1-L_t)\right). \tag{6}$$

The intuition for these two expressions is as follows. In the resource constraint, we need to modify the term associated with $\widetilde{K}_{t+1}$ to compensate for the fact that the value of the transformed capital goes down when technology improves. A similar argument holds for the term in front of the capital remaining after depreciation. In the utility function, we exploit its additive log form to write it in terms of $\widetilde{C}_t$.

The first order conditions for this transformed problem are an Euler equation:

$$\frac{e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}}}{\widetilde{C}_t} = \beta E_t\frac{1}{\widetilde{C}_{t+1}}\left(\alpha e^{\gamma+\varepsilon_{at+1}}\widetilde{K}_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha} + (1-\delta)\,e^{-\upsilon-\varepsilon_{\upsilon t+1}}\right) \tag{7}$$

and a labor supply condition:

$$\psi\frac{\widetilde{C}_t}{1-L_t} = (1-\alpha)\,e^{\gamma+\varepsilon_{at}}\widetilde{K}_t^{\alpha}L_t^{-\alpha}, \tag{8}$$

together with the resource constraint (5).

4. A Closed-Form Solution for the Case with Full Depreciation

The previous system of equations does not have a known analytical solution, and we need to use a numerical method to solve it. However, there is a case for which we can find a closed-form solution.

This happens when there is full depreciation, i.e., δ is equal to one. Then, the system boils down to:

$$\frac{e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}}}{\widetilde{C}_t} = \beta E_t\frac{1}{\widetilde{C}_{t+1}}\alpha e^{\gamma+\varepsilon_{at+1}}\widetilde{K}_{t+1}^{\alpha}L_{t+1}^{1-\alpha} \tag{9}$$

$$\psi \frac{\widetilde{C}_t}{1-L_t} = (1-\alpha)\, e^{\gamma+\varepsilon_{at}} \widetilde{K}_t^{\alpha} L_t^{-\alpha} \tag{10}$$

$$\widetilde{C}_t + e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}} \widetilde{K}_{t+1} = e^{\gamma+\varepsilon_{at}} \widetilde{K}_t^{\alpha} L_t^{1-\alpha} \tag{11}$$

4.1. Policy Functions in Transformed Variables

To see how this system has a closed-form solution we follow a "guess-and-verify" approach. First, we conjecture that the income and substitution effects of a real wage rate change offset each other. Then:

$$L_t = L = \frac{1-\alpha}{1-\alpha+\psi(1-\alpha\beta)}$$

Second, we postulate the following policy functions for capital:

$$\widetilde{K}_{t+1} = e^{-\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}} e^{\gamma+\varepsilon_{at}} \alpha\beta \widetilde{K}_t^{\alpha} L^{1-\alpha}.$$

As a consequence, and using the economy's resource constraint (11), consumption must be equal to:

$$\widetilde{C}_t = e^{\gamma+\varepsilon_{at}} \widetilde{K}_t^{\alpha} L^{1-\alpha} - e^{\gamma+\varepsilon_{at}} \alpha\beta \widetilde{K}_t^{\alpha} L^{1-\alpha}$$

or

$$\widetilde{C}_t = (1-\alpha\beta)\, e^{\gamma+\varepsilon_{at}} \widetilde{K}_t^{\alpha} L^{1-\alpha}.$$

To check that these are the correct policy functions, we substitute them into the Euler equation (9):

$$\frac{e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}}}{\widetilde{C}_t} = \beta E_t \frac{1}{\widetilde{C}_{t+1}} \alpha e^{\gamma+\varepsilon_{at+1}} \widetilde{K}_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} \Rightarrow$$

$$\frac{e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}}}{(1-\alpha\beta)\, e^{\gamma+\varepsilon_{at}} \widetilde{K}_t^{\alpha} L^{1-\alpha}} = \beta E_t \frac{1}{(1-\alpha\beta)\, e^{\gamma+\varepsilon_{at+1}} \widetilde{K}_{t+1}^{\alpha} L^{1-\alpha}} \alpha e^{\gamma+\varepsilon_{at+1}} \widetilde{K}_{t+1}^{\alpha-1} L^{1-\alpha} \Rightarrow$$

$$\frac{e^{\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}}}{(1-\alpha\beta)\, e^{\gamma+\varepsilon_{at}} \widetilde{K}_t^{\alpha} L^{1-\alpha}} = \beta E_t \frac{\alpha}{(1-\alpha\beta)\, e^{-\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}} e^{\gamma+\varepsilon_{at}} \alpha\beta \widetilde{K}_t^{\alpha} L^{1-\alpha}}$$

and in the labor supply condition (10):

$$\psi \frac{\widetilde{C}_t}{1-L} = (1-\alpha)\, e^{\gamma+\varepsilon_{at}} \widetilde{K}_t^{\alpha} L^{-\alpha} \Rightarrow$$

$$\frac{L}{1-L} = \frac{1-\alpha}{\psi(1-\alpha\beta)}$$

Since both equalities hold, our closed-form solution is correct.

4.2. Policy Functions in Levels

Often, it is useful to express the policy function in levels. Then, the policy function for capital is given by:

$$\begin{aligned}
\widetilde{K}_{t+1} &= e^{-\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}} e^{\gamma+\varepsilon_{at}} \alpha\beta \widetilde{K}_t^{\alpha} L^{1-\alpha} \Rightarrow \\
\frac{K_{t+1}}{Z_{t+1}V_t} &= e^{-\frac{\gamma+\alpha\upsilon+\varepsilon_{at}+\alpha\varepsilon_{\upsilon t}}{1-\alpha}} e^{\gamma+\varepsilon_{at}} \alpha\beta \left(\frac{K_t}{Z_t V_{t-1}}\right)^{\alpha} L^{1-\alpha} \Rightarrow \\
K_{t+1} &= e^{\gamma+\upsilon+\varepsilon_{at}+\varepsilon_{\upsilon t}} \left(A_{t-1}V_{t-1}\right) \alpha\beta K_t^{\alpha} L^{1-\alpha}
\end{aligned}$$

and for consumption:

$$\begin{aligned}
\widetilde{C}_t &= (1-\alpha\beta)\, e^{\gamma+\varepsilon_{at}} \widetilde{K}_t^{\alpha} L^{1-\alpha} \Rightarrow \\
\frac{C_t}{Z_t} &= (1-\alpha\beta)\, e^{\gamma+\varepsilon_{at}} \left(\frac{K_t}{Z_t V_{t-1}}\right)^{\alpha} L^{1-\alpha} \Rightarrow \\
C_t &= (1-\alpha\beta)\, e^{\gamma+\varepsilon_{at}} A_{t-1} K_t^{\alpha} L^{1-\alpha}
\end{aligned}$$

4.3. Policy Functions as a Backward Representation

If we take logs in the previous expressions, we get:

$$\begin{aligned}
\log K_{t+1} &= \Gamma_k + \gamma + \upsilon + \log A_{t-1} + \log V_{t-1} + \alpha \log K_t + \varepsilon_{at} + \varepsilon_{\upsilon t} \\
\log C_t &= \Gamma_c + \log A_{t-1} + \alpha \log K_t + \varepsilon_{at}
\end{aligned}$$

where $\Gamma_k = \log \alpha\beta L^{1-\alpha}$ and $\Gamma_c = \log(1-\alpha\beta)\, L^{1-\alpha}$ are constants.

If we substitute recursively the values of $\log A_{t-1}$ and $\log V_{t-1}$, and for simplicity assume that $A_0 = V_0 = 1$, we derive a representation of the behavior of the economy as a function of today's capital and past shocks:

$$\begin{aligned}
\log K_{t+1} &= \Gamma_k + \alpha \log K_t + \sum_{j=0}^{t} \Lambda^j \left(\gamma + \upsilon + \varepsilon_{at-j} + \varepsilon_{\upsilon t-j}\right) \\
\log C_t &= \Gamma_c + \alpha \log K_t + \sum_{j=0}^{t} \Lambda^j \left(\gamma + \varepsilon_{at-j}\right)
\end{aligned}$$

where Λ is the lag operator. In these two expression we can see how the unit roots in the technology processes imply that the effects of technology shocks are permanent. This representation also illustrates the importance of α as the key parameter accounting for the dynamics of the economy through the autoregressive component.

We can also express the log of capital as:

$$\log K_{t+1} = (1 - \alpha\Lambda)^{-1} \left(\Gamma_k + \sum_{j=0}^{t} \Lambda^j \left(\gamma + \upsilon + \varepsilon_{at-j} + \varepsilon_{\upsilon t-j} \right) \right)$$

Taking this formula, we can substitute in the expression for the log of consumption:

$$\log C_t = \Gamma_c + \alpha (1 - \alpha\Lambda)^{-1} \Lambda \left(\Gamma_k + \sum_{j=0}^{t} \Lambda^j \left(\gamma + \upsilon + \varepsilon_{at-j} + \varepsilon_{\upsilon t-j} \right) \right) + \sum_{j=0}^{t} \Lambda^j \left(\gamma + \varepsilon_{at-j} \right)$$

to obtain a backward representation.

Beyond describing the dynamic behavior of the economy, the backward representation is useful to build the likelihood of the model. This may be important, for example, to check the output of a procedure that evaluates the likelihood function by simulation methods (Fernández-Villaverde and Rubio-Ramírez, 2004).

References

[1] Fernández-Villaverde, J. and J.F. Rubio-Ramírez (2004). "Estimating Macroeconomic Models: a Likelihood Approach". *Mimeo*, University of Pennsylvania.

[2] Fisher, J. (2003). "The New View of Growth and Business Cycles". Federal Reserve Bank of Chicago Economic Perspectives 23:1, 35-56.

[3] Fisher, J. (2003). "Technology Shocks Matter". *Mimeo*, Federal Reserve Bank of Chicago.

[4] Greenwood, J., Z. Hercowitz, and P. Krusell (1997), "Long-Run Implications of Investment-Specific Technological Change". *American Economic Review* 87, 342-362.

[5] Greenwood, J., Z. Hercowitz, and P. Krusell (2000), "The Role of Investment-Specific Technological Change in the Business Cycle". *European Economic Review* 44, 91-115.

[6] Krusell, P., L.E. Ohanian, J.V. Ríos-Rull, and G. L. Violante (2000), "Capital-Skill Complementarity and Inequality: A Macroeconomic Analysis". *Econometrica* 68, 1029-53.