Technological Synergies, Heterogeneous Firms, and Idiosyncratic Volatility

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Abstract

This paper shows the importance of technological synergies among heterogeneous firms for aggregate fluctuations. First, we document six novel empirical facts using microdata that suggest the existence of important technological synergies between trading firms, the presence of positive assortative matching among firms, and their evolution during the business cycle. Next, we embed technological synergies in a general equilibrium model calibrated on firm-level data. We show that frictions in forming trading relationships and separation costs explain imperfect sorting between firms in equilibrium. In particular, an increase in the volatility of idiosyncratic productivity shocks significantly decreases aggregate output without resorting to non-convex adjustment costs.

Keywords: Technological synergies, heterogeneous firms, idiosyncratic uncertainty.

JEL classification: C63, C68, C78, E32, E37, E44, G12.

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1 Introduction

The premise of our analysis is that technological synergies —the cooperation among firms to increase the productivity of a production function— are prevalent in modern economies where final output results from the completion of complex operations that require strategic partnerships. For instance, producing a computer requires the integration of several highly sophisticated parts developed by different manufacturers, and the synergies across the several interlinked processes are critical for the final good. If one of the technologies fails the technical requirements for the manufacturing of the computer (or arrives below specifications), production suffers, and the relationship among firms may even dissolve with the entire loss of output.

Despite technological synergies being central to the failure or survival of modern business ventures, macroeconomists have overlooked their implications for aggregate fluctuations. Our study provides a first attempt to explore the role of technological synergies in the business cycle. We consider two key questions: How do technological synergies influence the sorting between producers with different productivity? How does a heightening in the volatility of idiosyncratic productivity affect aggregate output?

To answer these questions, we use Compustat fundamental annual data, Compustat Segment data, Factset Supply Chain Relationships data, and the BEA input-output tables to uncover six novel empirical facts.

Fact 1 is that the economic fundamentals of trading firms, measured as labor productivity, return on equity, and sales growth, are positively correlated. By focusing on the correlation in the year before the trading relationship is established, we argue that Fact 1 cannot be driven by common shocks to the firms.[1] Fact 2 is that a firm's output, conditional on the firm's productivity, is positively correlated with its partners' productivity and negatively correlated with its distance from its partners' labor productivity rankings. Fact 3 is that the degree of supermodularity is heterogeneous across industry pairs. Interestingly, industry pairs with strong supermodularity pay high wages, are located upstream in the production network, and are economically more relevant since they entail a larger Domar weight (i.e., gross output as a share of GDP). Fact 4 is that relationships with very different economic fundamentals between firms, which we refer to as mismatches, are less durable. Facts 1-4 indicate that positive assortative matching of trading

[1] To avoid repetition, we will drop "trading" from "trading relationship" when no ambiguity occurs.

relationships is prominent in the economy and is more stable than mismatching, which can be accounted for parsimoniously by technological synergies between trading firms.

Fact 5 is that a higher absolute value of idiosyncratic productivity shocks to either side of a relationship predicts a higher probability of separation in the subsequent years. We show that the dominating role of negative or positive shocks does not drive Fact 5. Instead, it is the magnitude of the shocks that leads to separation. Fact 5 can be rationalized by the destabilizing role of idiosyncratic shocks of both signs that make trading firms more different from each other, leading to less efficiency and endogenous separation. Fact 6 is that higher volatility of idiosyncratic productivity shocks in a sector is correlated with a fall in sectoral output paired with a fall in output in connected sectors, and industry pairs largely account for the negative effect with a positive degree of technological synergies. This can be explained by the volatility of idiosyncratic productivity shocks systematically altering the extent of mismatching in the economy. Facts 5 and 6 motivate us to investigate the role of technological synergies and idiosyncratic productivity shocks in a real business cycle model.

First, to build intuition, we develop a simple and static model with synergies in production input. We assume that manufacturing one unit of output requires a distinct input from firms in each sector whose productivity is either low- or high-type. Synergies in production technology entail a relationship with similarly productive firms to produce more output (our Fact 2). High-productivity firms prefer to form a relationship with partners of high productivity, a standard assumption in matching theory since the seminal study by Gale and Shapley (1962). Technological synergies indicate that sorting between firms with the same productivity type is the stable and efficient equilibrium (our Fact 1). Relationships between firms with different types of technology are unstable (our Fact 4) because the firm with high technology optimally terminates the relationship with a firm of low-type technology to seek to establish a new relationship with a firm of equally high-type technology that yields a larger payoff.

The simple model illustrates that idiosyncratic shocks destabilize relationships by transforming positive assortative matching into mismatching (our Fact 5). Technological synergies (our Fact 2) introduce a "bottleneck effect" that generates an asymmetry in output response to negative and positive idiosyncratic shocks. Adverse idiosyncratic shocks that reduce the productivity of one partner decrease output by impairing the production capacity of the relationship. In contrast, positive idiosyncratic shocks to a single partner exert limited benefits to the relationship since the firm with low productivity cannot exploit the improved technology.

The asymmetric effect of idiosyncratic shocks implies that the heightening in the volatility of idiosyncratic shocks depresses output on average (our Fact 6). In particular, an increase in the volatility of idiosyncratic productivity in one sector raises the number of relationships with a firm of different technology types, for which technological synergies imply sub-optimal production. Misallocation of relationships arises due to idiosyncratic shocks and technological synergies. It is intrinsic to business cycle fluctuations and not generated by exogenous distortions in goods or labor markets. Moreover, since technological synergies apply to all interlinked industries, the volatility of idiosyncratic shocks in one industry induces a local contraction in output and in the connected industries.

We embed the intuition of the simple static model into a quantitative and dynamic framework that allows us to quantify the relevance of the critical mechanisms at play for the importance of technological synergies for economic activity in a more realistic environment.

The quantitative model has four new features. First, frictions in the matching process across firms prevent the instantaneous and costless formation of relationships, which is motivated by the fact that sorting is far from perfect in the data. Second, we assume directed search from both sides of the market to form relationships.[2] We show theoretically that, with the above two features, log-supermodularity (which is stronger than supermodularity) of the surplus function is a sufficient condition for the stability of positive assortative matching. Third, the termination of relationships with different productivity types is staggered, motivated by the time-consuming separation of relationships observed in the data. Fourth, we propose a generalization of the production technology in which the degree of technological synergies is governed by a single parameter, which we estimate using firm-level data.

We use the extended model to assess the propagation channels for technological synergies quantitatively. We calibrate the novel parameters in the system that govern the degree of technological complementarity, search frictions, and endogenous separations using FactSet and Compustat firm-level data. We show that under the benchmark calibration, search frictions and delayed separations generate imperfect sorting of firms and a 21% drop in output in the

[2]Our assumptions of directed search from both sectors are more suitable for our analysis of inter-firm relationships than the conventional search models with two-sided heterogeneity, such as in Shimer and Smith (2000), who considered random search, or in Eeckhout and Kircher (2010), who assume that only one side of the matching market conducts a directed search, while the other side posts prices. Firms often have good knowledge of the potential partners in other industries: their main challenge is to get the right match for them.

stationary steady state, which is caused by a 12% decrease in the utilization rate of productive resources and an 11% decrease in the average production efficiency. The size of output losses is comparable to the output gap due to other types of frictions or misallocations.[3]

An increase of 34% in the standard deviation of idiosyncratic productivity shocks, which is of the same magnitude as the increase in uncertainty during the Great Recession, leads to a 1.2% drop in aggregate output. The fall in production is explained by an increase in the mismatch and the persistent rise in the separation rate. The effects of increased idiosyncratic uncertainty are persistent, since the termination of ongoing relationships with inefficient production is time-consuming and forming new relationships is costly.

Our analysis is related to three realms of research. First, we contribute to the literature on technological synergies. Kremer (1993) and Jones (2011) study the implications of technological synergies for economic development and the secular allocation of resources. Technological synergies are also central to the study of strategic mergers and acquisitions of firms (Rhodes-Kropf and Robinson, 2008; Xu, 2017; David, 2021), the magnification effect of technology adoption (Eslava et al., 2015), the slowdown in aggregate productivity (Acemoglu et al., 2023), international trade (Demir et al., 2023), and production networks (Acemoglu and Tahbaz-Salehi, 2020).[4] Compared to these studies, we investigate the role of technological synergies in aggregate fluctuationsand study how they evolve dynamically in response to exogenous disturbances.

Second, we add to the literature on misallocation. The literature attributes misallocation to distortions in physical (Banerjee and Duflo, 2005; Hsieh and Klenow, 2009; Restuccia and Rogerson, 2008) or human capital (Alder, 2016; Hsieh and Moretti, 2019; Baley et al., 2022). Instead, we study the dynamic misallocation in relationships, a new source of misallocation that yields significant output losses. Our misallocation originates from idiosyncratic shocks to technology rather than exogenous distortions such as financial frictions and distortionary taxes.

Finally, our work is linked with the literature on idiosyncratic uncertainty (Christiano et al., 2014; Bloom et al., 2018; Arellano et al., 2019) by showing that technological synergies trigger large output losses when idiosyncratic volatility increases, without the need to introduce non-convex adjustment costs in the model.

[3] For example, the output gap induced by financial constraints, through capital misallocation and inefficiently low capital accumulation, ranges from 35% to 70% (see the literature review by Hopenhayn, 2014).

[4] See Fernández-Villaverde et al. (2019, 2021) for alternative sources of complementarities based on the formation of vendor contracts to study fiscal policy and monopsony power in labor markets.

The rest of the paper is structured as follows. Section 2 documents the six novel facts about the technological synergies embedded in relationships. Section 3 develops a simple model to outline the interplay between technological synergies, idiosyncratic shocks, and sorting. Section 4 extends the simple model to a rich general equilibrium framework. Section 5 calibrates the model. Sections 6-8 explore its quantitative predictions. Section 9 concludes.

2 Empirical evidence

We document six facts about the assortative matching and technological synergies of trading relationships. These facts will motivate our model and offer a benchmark against which to evaluate it.

Data. We study the formation of relationships by combining two datasets. The first is the Compustat Customer Segment data, which provide information on inter-firm trading for the universe of publicly listed firms in the US. The data have a yearly frequency and cover 1976-2020, with approximately 18 thousand firms. Since publicly listed firms must supply the identity of trading partners that account for more than a 10% share of yearly sales, we can obtain 72,694 distinct customer-supplier relationships. The second dataset is the FactSet Supply Chain Relationships data, which comprises information on firms' relationships from public sources such as SEC 10-K annual filings, investor presentations, and press releases since 2003. Using the sample 2003-2020, we obtain 289,239 distinct customer-supplier relationships. The two datasets are merged with Compustat fundamental annual data, which provide information on firms' output, employment, and financial positions.

Figure 1 plots the cross-sectional distribution of the yearly duration of the relationship (the duration of a relationship that starts in year t_1 and ends in year t_2 is $t_2 - t_1 + 1$) in the combined dataset. Relationships are persistent, with a mean duration of about three years. Since the sample for the combined dataset ends in 2020, with many ongoing relationships, three years is a downward-biased estimate of the true persistence.

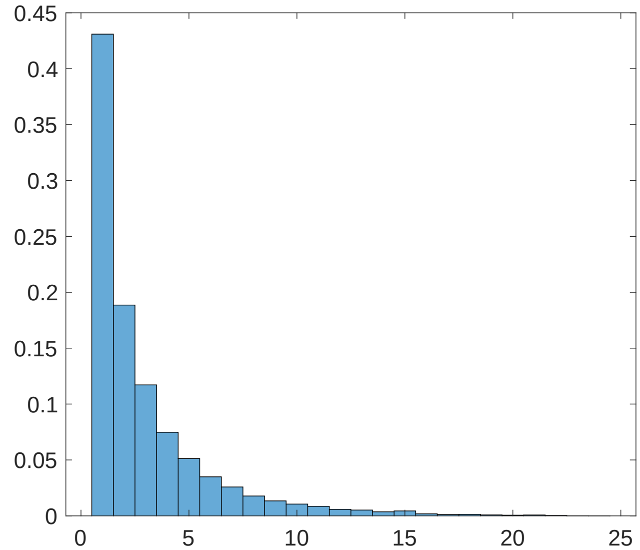


Figure 1: Distribution of duration of trading relationships (years)

Fact 1: Positive assortative matching of trading relationships

Fact 1 is that the economic fundamentals of trading firms are positively correlated. Since the correlation of economic fundamentals between trading firms could be driven by common shocks rather than positive assortative matching, we focus on the partners in a newly formed relationship and assess the correlation of economic fundamentals in the year *before* the formation of a relationship.[5]

More concretely, we use each distinct relationship as an observation and estimate:

$$decile\left(\pi_{j,k,t}\right) = \alpha + \beta \times decile\left(\pi_{j,k,t}^{cus}\right) + indu_{j,k} + \chi_t + \epsilon_{j,k}, \tag{1}$$

for $j \in \{1, 2, \cdots, J\}$ and $k \in \{1, 2, \cdots, J_j\}$.

The variable $\pi_{j,k,t}$ is firm j's economic fundamental in the year, t, before the start of its relationship at $t+1$ with its kth customer. We select three measures of fundamentals that are commonly used to gauge a firm's performance: labor productivity (the ratio of sales to employment), sales growth, and return on equity (ROE, the ratio of net income to net worth). We also consider Tobin's q (the ratio of market value to assets' replacement cost), which reflects how the financial market evaluates the firm and, hence, might capture the value of other fundamentals, such as intangibles, not easily measured. The regressand $decile\left(\pi_{j,k}\right)$ is the decile

[5]For robustness, Table A.1 in the Appendix reports the results when we use data for the years when the firms are in the relationship.

of $\pi_{j,k}$ within firm j's two-digit NAICS industry in the year before the start of the relationship, ranging from one (bottom 10%) to ten (top 10%). We use deciles to capture the relative position of a firm within its industry because absolute values of performance are hard to compare across industries. Our results are robust to ranking firms within three-digit NAICS industries.

The variable $\pi^{cus}_{j,k}$ is the economic fundamental of firm j's kth customer in the year before the start of the relationship. The regressor $decile\left(\pi^{cus}_{j,k}\right)$ is the decile of $\pi^{cus}_{j,k}$ within the customer firm's two-digit NAICS industry in the year before the start of the relationship. The other regressor $indu_{j,k}$ is the industry fixed effects for firm j and its kth customer, and the regressor χ_t is the year fixed effects. We include them in the regression to account for the potential difference in sorting across industries or over time.

Table 1: Assortative matching for ranking of economic fundamentals, one year before the match

	(1)	(2)	(3)	(4)
Meas. of fundamental	Labor productivity	Sales growth	ROE	Tobin's q
$decile\left(\pi^{cus}_{j,k,t}\right)$	0.06***	0.07***	0.03***	0.06***
	(0.01)	(0.01)	(0.01)	(0.01)
Industry Pair FE	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes
Adjusted R^2	0.07	0.02	0.03	0.08
Observations	23,829	24,374	28,034	23,131

Note: Sample: 1976-2020. Standard errors are in parentheses. *** denotes significance at the 1% level.

Table 1 shows that the estimate for β is positive and statistically significant for all four measures of economic fundamentals. The results imply that firms with stronger economic fundamentals (compared to other firms in the same industry over the same period) establish more relationships with customers with stronger economic fundamentals. Thus, Table 1 provides evidence for the existence of positive assortative matching.

Fact 2: Firm's sales comove with partner's labor productivity

Fact 2 is that a firm's output, controlling for the firm's labor productivity, is positively correlated with its partner's productivity. Also, a firm's output decreases with the degree of mismatch. These two observations are parsimoniously accounted for by a supermodular production function, like the one we will use in our model. Supermodularity implies that firms have an incentive to match with more productive partners, generating positive assortative matching.

To show these results, we first estimate the regression:

$$y_{j,t} = \beta z_{j,t} + \eta \times \overline{decile}\left(z^{cus}_{j,t}\right) + indu_j + \chi_t + \epsilon_{j,t}, \tag{2}$$

where $y_{j,t}$ and $z_{j,t}$ are firm j's log sales and log labor productivity, respectively (we remove firm-specific time trends from both variables). The regressor $\overline{decile}\left(z^{cus}_{j,t}\right) = \sum_k decile\left(z^{cus}_{j,k,t}\right)/N^{cus}_{j,t}$ is the average decile of firm j's partners' labor productivities within the partners' two-digit NAICS industries, where $N^{cus}_{j,t}$ is the number of firm j's partners.[6] The variables $indu$ and χ_t are the industry and year fixed effects, respectively. Our goal is to estimate the contribution of the partner's economic fundamentals to the firm's output rather than estimating the firm's production function, which would require more data than we have in our datasets.

Table 2: Sales comove with partner's labor productivity

	(1)	(2)
Dependent Variable :	Log sales	
$\overline{decile}\left(z^{cus}_{j,t}\right)$	0.03***	0.03***
	(0.01)	(0.01)
$\overline{\Delta}_{j,t}$		-0.05***
		(0.01)
$z_{j,t}$	0.88***	0.84***
	(0.01)	(0.02)
Time FE	Yes	Yes
Industry FE	Yes	Yes
Adjusted R^2	0.28	0.28
Observations	30,391	29,889

Note: Sample: 1976-2020. The dependent variables are the firm's log sales.

Column (1) in Table 2 shows the estimation results from regression (2). A firm's log sales are increasing in its log labor productivity. The firm's log sales are also positively correlated with the ranking of its partners' labor productivity. Conditional on the firm's labor productivity, increasing a firm's partners' labor productivity decile by one (e.g., from 5th to 6th) would increase the firm's sales by 3%, an economically significant move.

[6]We consider the average decile rather than the average labor productivity because the partners can be from different industries, making the levels of labor productivity hard to compare.

We extend our regression to include the average degree of mismatch between a firm and its partners. We measure the degree of mismatch between a firm and its kth customer with $\Delta_{j,k,t} = |decile\left(z_{j,t}\right) - decile\left(z^{cus}_{j,k,t}\right)|$, which takes values between zero and nine, with zero indicating no mismatch and a positive value indicating a mismatch. The average degree of mismatch is then computed as $\overline{\Delta}_{j,t} = \sum_k \Delta_{j,k,t}/N^{cus}_{j,t}$. We estimate the following regression.

$$y_{j,t} = \beta z_{j,t} + \eta \times \overline{decile}\left(z^{cus}_{j,t}\right) + \phi\overline{\Delta}_{j,t} + indu_j + \chi_t + \epsilon_{j,t}. \tag{3}$$

Column (2) in Table 2 shows that the estimate for ϕ is negative, indicating that a firm's log sales are higher when it has a ranking in labor productivity similar to that of its customers. This is consistent with the hypothesis that firms have a comparative advantage in trading with partners of similar labor productivity, and there is a bottleneck effect induced by the firm with a lower productivity ranking.

To see this, consider the simpler case where firm j has only one partner, firm k. The second and third terms on the RHS of equation (3) become:

$$\eta \times decile(z_{k,t}) + \phi|decile(z_{j,t}) - decile(z_{k,t})|. \tag{4}$$

If $decile(z_{k,t}) > decile(z_{j,t})$, i.e., firm j is the bottleneck, the term (4) becomes $(\eta+\phi)decile(z_{k,t}) - \phi decile(z_{j,t})$. Similarly, if $decile(z_{k,t}) \leq decile(z_{j,t})$, i.e., firm k is the bottleneck, the term (4) becomes $(\eta - \phi)decile(z_{k,t}) + \phi decile(z_{j,t})$. In either case, a more negative ϕ implies a higher weight of the lower-ranking firm and a lower weight of the higher-ranking firm.

Equation (3) can be motivated by a simple example. Imagine we have firm j and its intermediate goods supplier k. Supplier k produces intermediate goods at a unit cost of $e^{w_t - \zeta_{k,t}}$, where w_t and $\zeta_{k,t}$ are log wage and supplier k's efficiency. We use the term *efficiency* to distinguish this shifter of the cost function from the measured labor productivity in our empirical exercises. Firm i's output is determined by $Y_{j,t} = e^{x_t + f\left(\zeta_{j,t},\zeta_{k,t}\right)} L^{\alpha}_{j,t} M^{\gamma}_{j,t},\ \alpha + \gamma < 1$, where x_t is the aggregate TFP. The idiosyncratic TFP $f\left(\zeta_{j,t}, \zeta_{k,t}\right)$ can be potentially determined by both firms' efficiencies, $\zeta_{j,t}$ and $\zeta_{k,t}$. Finally, $L_{j,t}$ and $M_{j,t}$ are firm j's labor and intermediate inputs.

If factor markets are competitive, firm j's optimality conditions are $\alpha e^{x_t + f\left(\zeta_{j,t},\zeta_{k,t}\right)} L^{\alpha-1}_{j,t} M^{\gamma}_{j,t} = e^{w_t}$ and $\gamma e^{x_t + f\left(\zeta_{j,t},\zeta_{k,t}\right)} L^{\alpha}_{j,t} M^{\gamma-1}_{j,t} = e^{w_t - \zeta_{k,t}}$. Define $\bar{\epsilon} = \gamma^{\frac{\gamma}{1-\alpha-\gamma}} \alpha^{\frac{1-\gamma}{1-\alpha-\gamma}}$, a constant, possibly industry-specific. Then:

$$\log\left(Y_{j,t}\right) = \underbrace{\log\left(\frac{Y_{j,t}}{L_{j,t}}\right)}_{\text{log labor prod.}} + \underbrace{\frac{1}{1-\alpha-\gamma}\left[\gamma\zeta_{k,t} + f\left(\zeta_{j,t},\zeta_{k,t}\right) + x_t - w_t + \bar{\epsilon}\right]}_{\text{log labor input}}. \tag{5}$$

Equation (5) shows that firm j's log output can be decomposed among log labor productivity, the supplier k's efficiency ($\zeta_{k,t}$), a term that captures synergies ($f\left(\zeta_{j,t},\zeta_{k,t}\right)$), and the aggregate state ($x_t - w_t$), exactly the form of regression (3).

Fact 3: Heterogeneity in supermodularity across industry pairs

Fact 3 is that the degree of supermodularity, i.e., the extent to which mismatch hurts output, is heterogeneous across industry pairs. Furthermore, industry pairs with strong supermodularity pay high wages, are located upstream in the production network, and are economically more relevant since they entail a larger Domar weight (i.e., gross output as a share of GDP).

To study the heterogeneity in the degree of supermodularity across industry pairs, we separately estimate equation (3) for each industry pair. The point estimate for the coefficient of the degree of mismatch for the industry pair (p, q), denoted by $\phi_{p,q}$, encapsulates the degree of supermodularity.

These $\phi_{p,q}$ are reported in Figure 2. The x- and y-axes indicate the three-digit NAICS codes for the customer (q) and supplier (p) industries, respectively. A darker blue indicates a more negative $\phi_{p,q}$, and thus a stronger degree of supermodularity. In contrast, a lighter yellow color indicates a more positive $\phi_{p,q}$, and thus a weaker degree of supermodularity (or, equivalently, stronger *sub*modularity). The empty entry in white (indicated by NaN in the legend) reports the statistically insignificant estimates or those with insufficient observations.

Figure 2 reveals a wide dispersion in the degree of supermodularity across industry pairs. For instance, Mining Except for Oil and Gas (customer industry, $q = 212$, x-axis) and Petroleum and Coal Products Manufacturing (supplier industry, $p = 324$, y-axis) have one of the strongest degrees of supermodularity ($\phi_{324,212} = -5.81$). Similarly, the pair Chemical Products (customer industry, $q = 325$, x-axis) and Professional, Scientific, and Technical Services (supplier industry, $p = 541$, y-axis) have an estimated $\phi_{541,325} = -4.76$. In contrast, Computer and Electronic Product Manufacturing (customer industry, $q = 334$, x-axis) and Securities, Commodity Contracts,

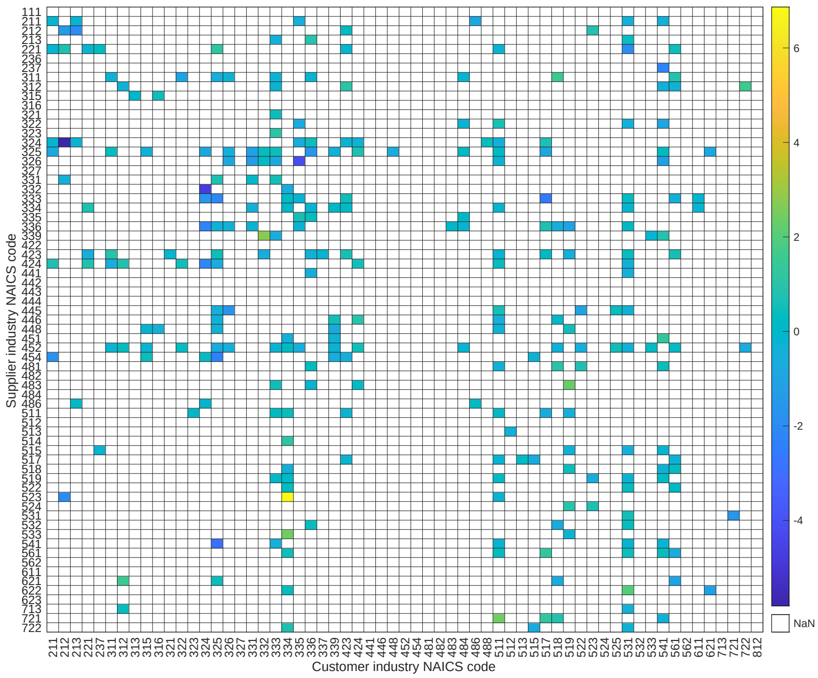


Figure 2: Degree of supermodularities across industry pairs

and Other Financial Investments and Related Activities (supplier industry, $p = 523$, y-axis) have one of the weakest degrees of supermodularity ($\phi_{523,334} = 6.87$).

Next, we show that a high degree of supermodularity is associated with higher wages (which may reflect the degree of complexity involved in the production processes) and that supermodularity is stronger for industry pairs positioned in the upstream part of the production network and generates larger sales. We unveil these new facts by estimating:

$$x_{p,q} = \alpha + \beta\phi_{p,q} + sector_{p,q} + \epsilon_{p,q}, \quad x_{p,q} \in \{w_{p,q}, ups_{p,q}, y_{p,q}\}, \tag{6}$$

where the variable $\phi_{p,q}$ is our constructed degree of supermodularity, and the variable $w_{p,q}$ is the log average wage for industries p and q in 2012, constructed by the BLS. The variable $ups_{p,q}$ measures the average upstreamness of industries p and q, as constructed by Antràs et al. (2012) using the distance of the industries from the final use. The variable $y_{p,q}$ represents the log average sales of industries p and q obtained from the 2012 input-output table. Industry sales are proportional to Domar weights across industries and proxy for the relative importance of industries in the transmission of sectoral shocks in the production network. Finally, $sector_{p,q}$ is the sector pair fixed effects. For example, the industry pair (325,541) belongs to the sector pair

(32,54). This approach enables us to compare industry pairs with similar types of inter-industry relationships. We focus on the trading of intermediate goods between different industries, i.e., focusing on $p \neq q$ and dropping the within-industries trading by the industry pairs on the main diagonal of the input-output table.

Table 3: Supermodularity is positively correlated with wages, upstreamness, and sales

	(1)	(2)	(3)
Dependent variable	Wages	Upstreamness	Sales
$\phi_{p,q}$	-0.27**	-0.27***	-0.10*
	(0.12)	(0.09)	(0.06)
Constant	2.90**	0.51***	1.15
	(1.30)	(0.19)	(0.72)
Adjusted R^2	0.01	0.02	0.01
Observations	341	483	317

Note: The dependent variables are the log of average wage, average upstreamness, and log of average sales of industries p and q, respectively. The independent variable $\phi_{p,q}$ is the coefficient of the mismatch term in equation (3) estimated using data in industry pair (p, q).

Table 3 shows that the estimated $\phi_{p,q}$ are negative and statistically significant, implying that supermodularity is associated with high wages, it is located upstream in the production network, and it is linked to larger sales.

Fact 4: Mismatches are less durable

Fact 4 is that the relationships of trading partners with very different economic fundamentals, which we refer to as mismatches, are less durable. More specifically, the larger the mismatch, the more likely the separation of trading firms.

As with Fact 2, we measure the degree of mismatch in a relationship as the distance between two partners' deciles in the distribution of economic fundamentals in the year before the start of the relationship (as defined in Facts 1 and 2), measured by $\Delta_{j,k} = |decile\left(\pi_{j,k}\right) - decile\left(\pi_{j,k}^{cus}\right)|$ and estimate:

$$\mathrm{dur}_{j,k} = \beta \times \Delta_{j,k} + indu_{j,k} + \chi_t + \epsilon_{j,k},$$

where $\mathrm{dur}_{j,k,t}$ is the annual duration of the relationship, $indu_{j,k}$ is the industry pair fixed effects for firm j and its kth customer, and χ_t is the year fixed effects.

Table 4: Relationship duration and the degree of mismatch

	(1)	(2)	(3)	(4)
Meas. of fundamental	Labor productivity	Sales growth	ROE	Tobin's Q
$\Delta_{j,k}$	-0.02*	-0.08***	-0.09***	-0.03**
	(0.01)	(0.01)	(0.01)	(0.01)
Time FE	Yes	Yes	Yes	Yes
Industry Pair FE	Yes	Yes	Yes	Yes
Adjusted R^2	0.18	0.14	0.15	0.14
Observations	23,829	24,374	28,034	23,131

Note: Sample: 1976-2020. Standard errors are in parentheses. *** denotes significance at the 1% level.

Table 4 shows that the duration of a relationship decreases with the degree of mismatch. The regression coefficient is statistically significant and quantitatively large for all measures of economic fundamentals. In contrast, relationships among firms with similar economic fundamentals are stable and durable. For robustness, Table A.2 in the Appendix shows the results when we focus on all years rather than the year preceding the start of matches.

Fact 5: Idiosyncratic shocks lead to separation of trading relationships

Fact 5 is that a higher absolute value of idiosyncratic productivity shocks to either side of a relationship predicts a higher probability of separation in the subsequent years.

We proxy idiosyncratic productivity shocks with the change in the labor productivity decile. Then, we study the relationship between the absolute value of idiosyncratic shocks to a firm and its trading partner and the subsequent separation of relationships by estimating:

$$\mathrm{sep}_{j,k,t} = \beta_1 \times |\Delta decile\,(\pi_{j,k,t-1})| + \beta_2 \times \left|\Delta decile\left(\pi^{cus}_{j,k,t-1}\right)\right| + indu_{j,k} + \gamma_t + \epsilon_{j,t},$$

where $\mathrm{sep}_{j,k,t}$ is a dummy variable equal to 1 if firm j terminates an existing relationship with customer k in year t, and the variables $|\Delta decile\,(\pi_{j,k,t-1})\,|$ and $|\Delta decile\left(\pi^{cus}_{j,k,t-1}\right)|$ are the absolute value of the change in firm j's and customer k's decile of fundamentals, while $indu_{j,k}$ is the industry pair fixed effects for firm j and its kth customer. Variable γ_t is a time fixed-effect that controls for the potential comovement between the time trends of separation and the magnitude of idiosyncratic shocks.

Column (1) of Table 5 shows the benchmark results. The estimation evinces a significant

Table 5: Absolute value of idiosyncratic shocks and trading relationship separation, panel

	(1)	(2)	(3)	(4)	(5)	(6)
Meas. of fundamental	Labor productivity			Sales growth		
$\lvert \Delta decile\,(\pi_{j,k,t-1}) \rvert$	0.01***			0.003***		
	(0.001)			(0.001)		
$\left\lvert \Delta decile\left(\pi^{cus}_{j,k,t-1}\right) \right\rvert$	0.01***			-0.0002		
	(0.001)			(0.002)		
$\lvert \Delta decile\,(\pi_{j,k,t-2}) \rvert$		0.01***			0.002***	
		(0.001)			(0.001)	
$\left\lvert \Delta decile\left(\pi^{cus}_{j,k,t-2}\right) \right\rvert$		0.01***			0.001**	
		(0.002)			(0.001)	
$\Delta decile\,(\pi_{j,k,t-2})$			-0.0003			-0.001*
			(0.001)			(0.0004)
$\Delta decile\left(\pi^{cus}_{j,k,t-2}\right)$			0.001			-0.001*
			(0.002)			(0.0005)
Time FE	Yes	Yes	Yes	Yes	Yes	Yes
Industry Pair FE	Yes	Yes	Yes	Yes	Yes	Yes
Adjusted R^2	0.14	0.14	0.13	0.13	0.13	0.13
Observations	89,236	85,927	85,927	92,308	88,966	88,966
	(7)	(8)	(9)	(10)	(11)	(12)
Meas. of fundamental	ROE			Tobin's q		
$\lvert \Delta decile\,(\pi_{j,k,t-1}) \rvert$	0.003***			0.003***		
	(0.001)			(0.002)		
$\left\lvert \Delta decile\left(\pi^{cus}_{j,k,t-1}\right) \right\rvert$	0.002***			0.01***		
	(0.001)			(0.001)		
$\lvert \Delta decile\,(\pi_{j,k,t-2}) \rvert$		0.002***			0.01***	
		(0.001)			(0.001)	
$\left\lvert \Delta decile\left(\pi^{cus}_{j,k,t-2}\right) \right\rvert$		0.002***			0.004***	
		(0.001)			(0.001)	
$\Delta decile\,(\pi_{j,k,t-2})$			0.0002			-0.002**
			(0.0005)			(0.001)
$\Delta decile\left(\pi^{cus}_{j,k,t-2}\right)$			-0.0008*			-0.001
			(0.0005)			(0.001)
Time FE	Yes	Yes	Yes	Yes	Yes	Yes
Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
Adjusted R^2	0.13	0.13	0.13	0.13	0.13	0.13
Observations	108,693	105,222	105,222	88,158	84,922	84,922

Note: Sample: 1976-2020. Standard errors are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% level, respectively.

positive correlation between idiosyncratic shocks to either side of the match and the separation of a relationship. Column (2), using the lagged absolute value of the change in the decile of labor productivity as the independent variable, delivers the same result. Our estimates support the assumption that separations are time-consuming (e.g., due to adjustment costs or long-term contracts) and positively correlated with past changes in productivity.

As a robustness check, we check whether the separation in relationships depends on the sign of idiosyncratic shocks (i.e., on whether a firm's ranking or that of its trading partner rises or falls). Column (3) shows that if we use the simple change in productivity in the regression instead of the absolute change, the coefficient is statistically insignificant, suggesting that the joint combination of positive and negative changes in profits is critical in accounting for the termination of relationships. That is, our results in Columns (1) and (2) do not point to large negative shocks dissolving relationships but toward the relevance of technological synergies. Columns (4)-(12) show that the results are similar when idiosyncratic productivity shocks are measured as the changes in the decile of alternative economic fundamentals.

Fact 6: Micro uncertainty decreases output in connected industries

Fact 6 is that higher volatility of idiosyncratic productivity shocks in a sector is correlated with a fall in sectoral output paired with a fall in output in connected sectors. We construct measures of the volatility of idiosyncratic productivity using Compustat fundamental annual data. We focus on the post-1998 period since data for that period is consistent with real output at the 3-digit NAICS industry level.[7]

We proxy the volatility in idiosyncratic productivity with the inter-quartile range (IQR) of labor productivity growth as in Bloom et al. (2018), denoted as $iqr_{p,t}$ for industry p. Our panel provides yearly measures of the volatility of idiosyncratic shocks and output growth for 84 industries from 1998 to 2020. Next, we construct an index for each industry that measures its downstream industries' volatility. For each industry, its downstream industries are identified from the BEA input-output tables, which report input-output values of intermediate goods for 66 private industries in 3-digit NAICS. On average, an industry sells intermediate goods to 45 downstream industries, and the three most connected downstream industries account for around 53% of the total sales of intermediate goods. For each industry p, we derive an index $iqr^{cus}_{p,t}$ that measures the volatilities in the downstream industries by weighting our volatility measures by the value of the input-output intermediate goods purchased from industry p:

$$iqr^{cus}_{p,t} = \frac{\sum_{q \in \mathbb{D}_{p,t}} iqr_{q,t} M_{p,q,t}}{\sum_{q \in \mathbb{D}_{p,t}} M_{p,q,t}},$$

[7]We use data on GDP by industry constructed by the BEA and available between 1998 and 2018. Historical data for 1947-1997 are not consistent with the latter data.

where $\mathbb{D}_{p,t}$ is the set of industry p's downstream industries, $iqr_{q,t}$ is the volatility in idiosyncratic productivity of downstream industry q, and $M_{p,q,t}$ is downstream industry q's value of intermediate goods purchased from industry p.

Then, we estimate:

$$\Delta y_{p,t} = \beta_1 \times iqr_{p,t} + \beta_2 \times iqr^{cus}_{p,t} + \chi_p + \gamma_t + \epsilon_{p,t}, \tag{7}$$

where $\Delta y_{p,t}$ is the growth rate of real gross output in each industry p at time t constructed using the BEA dataset, $iqr_{p,t}$ is the constructed index of the volatility of idiosyncratic shocks for industry p, $iqr^{cus}_{p,t}$ is the constructed measure of the volatility of idiosyncratic shocks for industry p's downstream industries, and χ_p and γ_t are industry and time fixed effects, respectively. The standard errors are clustered by industry.

Table 6: Volatility of idiosyncratic shocks in the connected industries is negatively correlated with output growth

	(1)	(2)	(3)
Dependent Variable :	Industry output growth		
$iqr_{p,t}$	-0.05*	-0.05*	-0.07
	(0.03)	(0.02)	(0.05)
$iqr^{cus}_{p,t}$	-0.19***	-0.14**	-0.01
	(0.07)	(0.06)	(0.09)
$\widehat{iqr}^{cus}_{p,t}$			-0.07*
			(0.04)
$\Delta y^{cus}_{p,t}$		0.92***	0.88***
		(0.08)	(0.15)
Time FE	Yes	Yes	Yes
Industry FE	Yes	Yes	Yes
Adjusted R^2	0.30	0.37	0.34
Observations	1,189	1,189	483

Note: Sample: 1998-2020. Standard errors are in parentheses. Standard errors are clustered at the industry level. *, **, and *** denote significance at the 10%, 5%, and 1% level, respectively.

Column (1) in Table 6 shows results for our benchmark estimation. The volatility of idiosyncratic shocks within an industry and in the industry's connected industries has a significant and contractionary effect on sectoral output growth. Also, the volatility of idiosyncratic shocks from other downstream industries has a larger negative impact on output growth than the volatility of idiosyncratic shocks originating within the same industry. This finding shows that the transmission of changes in the volatility of idiosyncratic shocks across industries is significant and hurts the industry's output.

Unfortunately, there is no standard measure of exogenous changes in the volatility of idiosyncratic shocks. Therefore, we cannot identify the causal effect of volatility on connected industries' output.[8] But to partially alleviate the issue induced by the lack of an exogenous shock or an instrumental variable, we include the output growth in industry p's downstream industries, $\Delta y^{cus}_{p,t}$, as a control variable in Column (2). The variable $\Delta y^{cus}_{p,t}$ is computed as the mean of the gross output growth in industry p's connected industries, weighted by the value of the intermediate goods input and output traded with industry p. The coefficients of $iqr_{p,t}$ and $iqr^{cus}_{p,t}$ are still estimated as negative and statistically significant conditional on $\Delta y^{cus}_{p,t}$. The finding suggests that the negative effect of the volatility of idiosyncratic shocks in connected industries on industrial output is not generated by the fall in output in the connected industries.

As documented in Fact 2, the degree of supermodularity is heterogeneous across industry pairs. Does the uncertainty of an industry entail a stronger spillover effect on connected industries among industry pairs with a stronger degree of supermodularity? To investigate, we select for each industry p the downstream industries that are associated with positive degrees of supermodularity (i.e., industries that have a negative value for the coefficient $\phi_{p,q}$ from the estimates in Figure 2), and construct a synergy-weighted volatility index for the downstream industries:

$$\widehat{iqr}^{cus}_{p,t} = \frac{\sum_{q \in \mathbb{D}^*_{p,t}} iqr_{q,t} M_{p,q,t} |\phi_{p,q}|}{\sum_{q \in \mathbb{D}^*_{p,t}} M_{p,q,t} |\phi_{p,q}|},$$

where $\mathbb{D}^*_{p,t}$ is the set of industry p's downstream industries, with $\phi_{p,q}$ being negative and significant at the 10% level. The variable $iqr_{q,t}$ is the volatility in the idiosyncratic productivity of downstream industry q, $M_{p,q,t}$ is downstream industry q's value of intermediate goods purchased from industry p, and $|\phi_{p,q}|$ is the absolute value of $\phi_{p,q}$, which measure the degree of supermodularity.

Column (3) in Table 6 shows the results when we include $\widehat{iqr}^{cus}_{p,t}$ as an additional independent variable in equation (7). The estimate for the coefficient $\widehat{iqr}^{cus}_{p,t}$ is negative and statistically significant, while the estimate for the coefficient $iqr^{cus}_{p,t}$ is insignificant. Our results suggest that the negative effect of the volatility of idiosyncratic shocks on output in connected industries is largely accounted for by industry pairs with a positive degree of technological synergies.

[8] See Fernandez-Villaverde and Guerron-Quintana (2020), Fernández-Villaverde et al. (2015), and Mumtaz and Zanetti (2013) for a discussion on the impact of volatility shocks as a measure of economic uncertainty.

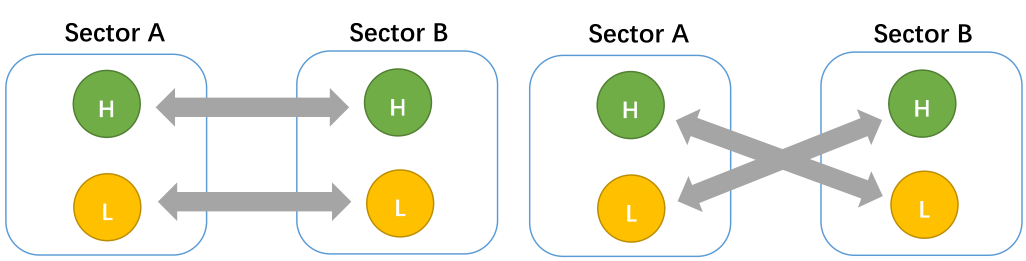
Taking stock

Facts 1-4 above suggest the existence of technological synergies between trading partners and that positive assortative matching of trading relationships is the stable equilibrium of a matching game. Furthermore, Fact 3 tells us that the synergies are the strongest where they matter: among firm pairs with a large Domar weight. Facts 5 and 6 motivate us to investigate the role of technological synergies and idiosyncratic productivity shocks in a business cycle model. We do so now in two steps: first, with a simple model that illustrates the main mechanisms at work and, second, with a quantitative model that replicates the facts we documented.

3 A simple model

We present a simple model that illustrates the interplay between inter-firm sorting, technological synergies, and idiosyncratic shocks. The economy is composed of two sectors, A and B. Each sector contains two firms, a firm H with high productivity, z^H, and a firm L with low productivity, z^L, where $z^H > z^L$.

Output $f\left(z^j, z^k\right)$ is produced by a trading relationship formed by two firms, each belonging to a different sector, where z^j is the productivity of the firm in sector A, and z^k is the productivity of the firm in sector B. The output from the relationship is divided between a payoff for the firm in sector A, $f_A\left(z^j, z^k\right)$, and a payoff for the firm in sector B, $f_B\left(z^j, z^k\right)$. Aggregate output is $f\left(z^j, z^k\right) + f\left(z^{-j}, z^{-k}\right)$ (where $-j$ and $-k$ denote the other firm in each sector).



(a) Positive assortative matching **(b)** Cross-matching

Figure 3: Alternative matching patterns

We call the relationships formed by firms of the same productivity *positive assortative matchings*, while we call the relationships of different productivity firms as *cross-matchings*.

Panels (a) and (b) in Figure 3 illustrate each of these cases. We assume that firms are always matched to focus on the key mechanisms, but we will relax this assumption in Section 4.

3.1 Positive assortative matching

First, we show that a standard assumption of monotonicity in the payoff functions produces positive assortative matching.

Assumption 1. *(Partial monotonicity). The payoff of a high-productivity firm strictly increases with the partner's productivity. Specifically,* $f_A\left(z^H, z^H\right) > f_A\left(z^H, z^L\right)$*, and* $f_B\left(z^H, z^H\right) > f_B\left(z^L, z^H\right)$*.*

Assumption 1 implies that a high-productivity firm strictly prefers forming a relationship with a high-productivity partner because it generates a larger payoff than matching with a low-productivity partner. Assumption 1 bundles a technological aspect and a distributional component: the share of output accrued to the high-productivity firm must be sufficiently low to make the relationship profitable for the low-productivity firm. Indeed, this will be the endogenous outcome under Nash bargaining rule for profit sharing in our quantitative model below. In our simple model, we assume that the profit-sharing rule is incentive-compatible for both partners. Under this weak condition, Assumption 1 generates positive assortative matching in equilibrium, since the H-type firm forms a relationship with an H-type partner and an L-type firm is forced to form a relationship with an L-type partner.

Assumption 1 also generates stable matches in the sense of Gale and Shapley (1962): H-type firms matched with other H-type firms do not want to switch partners. In comparison, cross-matching is unstable, since firms of H-type wish to separate from L-type firms and match with an H-type partner. Since cross-matching is unstable, we refer to it as *mismatch*.

Proposition 1 summarizes the effect of monotonicity on the sorting of firms across productivity types (the proof follows directly from Assumption 1).

Proposition 1. *(Positive assortative matching). Under the assumption of partial monotonicity, a trading relationship is stable if and only if it has positive assortative matching.*

3.2 Technological complementarity and its implications

Next, we show that technological complementarities make positive assortative matching the *efficient* equilibrium. But first, let us introduce the concept of supermodularity.

Definition 1. *(Supermodularity). A production function is supermodular and entails technological complementarity if* $f\left(z^H, z^H\right) + f\left(z^L, z^L\right) > f\left(z^H, z^L\right) + f\left(z^L, z^H\right)$.[9]

Definition 1 implies that the output of a relationship is greater with positive assortative matching than in a mismatch. Intuitively, supermodularity implies that firms have a comparative advantage in working with firms of the same productivity type. Supermodularity is embedded in standard production technologies and is widely used in economics (León-Ledesma and Satchi, 2019). For example, the Cobb-Douglas production function, $f\left(z^j, z^k\right) = \left(z^j\right)^{\alpha}\left(z^k\right)^{1-\alpha}$, is supermodular. Clearly, $\left(z^H\right)^{\alpha}\left(z^H\right)^{1-\alpha} + \left(z^L\right)^{\alpha}\left(z^L\right)^{1-\alpha} > \left(z^H\right)^{\alpha}\left(z^L\right)^{1-\alpha} + \left(z^L\right)^{\alpha}\left(z^H\right)^{1-\alpha}$ with $z^H > z^L$ and $0 < \alpha < 1$.[10]

Supermodularity delivers key results. For instance, assume that the economy starts from positive assortative matching with aggregate output $y = f\left(z^H, z^H\right) + f\left(z^L, z^L\right)$. Then, suppose that an unexpected idiosyncratic productivity shock hits sector A, changing the firm with H-type from z^H to z^L and the firm with L-type from z^L to z^H (but productivity in sector B remains unchanged). If firms cannot re-match, the new aggregate output is $y' = f\left(z^L, z^H\right) + f\left(z^H, z^L\right) < y$. In other words, shocks that change firms' idiosyncratic productivities translate into lower output under supermodularity if firms cannot rearrange their matches.

More pointedly, changes in the variance of the idiosyncratic shock generate movements in total output. To see this, assume that each sector is populated by a continuum of firms of size two (rather than two single firms). Half of the firms are H-type, and the other half is L-type. Also, the economy starts from positive assortative matching with a measure one of HH- and LL-type relationships, respectively. The total payoff in sectors A and B is $y_A = f_A\left(z^H, z^H\right) + f_A\left(z^L, z^L\right)$ and $y_B = f_B\left(z^H, z^H\right) + f_B\left(z^L, z^L\right)$.

[9] If the production function is twice differentiable, an equivalent definition of supermodularity is $\frac{\partial^2 f\left(z^j, z^k\right)}{\partial z^j \partial z^k} > 0$.

[10] A production technology can also be submodular, such that output is greater in mismatch than in positive assortative matching: $f\left(z^H, z^H\right) + f\left(z^L, z^L\right) < f\left(z^H, z^L\right) + f\left(z^L, z^H\right)$. An example of a submodular production function is $f\left(z^j, z^k\right) = log\left(z^j + z^k\right)$. Moreover, the production function can be neither supermodular nor submodular. For instance, $f\left(z^j, z^k\right) = \left(z^j\right)^{\alpha} + \left(z^k\right)^{\gamma}$ implies the same output under positive assortative matching and mismatch.

The idiosyncratic shock in sector A follows a Markov-switching process with a transition matrix:

$$\begin{pmatrix} 1-\rho & \rho \\ \rho & 1-\rho \end{pmatrix},$$

where ρ is the probability of changing technology type. We continue to assume that there is no shock in sector B. Assuming a law of large numbers, ρ is also the fraction of firms in sector A that change productivity type and, hence, the share of mismatched relationships. Thus, the expected payoff in the next period for firms in sector A is:

$$y'_A = (1-\rho)\left[f_A\left(z^H, z^H\right) + f_A\left(z^L, z^L\right)\right] + \rho\left[f_A\left(z^L, z^H\right) + f_A\left(z^H, z^L\right)\right], \tag{8}$$

and for firms in sector B:

$$y'_B = (1-\rho)\left[f_B\left(z^H, z^H\right) + f_B\left(z^L, z^L\right)\right] + \rho\left[f_B\left(z^L, z^H\right) + f_B\left(z^H, z^L\right)\right]. \tag{9}$$

We can rewrite equations (8) and (9) as $y'_A = y_A - \rho\Delta y_A$, and $y'_B = y_B - \rho\Delta y_B$, where $\Delta y_i = f_i\left(z^H, z^H\right) + f_i\left(z^L, z^L\right) - f_i\left(z^H, z^L\right) - f_i\left(z^L, z^H\right)$, with $i \in \{A, B\}$, and Δy_i represents the difference of total output between positive assortative matching and mismatch. Thus, $\Delta y_i > 0$ if and only if the payoff function is supermodular. In other words, the total payoff in both sectors is strictly decreasing with ρ if the payoff function is supermodular.

3.3 Takeaways

The simple model establishes four results. First, under the assumptions of monotonicity and supermodularity in technology, positive assortative matching is the stable and efficient equilibrium, corresponding to Facts 1 and 4. Second, mismatching is an unstable and inefficient equilibrium, corresponding to Fact 4. Third, idiosyncratic productivity shocks transform positive assortative matching to mismatching, predicting separation of the relationship, which relates to Fact 5. Fourth, an increase in the variance of idiosyncratic productivity shocks in one sector generates a fall in the output of both sectors, consistent with Fact 6.[11]

[11] Here and in our quantitative model below, we will not include different industry sectors in the economy. Otherwise, our models would become too complex to handle computationally. Thus, we cannot tackle Fact 3 directly. However, as we pointed out before, Fact 3 tells us that synergies are the strongest among firm pairs

While our simple model parsimoniously accounts for our empirical facts, it is unsuitable for quantitative analysis. For instance, sorting is far from perfect in the data, suggesting search frictions exist. Also, separation of a relationship in response to idiosyncratic shocks is staggered rather than instantaneous, implying another form of friction missing in the simple model. Lastly, as shown by Shimer and Smith (2000) and Eeckhout and Kircher (2010), the condition for positive assortative matching becomes more stringent once we have search frictions. The following section will address these issues with a fully-fledged general equilibrium model.

4 A general equilibrium model

In this section, we rebuild our simple model by adding households and firms that endogenously create and terminate relationships using directed search.

4.1 Households and firms

There is a representative household of unitary size with a continuum of members and utility function:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log\left(C_t\right) - N_t\right],$$

where $\mathbb{E}_0$ is the conditional expectation operator at time $t = 0$, C_t is consumption of final goods, N_t is labor input, and $\beta \in (0, 1)$ is the discount factor. For future reference, the household's stochastic discount factor is $\Lambda_{t+1} = \beta C_t / C_{t+1}$.

The household maximizes utility subject to the budget constraint $C_t = W_t N_t + \Pi_t$, where W_t is the wage rate set up in a competitive market, N_t is the total hours, and Π_t is the profit gained by the household from owning the firms.

There is a unitary measure of intermediate- and final-goods producers, indexed by $l_I \in [0, 1]$ and $l_F \in [0, 1]$, respectively. An intermediate goods producer must form a relationship with a final goods producer so that a final good can be manufactured. Such a relationship is indexed by (l_I, l_F). A firm not part of a relationship stays idle. We call it a "single firm."

At the start of each period t, firms experience aggregate and idiosyncratic productivity shocks and an exogenous separation shock with probability δ. Also, firms in a relationship might decide

with high Domar weights, thus fully motivating our investigation.

to separate from the current partner and become single. The firms produce if the relationship is not terminated (either exogenously or endogenously). Otherwise, single firms search to form new relationships with partners from the opposite sector. At the end of t, each relationship sells the produced goods to the households in a competitive market. Figure 4 summarizes this timeline.

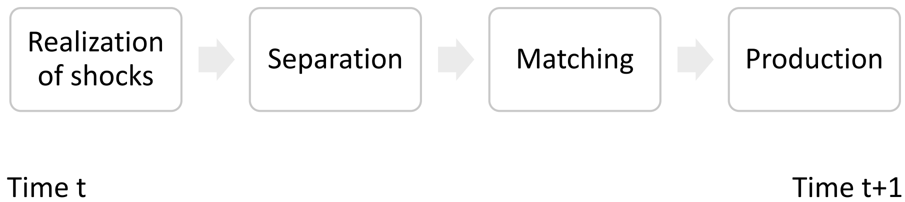


Figure 4: Timeline of firm events

The final output in the relationship (l_I, l_F) is $y_t(l_I, l_F) = e^{x_t + f\left(z_{I,t}(l_I), z_{F,t}(l_F)\right)} h_t(l_I, l_F)^{\alpha}$, where $y_t(l_I, l_F)$ is the final-goods output and $h_t(l_I, l_F)$ is labor input. We assume decreasing returns to scale (i.e., $0 < \alpha < 1$) to prevent the exclusive allocation of labor to the most productive firms. The variables $z_{I,t}(l_I)$ and $z_{F,t}(l_F)$ are the log idiosyncratic productivity (defined below) for the intermediate goods producer and the final goods producer, respectively. The log aggregate productivity x_t follows $x_t = \rho_x x_{t-1} + \sigma_x \epsilon_{x,t}$, where $0 < \rho_x < 1$, and $\epsilon_{I,t} \sim i.i.d.\ \mathcal{N}(0, 1)$.

The production function $f(z_{I,t}(l_I), z_{F,t}(l_F))$ determines the efficiency of a relationship in producing final goods. To encompass different degrees of technological complementarity, we consider a generalized technology function:

$$f(z_{I,t}(l_I), z_{F,t}(l_F)) = (1-\gamma)\left[z_{I,t}(l_I) + z_{F,t}(l_F)\right]/2 + \gamma \min\left[z_{I,t}(l_I), z_{F,t}(l_F)\right], \tag{10}$$

where γ encapsulates the degree of technological complementarity.[12] Equation (10) shows that the log productivity of the relationship is a weighted average of the distinct idiosyncratic productivities in each sector, $z_{I,t}(l_I)$ and $z_{F,t}(l_F)$. The weight assigned to the firm with a lower productivity increases with γ. When $\gamma = 0$, the log productivity of the relationship becomes the unweighted mean of the productivity of the two firms. In this case, the TFP of the relationship is $e^{x_t + f\left(z_{I,t}(l_I), z_{F,t}(l_F)\right)} = e^{x_t}\left(e^{z_{I,t}}\right)^{1/2}\left(e^{z_{F,t}}\right)^{1/2}$, which is the Cobb-Douglas function

[12] Another way to encompass different degrees of technological complementarity is to use the CES production function in Jones (2011): $\left[z_{I,t}(l_I)^{\gamma}/2 + z_{F,t}(l_F)^{\gamma}/2\right]^{\frac{1}{\gamma}}$, where a low γ indicates strong technological complementarity. This production function converges to a Leontief technology when $\gamma \to -\infty$. The CES function requires all inputs to be positive, while our generalized production function allows for a negative log productivity $z_{i,t}$.

of the idiosyncratic productivities of the two firms scaled by the aggregate productivity. When $\gamma > 0$, the log productivity function becomes supermodular by assigning a larger weight to the firm with the lowest productivity.[13] Equation (10) is equivalent to $f\left(z_{I,t}\left(l_I\right), z_{F,t}\left(l_F\right)\right) = z_{I,t}\left(l_I\right) + z_{F,t}\left(l_F\right) - \gamma|z_{I,t}\left(l_I\right) - z_{F,t}\left(l_F\right)|/2$, and similar to the regression equation (3) in Fact 2.

To study the interplay between the variance of the idiosyncratic shock and inter-firm sorting, we let the idiosyncratic productivities follow an AR(1) process with time-varying volatility, $z_{i,t}\left(l_i\right) = \rho_z z_{i,t-1}\left(l_i\right) + \sigma_{z,t}\epsilon_{i,t}\left(l_i\right)$, for $i \in \{I, F\}$, where $\epsilon_{i,t} \sim i.i.d.\ N\left(0,1\right)$, and $\sigma_{z,t}$ is the standard deviation of the idiosyncratic productivity shocks, which follows a Markov chain. See Fernandez-Villaverde and Guerron-Quintana (2020) for an empirical motivation.

Each relationship $\left(l_I, l_F\right)$ chooses the labor input to maximize profits $\pi_t\left(l_I, l_F\right) = y_t\left(l_I, l_F\right) - h_t\left(l_I, l_F\right) W_t$. Profit maximization yields:

$$y_t\left(l_I, l_F\right) = \left\{e^{x_t + f\left(z_{I,t}\left(l_I\right), z_{F,t}\left(l_F\right)\right)}\right\}^{\frac{1}{1-\alpha}} \left(\frac{W_t}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}. \tag{11}$$

Since output is identical across relationships with the same idiosyncratic productivities, we re-write equation (11) as $y_t\left(z_I, z_F\right) = \left\{e^{x_t + f\left(z_I, z_F\right)}\right\}^{\frac{1}{1-\alpha}} \left(\frac{W_t}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}$, and express the profit of a relationship as $\pi_t\left(z_I, z_F\right) = \left(1-\alpha\right) y_t\left(z_I, z_F\right)$. Thus for $\gamma > 0$, the production function and the profit function are log-supermodular, a condition that will play a key role in the next section.[14]

4.2 Directed search and relationship formation

To form a relationship, firms must search for a firm in the opposite sector. We assume directed search: firms in each sector choose the submarket with firms of the productivity type with which they want to match. The matching process is organized in a continuum of submarkets of productivity types, indexed by the idiosyncratic-productivity type of each sector, $\left(z_I, z_F\right) \in \mathbb{R}^2$. Specifically, single firms from sector I with a productivity of $\overline{z}_I$ can choose to enter any submarket $\left(\overline{z}_I, z_F\right)$, where $z_F \in \mathbb{R}$. Productivity is observable, i.e., firms in sector I with idiosyncratic productivity $\overline{z}_I$ cannot go to an alternative submarket $\left(z_I, z_F\right)$ with $z_I \neq \overline{z}_I$. Analogously, single

[13]For $\gamma = 1$, equation (10) becomes a Leontief technology, the special case that nests Kremer (1993). When $\gamma < 0$, the log productivity function is submodular.

[14]Notice how log-supermodularity holds here. For $\overline{z}_I > \underline{z}_I, \overline{z}_F > \underline{z}_F$, we have $log\left[y_t\left(\overline{z}_I, \overline{z}_F\right)\right] + log\left[y_t\left(\underline{z}_I, \underline{z}_F\right)\right] > log\left[y_t\left(\underline{z}_I, \overline{z}_F\right)\right] + log\left[y_t\left(\overline{z}_I, \underline{z}_F\right)\right]$. As we will show later, log-supermodularity is a sufficient condition for positive assortative matching under our benchmark calibration.

firms in sector F with productivity $\overline{z}_F$ can choose to enter any submarket $(z_I, \overline{z}_F)$ with $z_I \in \mathbb{R}$, but cannot enter a submarket (z_I, z_F) with $z_F \neq \overline{z}_F$.

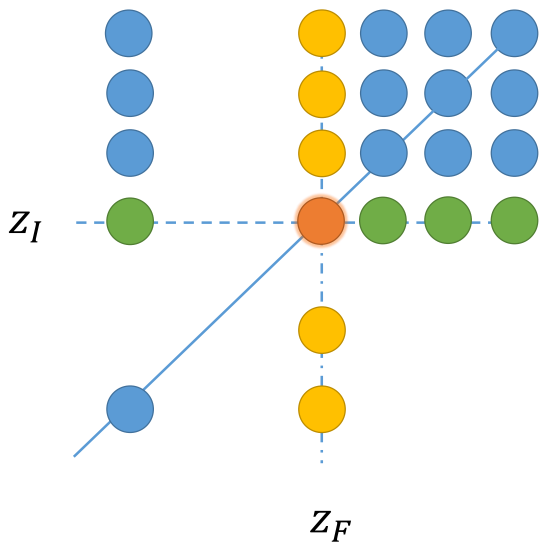


Figure 5: Organization of the submarkets

Figure 5 shows the organization of the submarkets. Each dot represents a submarket. A single firm F with productivity z_F can enter any submarket on the dash-dotted vertical line (the yellow and orange dots). Analogously, a single firm I with productivity z_I can enter any submarket on the dashed horizontal line (the green and orange dots). Under sectoral symmetry (i.e., the value functions and the distribution of firms are symmetric between the two sectors), positive assortative matching arises when firms only enter the submarket with firms of the same productivity type in the opposite sectors on the 45-degree line (the orange dot), and the alternative submarkets off the 45-degree line remain empty. We will establish later the sufficient condition for positive assortative matching to be a stable equilibrium, i.e., no firm prefers to form a relationship with a firm in a submarket off the 45-degree line.

To formalize directed search, we characterize the choice of a single firm I to enter a specific submarket as a function of its idiosyncratic productivity, $z_F^* = \mathsf{z}_{F,t}^*(z_I)$, where z_F^* is the productivity of the partner that firm I is targeting by entering the submarket (z_I, z_F^*). Analogously, a single firm F's optimal choice of entering a submarket is characterized by $z_I^* = \mathsf{z}_{I,t}^*(z_F)$, where z_I^* is the productivity of the partner that firm F is targeting by entering the submarket (z_I^*, z_F).

Under sectoral symmetry, positive assortative matching is a set of decision rules $\mathsf{z}_{I,t}^*(z)$ and

$\mathsf{z}^*_{F,t}(z)$ that satisfy $\mathsf{z}^*_{I,t}(z) = \mathsf{z}^*_{F,t}(z) = z$, for any z. The measure of single firms in sector I with productivity z_I is $\widetilde{n}_I(z_I)$ and the measure of single firms in sector F with productivity z_F is $\widetilde{n}_F(z_F)$. The measure of single firms from sector I with productivity z_I that choose to enter submarket (z_I, z_F) is $\widetilde{n}_I(z_I, z_F)$. Analogously, $\widetilde{n}_F(z_I, z_F)$ is the measure of single firms from sector F with productivity z_F that choose to enter submarket (z_I, z_F). Since single firms must choose one submarket to enter, the number of single firms in each submarket is equal to:

$$\widetilde{n}_{I,t}(z_I) = \int_{-\infty}^{\infty} \widetilde{n}_{I,t}(z_I, z_F)\, dz_F, \text{and } \widetilde{n}_F(z_F) = \int_{-\infty}^{\infty} \widetilde{n}_F(z_I, z_F)\, dz_I.$$

Under positive assortative matching, a firm enters the submarket with firms in the opposite sector that have the same productivity type, such that:

$$\widetilde{n}_{I,t}(z_I, z_F) = \begin{cases} \widetilde{n}_{I,t}(z_I) & \text{if } z_I = z_F \\ 0 & \text{if } z_I \neq z_F \end{cases} \text{ and } \widetilde{n}_{F,t}(z_I, z_F) = \begin{cases} \widetilde{n}_{F,t}(z_F) & \text{if } z_I = z_F \\ 0 & \text{if } z_I \neq z_F. \end{cases}$$

Our model differs from conventional models of frictional assignment with two-sided heterogeneity (Chade et al., 2017). For example, Shimer and Smith (2000) have symmetric buyers and sellers but assume random search. Eeckhout and Kircher (2010) consider directed search but with asymmetric buyers and sellers: the seller posts a price, and the buyer decides in which submarket to shop. Our model aims to represent the process of relationship formation among firms given that their location and productivity are public information (suggesting directed search) and where firms are not inherently different in terms of price setting. Nonetheless, our model still delivers positive assortative matching under the sufficient conditions we will discuss later, and the equilibrium allocations are similar to those achieved by the studies above.

The formation of relationships in each submarket depends on the measure of single firms from each sector searching in the submarket. A constant-returns-to-scale matching function determines new relationship formation, $M(\widetilde{n}_I(z_I, z_F), \widetilde{n}_F(z_I, z_F))$, where $\widetilde{n}_I(z_I, z_F)$ and $\widetilde{n}_F(z_I, z_F)$ are the measures of single firms in the two sectors.

Conditional on a submarket (z_I, z_F) having positive measures of visiting firms from both sectors (i.e., $\widetilde{n}_I(z_I, z_F) > 0$ and $\widetilde{n}_F(z_I, z_F) > 0$), the matching probability for firms in sector I

in the submarket (z_I, z_F) is:

$$\mu_I(z_I, z_F) = \frac{M(\widetilde{n}_I(z_I, z_F), \widetilde{n}_F(z_I, z_F))}{\widetilde{n}_I(z_I, z_F)} = M(1, \theta(z_I, z_F)),$$

and, similarly, the matching probability for firms in sector F in the same submarket (z_I, z_F) is

$$\mu_F(z_I, z_F) = \frac{M(\widetilde{n}_I(z_I, z_F), \widetilde{n}_F(z_I, z_F))}{\widetilde{n}_F(z_I, z_F)} = M(1/\theta(z_I, z_F), 1),$$

where $\theta(z_I, z_F) = \widetilde{n}_F(z_I, z_F)/\widetilde{n}_I(z_I, z_F)$ is the tightness ratio in submarket (z_I, z_F).

4.3 Firm value functions and Nash Bargaining

Next, we define the firms' Bellman equations. The value $J_{I,t}(z_I, z_F)$ of the intermediate-goods producer that starts period t in a relationship is:

$$J_{I,t}(z_I, z_F) = [\delta + \phi s_t(z_I, z_F)]\,\widetilde{J}_{I,t}(z_I) + [1 - \delta - \phi s_t(z_I, z_F)]\,\widehat{J}_{I,t}(z_I, z_F),$$

where $\widetilde{J}_{I,t}(z_I)$ is the value of a single intermediate-goods producer and $\widehat{J}_{I,t}(z_I, z_F) = \pi_{I,t}(z_I, z_F) + \mathbb{E}_t\left[\Lambda_{t+1} J_{I,t+1}\left(z_I', z_F'\right)\right]$ is the value of continuing the relationship, equal to the flow profit of $\pi_{I,t}(z_I, z_F)$ whose size is established by Nash bargaining (to be described below), plus the expected discounted continuation value $\mathbb{E}_t \Lambda_{t+1} J_{I,t+1}(z_I', z_F')$. The term $s_t(z_I, z_F)$ (derived below) is an indicator of the endogenous termination of a relationship, equal to one if at least one firm prefers to terminate the relationship. The probability of endogenous separation, ϕ, reflects the observed staggered separation process outlined in Section 2.

The value of the final-goods producer $J_{F,t}(z_I, z_F)$ in a relationship at the start of period t is:

$$J_{F,t}(z_I, z_F) = [\delta + \phi s_t(z_I, z_F)]\,\widetilde{J}_{F,t}(z_F) + [1 - \delta - \phi s_t(z_I, z_F)]\,\widehat{J}_{F,t}(z_I, z_F),$$

where $\widetilde{J}_{F,t}(z_F)$ is the value of a single intermediate-goods producer and $\widehat{J}_{F,t}(z_I, z_F) = \pi_{F,t}(z_I, z_F) + \mathbb{E}_t\left[\Lambda_{t+1} J_{F,t+1}\left(z_I', z_F'\right)\right]$ is the value of continuing the relationship.

The value of a single firm in sector I is:

$$\widetilde{J}_{I,t}(z_I) = \mu_{I,t}(z_I, z_F^*)\left\{\pi_{I,t}(z_I, z_F^*) + \mathbb{E}_t\left[\Lambda_{t+1} J_{F,t+1}\left(z_I', z_F^{*\prime}\right)\right]\right\},$$

where z_F^* is the productivity of the partner chosen by the single firm in sector I, $\mu_{I,t}(z_I, z_F^*)$ is the probability of forming a relationship in the chosen submarket, $\pi_{I,t}(z_I, z_F^*)$, and $\mathbb{E}_t\left[J_{I,t+1}\left(z_I^{'}, z_F^{*'}\right)\right]$ are the profit and the expected value conditional on relationship formation, respectively.

Similarly, the value of a single firm in sector F is:

$$\widetilde{J}_{F,t}(z_F) = \mu_{F,t}(z_I^*, z_F)\left\{\pi_{F,t}(z_I^*, z_F) + \mathbb{E}_t\left[\Lambda_{t+1}J_{F,t+1}\left(z_I^{*'}, z_F^{'}\right)\right]\right\},$$

where z_I^* is the productivity of the partner that the single firm F is targeting.

Finally, we derive the indicator variable for the termination of a relationship. A relationship endogenously terminates if the value of becoming a single firm for any of the partners in the relationship exceeds the value of continuing with the relationship:

$$s_t(z_I, z_F) = \begin{cases} 0 & \text{if } \widehat{J}_{I,t}(z_I, z_F) \geq \widetilde{J}_{I,t}(z_I) \text{ and } \widehat{J}_{F,t}(z_I, z_F) \geq \widetilde{J}_{F,t}(z_F) \\ 1 & \text{if } \widehat{J}_{I,t}(z_I, z_F) < \widetilde{J}_{I,t}(z_I) \text{ or } \widehat{J}_{F,t}(z_I, z_F) < \widetilde{J}_{F,t}(z_F) \end{cases}.$$

The division of profits from the relationship is negotiated after the separation decision and before production. The total surplus of the relationship, $TS_t(z_I, z_F)$, is equal to the sum of the surpluses obtained by each firm in forming a relationship versus remaining a single firm, such that $TS_t(z_I, z_F) = \left[\widehat{J}_{I,t}(z_I, z_F) - \widetilde{J}_{I,t}(z_I)\right] + \left[\widehat{J}_{F,t}(z_I, z_F) - \widetilde{J}_{F,t}(z_F)\right]$.

This surplus is split according to Nash bargaining, and the bargained profits, $\pi_{i,t}$, satisfy $\widehat{J}_{I,t}(z_I, z_F) - \widetilde{J}_{I,t}(z_I) = \tau TS_t$, and $\widehat{J}_{F,t}(z_I, z_F) - \widetilde{J}_{F,t}(z_F) = (1-\tau)TS_t$, where τ is the bargaining share of the intermediate-goods producer. Thus, a firm terminates a relationship if the total surplus becomes negative, and the indicator variable for endogenous termination becomes:

$$s_t(z_I, z_F) = \begin{cases} 0 & \text{if } TS(z_I, z_F) \geq 0 \\ 1 & \text{if } TS(z_I, z_F) < 0 \end{cases}.$$

4.4 Flow motion of firms

The measure of relationships after the realization of shocks and before separation and matching is:

$$m_t(z_I, z_F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n_{t-1}(\widehat{z}_I, \widehat{z}_F) \times g_{I,t}(z_I \mid \widehat{z}_I)\, g_{F,t}(z_F \mid \widehat{z}_F)\, d\widehat{z}_I d\widehat{z}_F,$$

where $\widehat{z}_I$ and $\widehat{z}_F$ are the productivities in $t-1$, and $n_{t-1}(\widehat{z}_I, \widehat{z}_F)$ is the measure of relationships from $t-1$ with productivities $(\widehat{z}_I, \widehat{z}_F)$. The conditional density $g_{j,t}(z_j \mid \widehat{z}_j)$ is the transition probability of idiosyncratic productivity in sector j. The transition probability functions change over time due to time-varying volatility in the idiosyncratic shocks.

Define $suv_t = [1 - \delta - \phi s_t(z_I, z_F)]$ as the fraction of relationships that survive separation and $M_t(z_I, z_F)$ as the measure of new relationship formation in the submarket (z_I, z_F). Then, the measure of relationships after separation and matching is $n_t(z_I, z_F) = suv_t m_t(z_I, z_F) + M_t(z_I, z_F)$ while the measure of single firms in sector I after the realization of shocks and before separation and matching is:

$$\widetilde{m}_{I,t}(z_I) = \int_{-\infty}^{\infty} \widetilde{n}_{I,t-1}(\widehat{z}_I) \times g_{I,t}(z_I \mid \widehat{z}_I)\, d\widehat{z}_I,$$

where $\widetilde{n}_{I,t-1}(\widehat{z}_I)$ is the measure of single firms in the previous period $t-1$ with productivity $\widehat{z}_I$.

After separation and matching, the measure of single firms in sector I is:

$$\widetilde{n}_{I,t}(z_I) = [1 - \mu_I(z_I, z_F^*(z_I))]\, \widetilde{m}_{I,t}(z_I) + \int_{-\infty}^{\infty} [\delta + \phi s_t(z_I, z_F)]\, m_t(z_I, z_F)\, dz_F, \tag{12}$$

where $\mu_I(z_I, z_F^*(z_I))$ is the probability of forming a relationship in the optimal submarket $(z_I, z_F^*(z_I))$ for the z_I-type single firms. The integrated term on the right-hand side of equation (12) is the measure of z_I-type single firms newly separated from relationships.

Similarly, the measure of single firms in sector F is:

$$\widetilde{m}_{F,t}(z_F) = \int_{-\infty}^{\infty} \widetilde{n}_{F,t-1}(\widehat{z}_F) \times g_{F,t}(z_F \mid \widehat{z}_F)\, d\widehat{z}_F$$

and

$$\widetilde{n}_{F,t}(z_F) = [1 - \mu_F(z_I^*(z_F), z_F)]\, \widetilde{m}_{F,t}(z_F) + \int_{-\infty}^{\infty} [\delta + \phi s_t(z_I, z_F)]\, m_t(z_I, z_F)\, dz_I,$$

where $\widetilde{m}_{I,t}(z_I)$ and $\widetilde{n}_{I,t}(z_I)$ are the measure of single firms in sector I before and after separation and matching, respectively.

4.5 Positive assortative matching

Next, we establish sufficient conditions for positive assortative matching to be the stable equilibrium (i.e., no firms prefer to meet in a submarket off the diagonal in Figure 5). We focus our analysis on the case of sectoral symmetry in which the two sectors have the same distribution of single firms, that is, $\widetilde{n}_{I,t}(z) = \widetilde{n}_{F,t}(z)$ for any t and z.[15]

Assortative matching without search frictions. Becker (1973) shows that in markets without search frictions, a supermodular surplus function $TS_t(z_I, z_F)$ is sufficient for positive assortative matching to be the stable equilibrium. To see this, suppose the economy begins from an equilibrium with assortative matching. A pair of firms with idiosyncratic productivity $z_I = z$, $z_F = z'$ ($z \neq z'$) prefer to depart from positive assortative matching and establish a new relationship together if the new total surplus is larger than the total surplus in the ongoing relationship, which occurs if:

$$\tau TS(z, z') > \tau TS(z, z)\,, \text{ and } (1-\tau)\, TS(z, z') > (1-\tau)\, TS(z', z') \tag{13}$$

hold simultaneously. Equation (13) implies $2TS(z, z') > TS(z, z) + TS(z', z')$, which cannot be satisfied when TS is supermodular. In other words, supermodularity ensures that no pair of firms prefer to deviate from the equilibrium with positive assortative matching. Although we assume for simplicity Nash bargaining between the firms to split the output, the result of Becker (1973) applies to any bargaining rule.

Assortative matching with search frictions. The main intuition of Becker (1973) continues to hold in our model. However, search frictions mean that a firm may prefer to enter a submarket with lower-productivity firms but with a higher probability of forming a mutually beneficial relationship. Thus, for positive assortative matching to arise as a stable equilibrium, we need more stringent conditions on the supermodularity of the total surplus.

Formally, an intermediate-goods producer with idiosyncratic productivity $z_I = z$ would

[15] The existence of a symmetric equilibrium depends on three conditions: (1) same transition probability functions (i.e., $g_{I,t}(z \mid z') = g_{F,t}(z \mid z')$); (2) symmetric matching functions (i.e., $M(n, n') = M(n', n)$); (3) symmetric decision rules (i.e., $z^*_{I,t}(z) = z^*_{F,t}(z)$, and $s_t(z, z') = s_t(z', z)$). We have already assumed condition (1) and will assume condition (2) in our calibration. Condition (3) holds under conditions (1) and (2), and the joint surplus is split according to Nash bargaining.

invite θ measure of final-goods producers with idiosyncratic productivity $z_F = z'$ to meet in the submarket (z, z') (θ is also the tightness ratio for that submarket) if:

$$\tau \mu_I(\theta) TS(z, z') > \tau \mu_I(\theta(z, z)) TS(z, z), \tag{14}$$

where the sides of equation (14) are the expected surplus of the intermediate-goods producer in submarkets (z, z') and (z, z), respectively. The final-goods producers with productivity z' would accept the invitation by going to the new submarket (z, z') if:

$$(1 - \tau) \mu_F(\theta) TS(z, z') > (1 - \tau) \mu_F(\theta(z', z')) TS(z', z'), \tag{15}$$

where the sides of equation (15) are the expected surplus of the final-goods producer in submarkets (z, z') and (z', z'), respectively.

To give both firms an incentive to deviate from positive assortative matching, equations (14) and (15) must hold simultaneously, which implies that:

$$\frac{\mu_I(\theta) \mu_F(\theta)}{\mu_I(\theta(x, x)) \mu_F(\theta(y, y))} TS^2(z, z') > TS(z, z) TS(z', z'). \tag{16}$$

Under sectoral symmetry, we have that $\theta(z, z) = \theta(z', z') = 1$, and equation (16) becomes:

$$\frac{\mu_I(\theta) \mu_F(\theta)}{\mu_I(1) \mu_F(1)} TS^2(z, z') > TS(z, z) TS(z', z'). \tag{17}$$

Equation (17) cannot be satisfied, which implies that equations (14) and (15) cannot hold simultaneously, if $\log[TS(z, z)] + \log[TS(z', z')] > \log(\mu_0) + 2\log[TS(z, z')]$, where we define:

$$\mu_0 = \max_{\theta} \frac{\mu_I(\theta) \mu_F(\theta)}{\mu_I(1) \mu_F(1)}, \text{ s.t. } \mu_I(\theta) < 1, \ \mu_F(\theta) < 1.$$

Note that $\log(\mu_0)$ can be zero or positive depending on the matching function. Under our benchmark calibration below with $\mu_I(\theta) = \psi\theta^{1/2}$ and $\mu_F(\theta) = \psi\theta^{-1/2}$, we have that $\mu_I(\theta) \mu_F(\theta) = \psi^2$ for any θ, and hence $\log(\mu_0) = 0$. In this case, log-supermodularity is a sufficient condition for positive assortative matching. Log-supermodularity is stronger than supermodularity: the former implies the latter, but the opposite does not hold. This is consistent with our previous argument that search frictions make positive assortative matching more difficult to achieve. Interestingly, log-supermodularity is also identified as a sufficient condition for positive assortative matching by Shimer and Smith (2000) and Eeckhout and Kircher (2010), who study alternative models of sorting whose market structures are very different from ours.[16]

4.6 Market-clearing conditions and equilibrium

Given a set of relationships $\Omega_t \subseteq [0,1] \times [0,1]$, aggregate output, $Y_t = \int_{\Omega_t} y_t\left(l_I, l_F\right) d\left(l_I, l_F\right)$, is the sum of final goods output produced by all the relationships in the economy. Notice that $\Omega_t \neq [0,1] \times [0,1]$ due to the presence of single firms that remain idle. Consumption equals output, $C_t = Y_t$, and the labor market clears when $N_t = \int_{\Omega_t} h_t\left(l_I, l_F\right) d\left(l_I, l_F\right)$. The definition of equilibrium is standard, and we omit it in the interest of space.

5 Calibration

We calibrate the model by matching the steady state of the model to US data at a quarterly frequency. Table 7 summarizes the calibration.

Conventional parameters. The discount factor, β, equals 0.987 (equivalent to 0.95 at a yearly frequency) to replicate the average annual interest rate of 5% over the sample period. The labor share, α, is set to 0.66 to match the labor share of income.

Following Khan and Thomas (2008), we set the persistence of the AR(1) processes for aggregate and idiosyncratic productivity, x_t and $z_{i,t}\left(l_i\right)$, to 0.95. The standard deviation of the aggregate productivity shock is 0.006, which implies that the quarterly standard deviation of aggregate productivity is 0.02, consistent with the estimates in Zanetti (2008).

We assume that the volatility of the idiosyncratic productivity shock, $\sigma_{z,t}$, follows a two-state Markov chain, $\sigma_{z,t} \in \left\{\sigma_z^L, \sigma_z^H,\right\}$, where $\Pr\left(\sigma_{z,t+1} = \sigma_z^k \mid \sigma_{z,t} = \sigma_z^j\right) = \pi^{kj}$. Following Bloom et al. (2018), we set $\sigma_z^L = 0.039$. Since the variance of plant-level TFP shocks increased by 76% during the Great Recession of 2008 (an increase of 34% in the standard deviation), we set $\sigma_z^H = 0.052$. Also, after Bloom et al. (2018), we calibrate the transition probability from low to

[16] In contrast, Lentz (2010) shows that supermodularity is sufficient for positive assortative matching when search intensity is endogenous.

Table 7: Calibration

Description	Parameter	Value
Preference and technology		
Discount factor	β	0.987
Labor share	α	0.66
Degree of supermodularity	γ	0.86
Matching, separation, and bargaining		
Matching efficiency	ψ	0.38
Matching elasticity	ι	0.5
Exogenous separation rate	δ	6.8%
Staggerness of endogenous separation	ϕ	0.25
Bargaining share of intermediate-goods producers	τ	0.5
Shock process		
Persistence of aggregate productivity (prod.) shock	ρ_x	0.95
Standard deviation (std.) of aggregate prod. shock	σ_x	0.006
Persistence of idiosyncratic prod. shock	ρ_z	0.95
Std of idiosyncratic prod. shock (low uncertainty)	σ_z^L	0.039
Std of idiosyncratic prod. shock (high uncertainty)	σ_z^H	0.052
Transition prob. from low to high uncertainty	$\pi_{L,H}$	0.05
Transition prob. of remaining in high uncertainty	$\pi_{H,H}$	0.92

high uncertainty equal to 0.05 and the probability of remaining in high uncertainty equal to 0.92. Conditional on receiving an idiosyncratic shock that makes a relationship mismatched, Section 2 documents that it takes one year on average for firms to separate, which implies $\phi = 0.75$. We let the firms in a relationship split the surplus evenly by setting $\tau = 0.5$.

We assume a standard Cobb-Douglas matching function $M(\widetilde{n}_I, \widetilde{n}_F) = \psi (\widetilde{n}_I)^{1-\iota} (\widetilde{n}_F)^{\iota}$, where ψ is the matching efficiency. We set $\iota = 0.5$ to have sectoral symmetry, implying that $\mu_I = \psi\theta^{\iota} = \psi\theta^{0.5}$ and $\mu_F = \psi\theta^{\iota-1} = \psi\theta^{-0.5}$. Hence $\mu_I\mu_F = \psi^2$ for any tightness ratio θ. Under this calibration, log-supermodularity (achieved for $\gamma > 0$) is a sufficient condition for the equilibrium with positive assortative matching.

Model-specific parameters. Three parameters are new in our analysis: the degree of technological complementarity in the production function, γ, the efficiency in the matching function, ψ, and the rate of exogenous separation of relationships, δ. We calibrate them to replicate three moments in the data: the average duration of relationships, the idleness rate, and the correlation between matches' expected durations and the degree of mismatch measured as the distance in the decile of productivity between partners.

We target the average duration of a relationship to 12 quarters, consistent with the findings from the Compustat Segment and FactSet data documented in Section 2. We target the fraction of single firms to the observed 12% average idleness rate in the US before the Great Recession (Michaillat and Saez, 2015, and Ghassibe and Zanetti, 2022). We target the correlation between expected duration and the degree of mismatch to -0.07. The correlation in the data between expected durations and the decile gaps between partners' economic fundamentals is about -0.035. However, the correlation is underestimated due to measurement errors. For example, Bils et al. (2021) establish that measurement errors are as important as productivity shocks in the measured productivity dispersion, implying that the correlation between expected duration and the degree of mismatch is, on average, 1/2 underestimated.[17] Moreover, firms are likely to pick their partners based on weakly correlated (i.e., almost uncorrelated) factors with measured productivity. For example, the producer of a mobile device might choose to work with a software company with an operating system with a high potential to build a rich ecosystem for app developers but low current measured productivity. Targeting the correlation between expected duration and the degree of mismatch to -0.035 would understate the sensitivity of endogenous separation to mismatch while overstating the extent of misallocation and output loss. Thus, we adjust the original estimate from the Compustat Segment and FactSet data to -0.07.

To match these targeted moments, we set $\gamma = 0.86$. In particular, using equation (10), the weight of the high-productivity firm in the log-productivity of a relationship is 7%, while the weight of the low-productivity firm is 93%. We set $\psi = 0.38$, implying that forming a relationship takes 2.6 quarters. We calibrate $\delta = 6.8\%$. Given that 1.4% of relationships separate endogenously in each period, the gross separation rate is 8.2%, which implies an average duration of a relationship of 12 quarters (see Hamano and Zanetti, 2017, for a discussion on the empirical estimates of the plant separation rate). Since the number of moments equals the number of unknown parameters, we can match our target exactly.

To illustrate how each parameter is identified, Table 8 and Figure 6 display the comparative statics on the effect of each parameter on the endogenous variables. In each panel of Figure 6, we fix two parameters and let the other parameter move around its calibrated value. The

[17]Suppose $\widetilde{z}_{i,t} = z_{i,t} + \widetilde{\epsilon}_{i,t}$, where $\widetilde{z}_{i,t}$ is measured productivity, $\widetilde{\epsilon}_{i,t}$ is measurement error. By assuming that $\sigma\left(\widetilde{\epsilon}_{i,t}\right) = \sigma\left(z_{i,t}\right)$ and $corr\left(z_{i,t}, \widetilde{\epsilon}_{i,t}\right) = 0$, simulation yields that $corr\left(\widetilde{\Delta}_{I,F,t}, dur_{I,F,t}\right) \geq 2corr\left(\Delta_{I,F,t}, dur_{I,F,t}\right)$, where $\widetilde{\Delta}_{I,F,t} = |decile(\widetilde{z}_{I,t}) - decile(\widetilde{z}_{F,t})|$ is the measured mismatch level, $\Delta_{I,F,t} = |decile(z_{I,t}) - decile(z_{F,t})|$ is the actual mismatch level, and $dur_{I,F,t}$ is the expected duration.

Table 8: The effect of γ, ψ, and δ on selected moments

	γ	ψ	δ
Duration of trading relationship	$\downarrow$	$\downarrow$	$\downarrow$
Correlation between duration and mismatch	$\uparrow$	$\uparrow$	$\downarrow$
Idleness rate	$\uparrow$	$\downarrow$	$\uparrow$

Note: The symbols $\downarrow$, $\uparrow$, and $\sim$ indicate a decrease, increase, and unchanged effect of γ, ψ, and δ on the moment.

x-axis is the value of the moving parameter. The y-axis is the ratio of the moment implied by the model with the parameter value on the x-axis to the targeted value of the moment, which is equal to one when moment matching is successful (thus, the crossing of the three curves in each panel corresponds to our benchmark calibration).

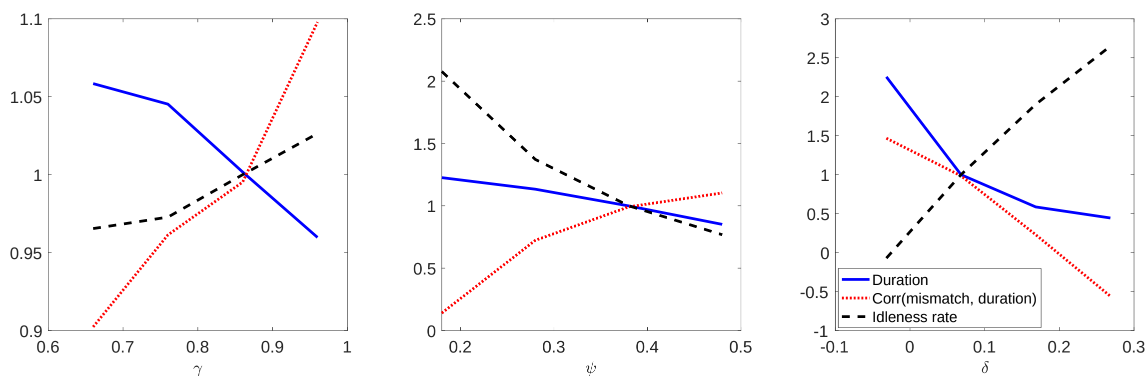


Figure 6: The effect of moving γ, ψ, and δ on selected moments

We start with γ (the upper row of Table 8 and the left panel of Figure 6). A higher γ raises the negative impact of mismatching on expected duration by making firms less tolerant of mismatching. It also induces more endogenous separations, increasing the idleness rate and decreasing the relationships' average duration. Next, we discuss the role of ψ (the middle row of Table 8 and the middle panel of Figure 6). A higher ψ makes the reallocation of relationships easier, decreasing their duration and strengthening the negative correlation between mismatching and expected duration. It also improves the matching speed, leading to a lower idleness rate. Lastly, the bottom row of Table 8 and the right panel of Figure 6 show that a higher δ increases the idleness rate, decreases the average duration of a relationship, and weakens the negative impact of mismatching on expected duration.

6 Quantitative analysis I: Steady state

This section studies the steady state of the model by fixing the aggregate productivity (x) at the normalized value of one. However, we still have idiosyncratic productivity shocks and compute the stationary distribution for relationships, single firms, and aggregate variables by simulating the model for 100,000 periods (we checked that those were more than enough for convergence).

We also consider three alternative calibrations that abstract from search frictions and staggered separation. In calibration A, we assume that the parameter for matching efficiency ψ is one and the rate of staggered separation ϕ equals $1-\delta$ (since the exogenous separation rate is δ, the gross probability of separation is $\delta+\phi$ for firms that want to separate). This frictionless calibration entails perfect sorting and no single firms. In calibration B, we set $\phi = 1-\delta$ (search frictions only), and in calibration C, we set $\psi = 1$ (staggered separation only). By comparing our benchmark calibration with the alternative calibrations, we measure the role of search frictions and staggered separation for (i) the separation policy of relationships, (ii) the stationary distribution of relationships, and (iii) the level of aggregate output.

6.1 The separation policy

We first investigate the separation of relationships in the steady state by plotting, in grey, the values of the productivity of firms in sectors F (x-axes) and I (y-axes) where an existing relationship continues. For other values, the relationship is dissolved.

The top panel in Figure 7 shows the separation policy for our benchmark calibration. The grey area is wide: we have imperfect sorting because firms endogenously prefer to remain in a relationship with a firm of a different productivity type. Search frictions reduce the likelihood of forming a relationship upon separation and thereby lower the expected profits of re-matching. Although we do not have non-convex adjustment costs, the optimizing behavior of firms leads to endogenous separation of relationships that is reminiscent of the Ss policy rules outlined in Scarf (1963) and exploited in general equilibrium by Bloom (2009).

In comparison, in calibration A (top-right panel), firms only stay in the relationship if they achieve perfect sorting: the region where the relationship continues is the 45-degree line. In calibration B (bottom-left panel), the grey-shaded area remains sizable and looks similar to the benchmark case. That is, search frictions explain the bulk of imperfect sorting across firms in

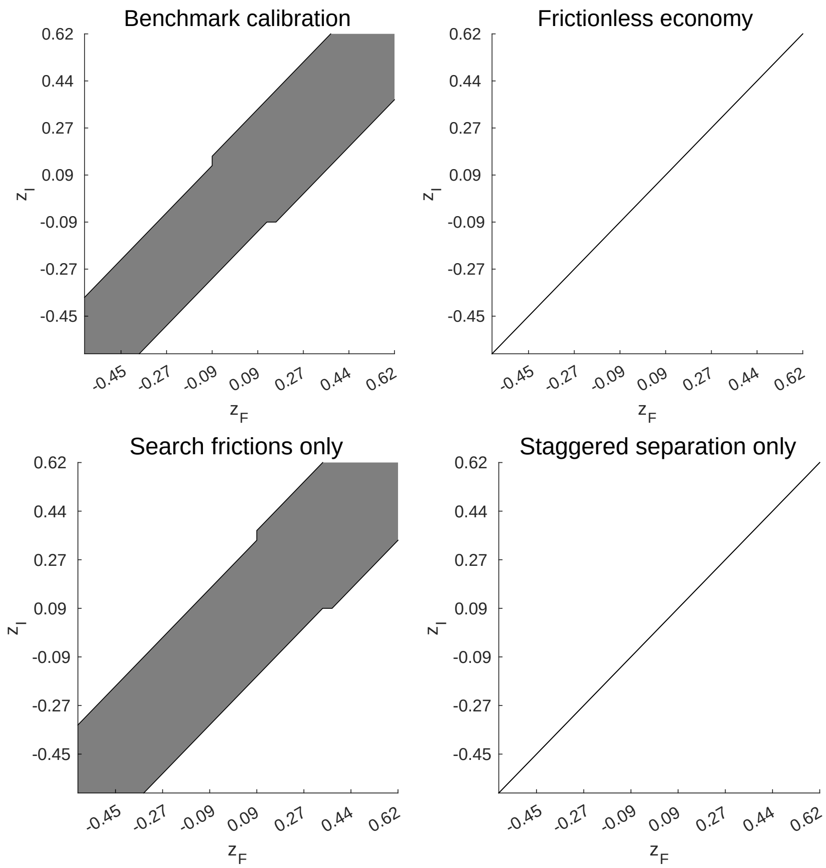


Figure 7: Separation policy: Benchmark and alternative calibrations

a relationship. In calibration C (bottom-right panel), we are back to a continuation region of just the 45-degree line: while staggered separation will hinder the realization of separation by construction, it does not discourage firms' separation decisions without search frictions.

6.2 Stationary distribution of relationships

Figure 8 plots the stationary distribution of relationships across different productivity levels for firms in sectors I and F implied by the separation policies above.

The top-left panel shows the distribution of relationships in the benchmark calibration. Despite perfect sorting being predominant in the steady state, as displayed by the larger density in the distribution of relationships along the 45-degree line, the economy generates a sizable fraction of mismatches, exhibited by the positive density off the 45-degree line. The correlation of productivities between partners is 0.61. The top-right panel shows that calibration A begets perfect sorting: the distribution of relationships retains a positive mass on the 45-degree line

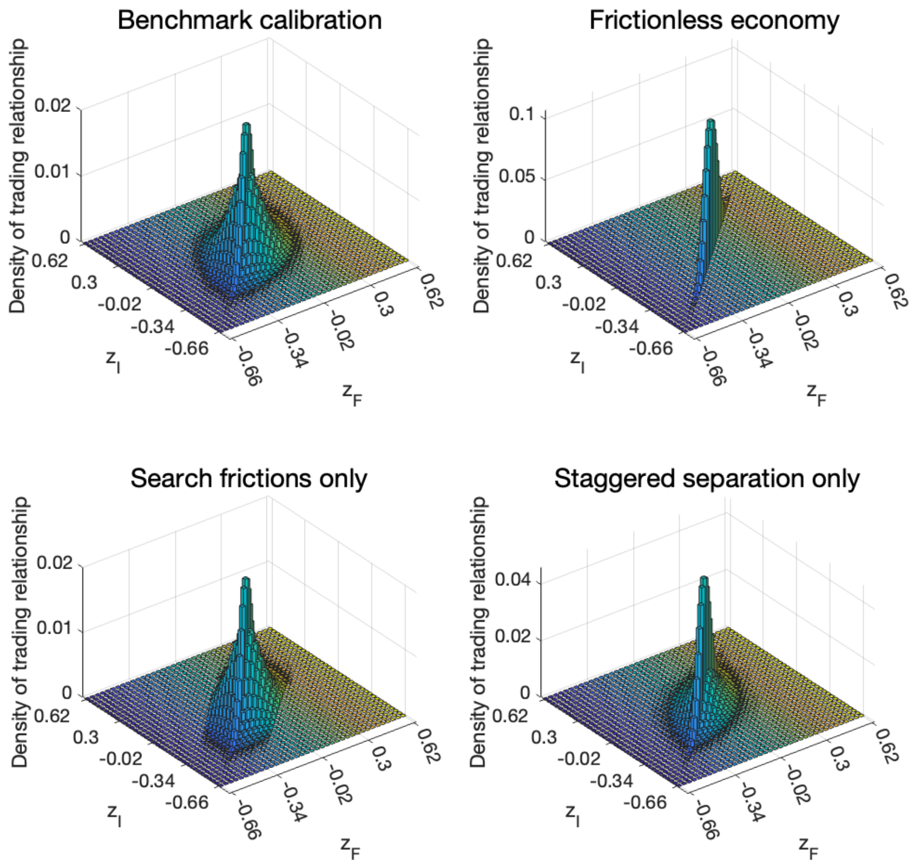


Figure 8: Distribution of relationships

of productivity and zero mass elsewhere. In this case, the degree of sorting equals 1. The bottom-left panel shows calibration B, with a degree of sorting of 0.69, and the bottom-right panel shows calibration C, with a degree of sorting of 0.82. Thus, we learn that the bulk of the mismatch in the steady state comes from search frictions, not staggered separations.

6.3 Comparing model predictions to firm-level data

Next, we re-conduct the empirical analysis in Section 2 with the data simulated by our model and show that our model generates the same regressions as in the data. We simulate 18,500 firms for 160 quarters.[18] Then, we convert the remaining quarterly data to yearly series (the time-frequency of Compustat Customer Segment and FactSet data) with time averaging. Appendix B explains the details of the simulation.

[18]The simulated model has 37 productivity grids. Each grid accommodates 500 firms in the starting period.

Distribution of trading relationships' duration. Figure 9 plots the histogram of match duration (in years) for the model's simulation and the data (the same as Figure 1), respectively. The model generates a cross-sectional distribution of match duration close to the data.

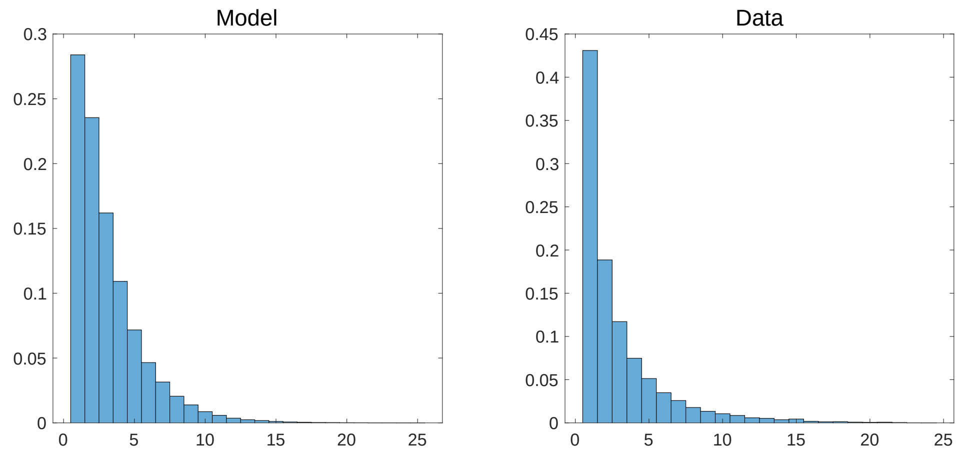


Figure 9: Distribution of trading relationships' duration

Positive assortative matching of relationships. Does our model match the positive assortative matching in the data documented in Section 2? Table 9 reports the results of estimation for equation (1) with our simulated data and the actual data, respectively.

Table 9: Assortative matching for ranking of economic fundamentals, one year before the match

	(1)	(2)	(3)	(4)	(5)
	Model		Data		
		Labor productivity	Sales growth	ROE	Tobin's q
$decile\left(\pi^{cus}_{j,k,t}\right)$	0.66***	0.06***	0.07***	0.03***	0.06***
	(0.003)	(0.01)	(0.01)	(0.01)	(0.01)
Industry FE	No	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Adjusted R^2	0.35	0.07	0.02	0.03	0.08
Observations	63,356	23,829	24,374	28,034	23,131

Note: Standard errors are in parentheses. *** denotes significance at the 1% level.

In Column (1), the dependent variable, $decile\,(z_{F,t})$, is firm F's decile of productivity in the year before the start of the relationship simulated from the model. The independent variable, $decile\,(z_{I,t})$, is firm I's decile of productivity in the year before the start of the relationship.

Columns (2)-(5) show the results estimated with actual data, which are the same as in Table 1. Both the model and the data indicate positive assortative matching in relationship formation. However, our model generates a stronger degree of sorting than the data as the coefficient is estimated higher in Column (1) than in Columns (2)-(5). The high degree of sorting in the year before match formation is driven by construction. In particular, our directed search model predicts a perfect sorting in the quarter in which the relationship is formed.[19] Since the volatility of idiosyncratic shocks is calibrated to a low value, the degree of sorting must be high the year before the start of the relationship. As we argued earlier, the relatively low degree of sorting measured in the data is likely explained by unobserved firm characteristics and measurement errors. Consequently, we consider the model's prediction empirically plausible.

Firms' output comoves with partners' productivity and entails technological synergies. Next, we verify that our model implies that firms' output comoves with partners' productivity and entails technological synergies, a direct prediction of the production function.

Table 10: Sales comove with partner's labor productivity and entail technological synergies

	(1)	(2)
Dependent Variable :	Log sales	
	Model	Data
z_I	1.47***	0.84***
	(0.00)	(0.02)
$decile\,(z_F)$	0.19***	0.03***
	(0.00)	(0.01)
$\Delta_{I,F}$	-0.15***	-0.05***
	(0.00)	(0.01)
Time FE	Yes	Yes
Industry FE	Yes	Yes
Adjusted R^2	0.99	0.28
Observations	201,259	29,889

Note: Standard errors are in parentheses. *** denotes significance at the 1% level.

Table 10 shows the estimation result for equation (3) with our simulated data and the actual data, respectively. Column (1) shows the model's prediction. The dependent variable is the

[19] While our model is purposely parsimonious, extending the framework to allow for multiple relationships across firms or assuming random search will decrease the degree of sorting.

log output. The independent variables include the intermediate-goods producer's productivity, the final-goods producer's productivity decile, and $\Delta_{I,F} = |decile\,(z_I) - decile\,(z_F)\,|$. The $\Delta_{I,F}$ coefficient is estimated as negative and statistically significant, indicating that the firm's output is penalized by mismatching. Column (2) reports the empirical result estimated with the actual data, which is the same as Column (2) of Table 2. The empirical results are close to the prediction of our model. The consistency of our model with the data is encouraging, given that our calibration does not use any information from this regression.

Mismatches are less durable. Next, we examine whether our model implies that mismatches are less durable, an empirical fact documented in Section 2. Table 11 shows the estimation result for equation (2) with our simulated data and the actual data, respectively.

Table 11: Relationship duration and the degree of mismatch

	(1)	(2)	(3)	(4)	(5)
	Model	Data			
		Labor productivity	Sales growth	ROE	Tobin's Q
$\Delta_{j,k}$	-0.09***	-0.02*	-0.08***	-0.09***	-0.03**
	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)
Industry FE	No	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Adjusted R^2	0.05	0.18	0.18	0.19	0.18
Observations	63,356	23,829	24,374	28,034	23,131

Note: Standard errors are in parentheses. *** denotes significance level at the 1% level.

Column (1) shows the model's prediction. The dependent variable is the duration of the match. The independent variable is the distance between the two partners' deciles in the distribution of productivity in the year preceding the formation of the relationship, measured by $\Delta_{I,F} = |decile\,(z_I) - decile\,(z_F)\,|$. The $\Delta_{I,F,t}$ coefficient is estimated as negative and statistically significant, indicating that mismatches are less stable. Columns (2)-(5) report the estimates with the actual data, which is the same as in Table 4. The empirical results are close to the prediction of our model.

Idiosyncratic shocks lead to separation of relationships. Lastly, we examine if our model implies that idiosyncratic shocks predict the separation of relationships, an important observation documented in Section 2. Table 12 shows the estimation results for equation (2) with our simulated data and the actual data, respectively.

Table 12: Changes in productivity and match separation

	(1)	(2)	(3)	(4)	(5)
	Model		Data		
		Labor productivity	Sales growth	ROE	Tobin's Q
$\lvert\Delta decile\,(z_{I,,t-1})\rvert$	0.01***	0.01***	0.003***	0.003***	0.003***
	(0.001)	(0.001)	(0.002)	(0.001)	(0.002)
$\lvert\Delta decile\,(z_{I,,t-1})\rvert$	0.01***	0.01***	-0.0002	0.002***	0.01***
	(0.001)	(0.001)	(0.002)	(0.001)	(0.001)
Industry FE	No	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Adjusted R^2	0.001	0.13	0.13	0.13	0.13
Observations	288,600	89,236	92,308	105,222	88,158

Note: Standard errors are in parentheses. *** denotes significance at the 1% level.

In Column (1), the dependent variable is a dummy variable, $sep_{I,F,t}$, which equals one if firm I terminates an existing relationship with firm F in year t. The independent variables are the absolute values of the change in firm F's and firm I's decile of productivity.[20] We find a significant and positive correlation between idiosyncratic shocks to either side of the match and the separation of the match. Columns (2) and (3) show the estimation result using the actual data, which is the same as Columns (1) and (4) in Table 5. Our model prediction is consistent with the data even when our calibration does not use the estimated regression coefficient.

6.4 Aggregate output

Finally, we gauge the aggregate implications of the model. Aggregate output can be decomposed as the product of the measure of relationships and the output per relationship:

$$Y_t = \underbrace{\int_{\Omega_t} d\left(l_I, l_F\right)}_{\text{Measure of relationships}} \times \underbrace{\frac{\int_{\Omega_t} y_t\left(l_I, l_F\right) d\left(l_I, l_F\right)}{\int_{\Omega_t} d\left(l_I, l_F\right)}}_{\text{Output per relationship}}, \tag{18}$$

[20] In particular, $\lvert\Delta decile\,(z_{I,t-1})\rvert = \lvert decile\,(z_{I,t-1}) - decile\,(z_{I,t-2})\rvert$ and $\lvert\Delta decile\,(z_{F,t-1})\rvert = \lvert decile\,(z_{F,t-1}) - decile\,(z_{F,t-2})\rvert$.

where Ω_t is the set of relationships. If all firms are matched in relationships, the rate of idleness is zero, and the measure of partnerships is one. In contrast, if some firms fail to form a relationship, the rate of idleness, idleness_t, is positive, and the measure of relationships is equal to one minus the rate of idleness. Thus, we can rewrite equation (18) with the more intuitive notation:

$$Y_t = (1 - \text{idleness}_t) \times \overline{y}_t, \tag{19}$$

where $\overline{y}_t$ is the output per relationship from equation (18).

Frictions generate an output gap by introducing a positive rate of idleness or reducing the output per relationship:

$$Y^n - Y_t \approx (\overline{y}^n - \overline{y}_t) + \overline{y}^n \times \text{idleness}_t, \tag{20}$$

where Y^n and $\overline{y}^n$ are the natural total output and natural output per relationship achieved in the steady state in the frictionless economy.[21]

Table 13: The effect of frictions on the the aggregate output

	(1) Benchmark calibration	(2) Frictionless economy	(3) Search frictions only	(4) Staggered separation only
Output	0.808	1.029	0.811	0.962
Output per trading relationship	0.918	1.029	0.921	0.962
Idleness rate	0.120	0	0.120	0

To study the contribution of the different components of output, Table 13 shows the idleness rate and the output per relationship for alternative calibrations of the model. Column (1) shows the stationary steady state for output, output per relationship, and the idleness rate in the benchmark calibration. Column (2) shows the results for calibration A. Thus, the first and second rows correspond to Y^n and $\overline{y}^n$ in equation (20), respectively. The entries reveal that output is 21.5% higher in the frictionless economy. This output loss is the joint effect of search frictions and staggered separation, which generate a reduction of 10.8% in the output per relationship and an increase in the idleness rate of 12%.

To disentangle the role of staggered separation and search frictions in determining output losses, we simulate the economy abstracting from each of the two frictions in turn. Column (3)

[21] To derive equation (20), take the total derivative of equation (19) at Y^n, $\overline{y}^n$, and idleness^n (the natural idleness rate): $Y_t - Y^n \approx (1 - \text{idleness}^n) \times (\overline{y}_t - \overline{y}^n) - \overline{y}^n\,(\text{idleness}_t - \text{idleness}^n)$, and impose $\text{idleness}^n = 0$ (i.e., the idleness rate is equal to zero in the frictionless economy).

shows the results for calibration B, while column (4) shows the results for calibration C. Search frictions alone explain most of the output loss in the benchmark case. In comparison, staggered separation plays a limited role in explaining output losses.

7 Quantitative analysis II: Aggregate TFP shocks

We move now to study the effect of aggregate TFP shocks. Figure 10 shows the impulse-response functions (IRFs) to a negative 10% TFP shock for aggregate output (left panel), the correlation of productivity within relationships, which measures the degree of sorting (middle panel), and the separation rate of relationships (right panel).

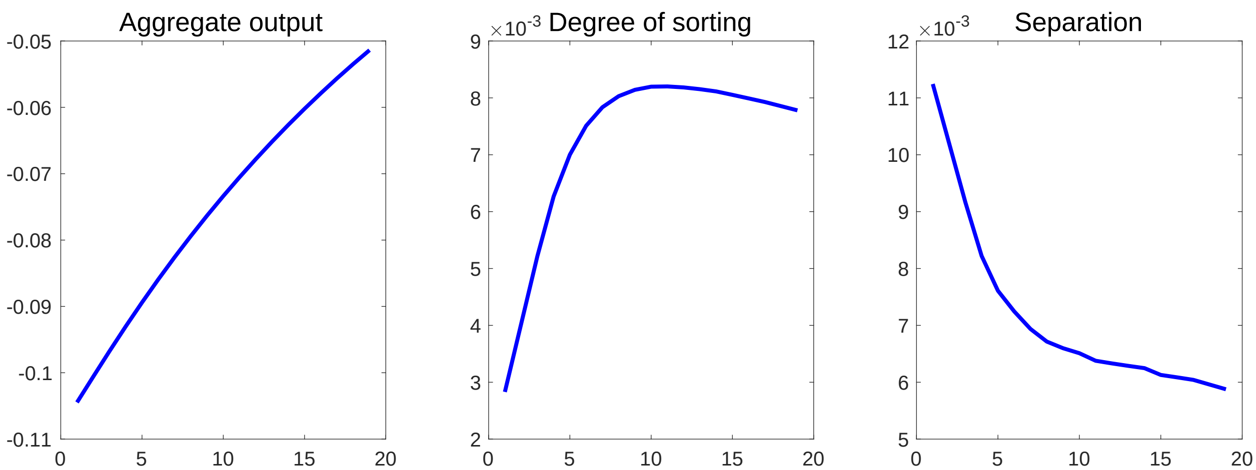


Figure 10: IRFs to a negative 10% TFP shock

The left panel shows that aggregate output decreases in response to the decline in TFP. The middle and right panels show a slight improvement in the degree of sorting and a mild increase in separations, indicating a small cleansing effect of the decreasing TFP. The cleansing effect is as follows: because single firms cannot produce, unsuccessful matching after separation entails the loss of a cash flow stream. These cash flows are lower when TFP is low. Hence, a lower TFP implies a lower opportunity cost of separation. As a result, firms are more willing to separate and search for more efficient matches.

Figure 11 illustrates the cleansing effect by displaying the policy rules for separation in high TFP (10% above SS, dark-shadowed area) and low TFP (10% below SS, light-shadowed area) states, respectively. The figure shows that more mismatches dissolve in the low-TFP state than

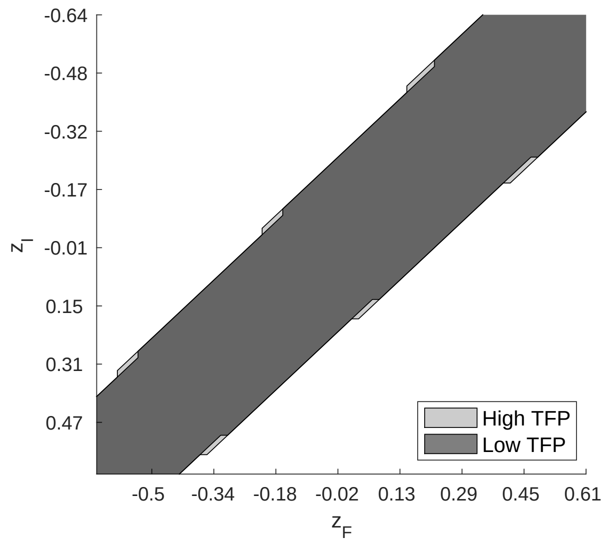


Figure 11: TFP and separation decision

in the high-TFP state. This will lead to a more efficient allocation of matches. However, the cleansing effect is negligible: the two shadowed areas almost overlap. Matches with different degrees of sorting are affected by aggregate TFP shocks almost uniformly, making the value gap between different matches relatively inelastic to aggregate TFP. The cleansing effect can be much stronger once we introduce mechanisms that make the value of mismatches more sensitive to aggregate TFP shocks than that of positive assortative matchings.

8 Quantitative analysis III: Uncertainty shocks

Our next step is studying the effect of uncertainty shocks that result from an increase in the variance of idiosyncratic productivity shocks. We follow the approach in Bloom et al. (2018) and simulate 400 economies independently for 200 periods. We let each economy have low uncertainty in the first 100 periods (to settle the distribution toward an area where low uncertainty has been prevalent for some time), increase uncertainty to a higher level from period 101 onwards, and let the system evolve according to the Markov-transition process described in Section 3 from period 101 onwards. Then, we take the mean of the time series across the simulated economies. Since the stochastic discount factor and the wage rate are functions of the time-varying distribution of firms, we implement a dimensionality reduction algorithm inspired by Krusell and Smith (1998). See Appendix C for details.

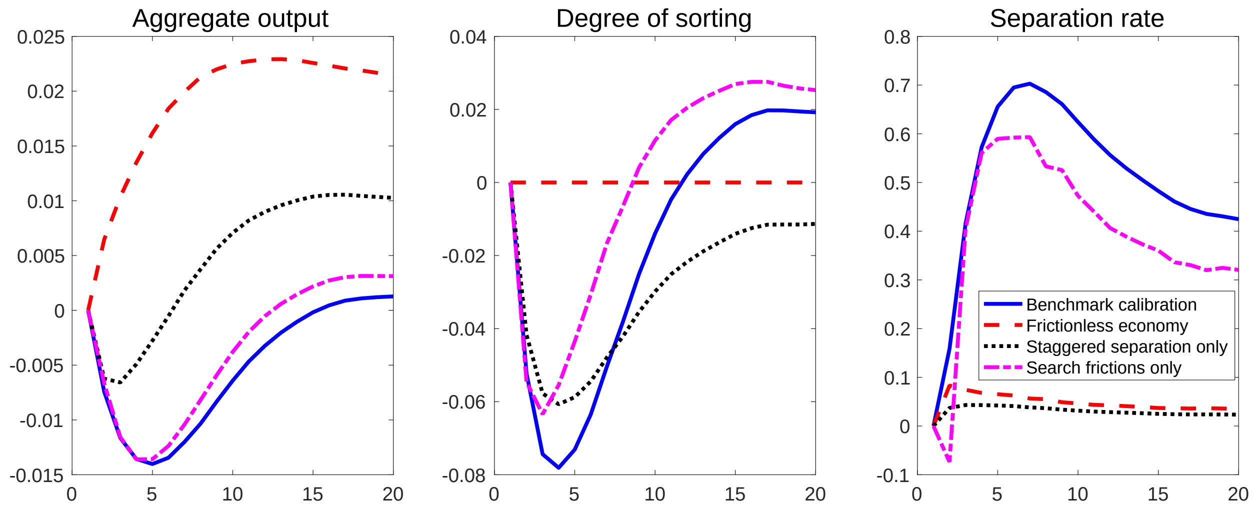


Figure 12: IRFs to an uncertainty shock

Figure 12 shows the IRFs to the uncertainty shock from period 100 to period 150 or aggregate output (left panel), the correlation of productivity within relationships (middle panel), and the separation rate of relationships (right panel). Responses are represented in percentage deviations. The solid blue line and the dashed red line show the responses for the benchmark and frictionless models, and the dotted black and dashed-dotted magenta lines show the responses for the staggered separation only and search frictions only, respectively.

We start by focusing on the benchmark model. The increase in uncertainty reduces aggregate output by 1.2% in the four periods after the shock, before the economy starts recovering. The initial drop in output is driven by the increase in the measure of relationships with mismatched productivity types that generate inefficient production.

In the frictionless economy, output increases in response to the rise in uncertainty. Without frictions, higher uncertainty generates a rise in the mass of firms that manufacture output with high idiosyncratic productivity, and mismatched relationships instantaneously dissolve, and firms are reallocated to efficient matches. The fall in output is primarily determined by search frictions, since the output drop remains large in the absence of delayed separation of partnerships. The economy with delayed separation but without search frictions shows a mild initial drop in output, followed by a persistently higher level of output as in the frictionless economy.

The middle panel in Figure 12 studies the degree of sorting, represented as before by the correlation of productivity between partners within relationships. The blue line shows that uncertainty sharply and persistently reduces sorting. The dashed red line shows that sorting remains unchanged in the frictionless economy as firms instantaneously establish relationships between equally productive firms. The comparison between the economy with search frictions only and staggered separation only illustrates that the two frictions evenly contribute to the overall drop in the degree of sorting.

The right panel in Figure 12 plots the rate of relationship separation. The uncertainty shock generates a persistent increase in separation. In the frictionless economy, separation mildly increases, driven by the technological complementarity that induces firms with different productivity types to separate. With search frictions only, separations rise substantially, close to the benchmark case. Finally, staggered separations have less effect on the rate of separation than in the frictionless case.

9 Conclusion

In this paper, we documented six empirical facts about the creation of trading relationships among firms. These facts suggest the existence of technological synergies between trading partners that lead to positive assortative matching among firms and their potential impact on aggregate fluctuations.

Then, we built a general equilibrium model with heterogeneous firms calibrated on new firm-level data. We have shown that frictions in forming relationships and separation costs explain imperfect sorting between firms by matching the model's predictions with the data. Among the most interesting quantitative implications of the model, we illustrated how an increase in the volatility of idiosyncratic productivity shocks significantly decreases aggregate output without resorting to non-convex adjustment costs.

Our investigation opens many doors for future research, including extending the model to multiple-firm production networks (as suggested by Fact 3), exploring the consequences of relationship-specific capital, and the effects of IT and automation on technological synergies. For instance, Ghassibe (2021) provides a promising framework for this extension. We hope to follow up on some of these ideas shortly.

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Appendix

This appendix provides extra robustness exercises related to our empirical findings and additional details about the computation of the paper.

A Alternative specifications

In the main text, we studied the impact of economic fundamentals on assortative matching using data from the year before the relationship was formed to control for the effect of common shocks. Here, Table A.1 shows that our results remain the same if we use data for the years when the firms are in the relationship.

Table A.1: Assortative matching for ranking of economic fundamentals, during match

	(1)	(2)	(3)	(4)
	Labor productivity	Sales growth	ROE	Tobin's q
$decile\left(\pi_{j,k}^{cus}\right)$	0.09***	0.09***	0.02***	0.06***
	(0.003)	(0.004)	(0.003)	(0.003)
Industry FE	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes
Adjusted R^2	0.09	0.02	0.03	0.09
Observations	98,619	100,028	113,537	97,638

Note: Sample: 1976-2020. Standard errors are in parentheses. *** denotes significance at the 1% level.

Table 4 in the main text proved that the duration of a relationship decreases with the degree of mismatch. Here, Table A.2 demonstrates the robustness of our results when we focus on the year preceding the start of matches.

Table A.2: Relationship duration and the degree of mismatch, during match

	(1)	(2)	(3)	(4)
	Labor productivity	Sales growth	ROE	Tobin's Q
$\Delta_{j,k,t}$	-0.04***	-0.20***	-0.22***	-0.07**
	(0.01)	(0.01)	(0.01)	(0.01)
Time FE	Yes	Yes	Yes	Yes
Industry FE	Yes	Yes	Yes	Yes
Adjusted R^2	0.10	0.11	0.12	0.11
Observations	99,702	101,978	224,787	100,166

Note: Sample: 1976-2020. Standard errors are in parentheses. *** denotes significance level at the 1% level.

B Simulation with a finite number of firms

Next, we explain how we simulate the model with idiosyncratic shocks to the firms. A key variable of the model is the allocation of single firms across submarkets. In particular, one needs to solve the matrix of $\widetilde{n}_{I,t}(z_I, z_F)$ such that no firm wants to deviate from the allocation given the matrix of tightness ratios, $\theta_t(z_{I,}z_F)$, which are implied by $\widetilde{n}_{I,t}(z_I, z_F)$.

Given the large number of unknowns, this is a difficult numerical problem. Fortunately, we can show that when the distribution of firms is symmetric between the two sectors and when log-supermodularity holds, the model entails positive assortative matching with the following structure:

$$\widetilde{n}_{I,t}(z_I, z_F) = \begin{cases} \widetilde{n}_{I,t}(z_I) & \text{if } z_I = z_F \\ 0 & \text{if } z_I \neq z_F \end{cases} \text{ and } \widetilde{n}_{F,t}(z_I, z_F) = \begin{cases} \widetilde{n}_{F,t}(z_F) & \text{if } z_I = z_F \\ 0 & \text{if } z_I \neq z_F \end{cases},$$

which further implies that: $\theta_t(z_{I,}z_F) = 1$ and $\mu(z_I, z_F) = \psi$ if $z_I = z_F$, where μ is the matching probability.

These theoretical results simplify our numerical analysis because the transition rule of $\widetilde{n}_{I,t}(z_I, z_F)$ and the equilibrium level of $\theta_t(z_{I,}z_F)$ and $\mu(z_{I,}z_F)$ are all analytically given. However, when we simulate the model with a finite number of firms in both sectors (as one is forced to do in practice), it is impossible to achieve sectoral symmetry, which prevents us from using the above theoretical results. We simulate the model by imposing sectoral symmetry as follows.

In period 0, we randomly draw N firms I. We assume each is matched to a firm F with the same productivity.

In period t:

- Step 1: Each firm I is hit by an idiosyncratic shock.
- Step 2: Each F that is matched to a firm I is hit by an idiosyncratic shock.
- Step 3: Firms in relationships decide whether to separate. If they continue to match, they jump to Step 6. If they separate, they proceed to Step 4.
- Step 4: Every single firm I is matched to a firm F with the same productivity with probability ψ. If they fail to match, they jump to period t+1. If they form a new match, they proceed to Step 5. Here we are imposing sectoral symmetry and positive assortative matching on the single firms.
- Step 5: For every single firm I that forms a new match in period t, we simulate the history of productivities for its new partner firm F.
- Step 6: Relationships produce according to the production function.

C Solution with uncertainty shocks

When the volatility of idiosyncratic shocks is stochastic, the distribution of firms, Ω_t, becomes a time-varying state variable in a relationship's value and policy functions. In particular, a relationship's state space consists of $(z_{I,t}, z_{F,t}, \sigma_t, \Omega_t)$, which is infinitely dimensional. The intuition is that the stochastic discount factor Λ_{t+1} and the wage W_t, which are used to discount future utility and to determine labor demand, depend on aggregate consumption. And since aggregate consumption depends on the distribution of firms, firms need to keep track of the transition of Ω_t to make decisions.

We simplify the model solution with a set of forecasting rules:

$$\Lambda_t = \alpha_{1,\Lambda} + \alpha_{1,\Lambda}\Lambda_{t-1} + \alpha_{1,\Lambda}\sigma_{t-1} + \alpha_{1,\Lambda}\sigma_t$$

$$\Lambda_{t+1}(\sigma_{t+1}) = \alpha_{1,\Lambda} + \rho_{2,\Lambda}\Lambda_{t-1} + \beta_{2,\sigma}\sigma_{t-1} + \gamma_{2,\sigma}\sigma_t + \phi_\sigma\sigma_{t+1}$$

$$W_t = \alpha_W + \rho_W\Lambda_{t-1} + \beta_W\sigma_{t-1} + \gamma_W\sigma_t$$

where $A = (\alpha_{1,\Lambda}, \alpha_{1,\Lambda}, \alpha_{1,\Lambda}, \alpha_{1,\Lambda}, \alpha_{1,\Lambda}, \rho_{2,\Lambda}, \beta_{2,\sigma}, \gamma_{2,\sigma}, \phi_{\sigma}, \alpha_W, \rho_W, \beta_W, \gamma_W)$ is the vector of coefficients to be determined. The second forecasting rule for $\Lambda_{t+1}(\sigma_{t+1})$ is contingent on the realization of σ_{t+1}. Intuitively, firms do not need to know the transition process of Ω_t to make decisions if the forecast rule is accurate, which reduces the dimension of the space to a finite number. In particular, the new state space of the relationship is $(z_{I,t}, z_{F,t}, \sigma_t, \sigma_{t-1}, \Lambda_{t-1})$.

To do so, we proceed as follows:

- Step 1: We initialize the forecasting rule with some initial guesses:

$$A^{(0)} = \left(\alpha_{1,\Lambda}^{(0)}, \alpha_{1,\Lambda}^{(0)}, \alpha_{1,\Lambda}^{(0)}, \alpha_{1,\Lambda}^{(0)}, \alpha_{1,\Lambda}^{(0)}, \rho_{2,\Lambda}^{(0)}, \beta_{2,\sigma}^{(0)}, \gamma_{2,\sigma}^{(0)}, \phi_{\sigma}^{(0)}, \alpha_W^{(0)}, \rho_W^{(0)}, \beta_W^{(0)}, \gamma_W^{(0)}\right)$$

- Step 2: We solve for the value functions, J_F, J_I, $\widehat{J}_F$, $\widehat{J}_I$, $\widetilde{J}_F$, $\widetilde{J}_I$, and the policy functions, s_F and s_I.

- Step 3: We simulate the model for 10,000 periods (disregarding the first 2,000 as a burn-in) with random draws of $(z_{I,t}, z_{F,t}, \sigma_t)$. Then, we compute the series of Λ_t and W_t.

- Step 4: Based on the simulated data, we update the coefficient of the forecast rule $A^{(q)}$ with $A^{(q+1)}$ using ordinary least squares. If $A^{(q)}$ and $A^{(q+1)}$ are sufficiently close to each other, we stop the iteration. Otherwise, we return to Step 2.

The converged forecasting rules explain the fluctuations of Λ_t, $\Lambda_{t+1}(\sigma_{t+1})$, and W_t well, with R^2 of 0.92, 0.95, and 0.87, respectively.