Perturbation Methods IV: Perturbing the Value Function

(Lectures on Solution Methods for Economists VIII)

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Perturbing the value function

- We worked with the equilibrium conditions of the model.
- Sometimes we may want to perform a perturbation on the value function formulation of the problem.
- Possible reasons:
  1. Gain insight.
  2. Difficulty in using equilibrium conditions.
  3. Evaluate welfare.
  4. Initial guess for VFI.
- More general point: we can perturb any operator problem that we find useful.

Basic problem

- The previous problem in recursive form:

$$V\left(k_t, z_t\right) = \max_{c_t} \left[\log c_t + \beta \mathbb{E}_t V\left(k_{t+1}, z_{t+1}\right)\right]$$

$$\text{s.t. } c_t + k_{t+1} = e^{z_t} k_t^{\alpha}$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t,\ \varepsilon_t \sim \mathcal{N}\left(0, 1\right)$$

- Write it as:

$$V\left(k_t, z_t; \lambda\right) =$$

$$\max_{c_t} \left[\log c_t + \beta \mathbb{E}_t V\left(e^{z_t} k_t^{\alpha} - c_t, \rho z_t + \lambda \sigma \varepsilon_{t+1}; \lambda\right)\right]$$

- The solution of this problem is value function $V\left(k_t, z_t; \lambda\right)$ and a policy function for consumption $c\left(k_t, z_t; \lambda\right)$.

Expanding the value function

The second-order Taylor approximation of the value function around the deterministic steady state $(k, 0; 0)$ is:

$$
\begin{gathered}
V\left(k_t, z_t; \lambda\right) \simeq \\
V_{ss} + V_{1,ss}\left(k_t - k_{ss}\right) + V_{2,ss} z_t + V_{3,ss}\lambda \\
+\frac{1}{2} V_{11,ss}\left(k_t - k\right)^2 + \frac{1}{2} V_{12,ss}\left(k_t - k\right) z_t + \frac{1}{2} V_{13,ss}\left(k_t - k\right)\lambda \\
+\frac{1}{2} V_{21,ss} z_t\left(k_t - k\right) + \frac{1}{2} V_{22,ss} z_t^2 + \frac{1}{2} V_{23,ss} z_t \lambda \\
+\frac{1}{2} V_{31,ss}\lambda\left(k_t - k\right) + \frac{1}{2} V_{32,ss}\lambda z_t + \frac{1}{2} V_{33,ss}^2 \lambda^2
\end{gathered}
$$

where

$$
\begin{aligned}
V_{ss} &= V\left(k, 0; 0\right) \\
V_{i,ss} &= V_i\left(k, 0; 0\right) \text{ for } i = \{1, 2, 3\} \\
V_{ij,ss} &= V_{ij}\left(k, 0; 0\right) \text{ for } i, j = \{1, 2, 3\}
\end{aligned}
$$

Expanding the value function

- By certainty equivalence, we will show below that:

$$V_{3,ss} = V_{13,ss} = V_{23,ss} = 0$$

- Taking advantage of the equality of cross-derivatives, and setting $\lambda = 1$, which is just a normalization:

$$\begin{aligned} V\left(k_t, z_t; 1\right) \quad \simeq \quad & V_{ss} + V_{1,ss}\left(k_t - k\right) + V_{2,ss} z_t \\ & + \frac{1}{2} V_{11,ss}\left(k_t - k\right)^2 + \frac{1}{2} V_{22,ss} z_{tt}^2 \\ & + V_{12,ss}\left(k_t - k\right) z + \frac{1}{2} V_{33,ss} + \ldots \end{aligned}$$

- Note that $V_{33,ss} \neq 0$, a difference from the standard linear-quadratic approximation to the utility functions.

Expanding the consumption function

- The policy function for consumption can be expanded as:

$$c_t = c\left(k_t, z_t; \lambda\right) \simeq c_{ss} + c_{1,ss}\left(k_t - k\right) + c_{2,ss} z_t + c_{3,ss}\chi + \ldots$$

  where:

$$\begin{aligned} c_{1,ss} &= c_1\left(k_{ss}, 0; 0\right) \\ c_{2,ss} &= c_2\left(k_{ss}, 0; 0\right) \\ c_{3,ss} &= c_3\left(k_{ss}, 0; 0\right) \end{aligned}$$

- Since the first derivatives of the consumption function only depend on the first and second derivatives of the value function, we must have that $c_{3,ss} = 0$ (precautionary consumption depends on the third derivative of the value function, Kimball, 1990).

Linear components of the value function

- As before, we first find the steady state of the model:

$$k = (\alpha\beta)^{\frac{1}{1-\alpha}}$$
$$c = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}$$
$$V_{ss} = \frac{\log c}{1-\beta}$$

- We substitute the decision rules into the value function and drop the max operator:

$$V\left(k_t, z_t; \lambda\right) - \log c\left(k_t, z_t; \lambda\right)$$
$$+\beta\mathbb{E}_t V\left(e^{z_t} k_t^{\alpha} - c\left(k_t, z_t; \lambda\right), \rho z_t + \lambda\sigma\varepsilon_{t+1}; \lambda\right) = 0$$

- We take derivatives of the value function with respect to the control (c_t), the states (k_t, z_t), and the perturbation parameter λ.

Derivatives

- Derivative with respect to c_t:

$$c_t^{-1} - \beta \mathbb{E}_t V_{1,t+1} = 0$$

- Derivative with respect to k_t:

$$V_{1,t} = \beta \mathbb{E}_t V_{1,t+1} \left(\alpha e^{z_t} k_t^{\alpha-1} \right)$$

- Derivative with respect to z_t:

$$V_{2,t} = \beta \mathbb{E}_t \left[V_{1,t+1} e^{z_t} k_t^{\alpha} + \rho V_{2,t+1} \right]$$

- Derivative with respect to λ:

$$V_{3,t} = \beta \mathbb{E}_t \left[V_{2,t+1} \sigma \varepsilon_{t+1} + V_{3,t+1} \right]$$

- We apply the envelope theorem to eliminate the derivatives of consumption with respect to k_t, z_t, and λ.

System of equations I

Now, we have the system:

$$
\begin{gathered}
c_t^{-1} - \beta \mathbb{E}_t V_{1,t+1} = 0 \\
V_{1,t} = \beta \mathbb{E}_t V_{1,t+1} \alpha e^{z_t} k_t^{\alpha-1} \\
V_{2,t} = \beta \mathbb{E}_t \left[V_{1,t+1} e^{z_t} k_t^{\alpha} + \rho V_{2,t+1} \right] \\
V_{3,t} = \beta \mathbb{E}_t \left[V_{2,t+1} \sigma \varepsilon_{t+1} + V_{3,t+1} \right] \\
z_t = \rho z_{t-1} + \lambda \sigma \varepsilon_t
\end{gathered}
$$

System of equations II

If we set $\lambda = 0$ and compute the steady state, we get a system of four equations on four unknowns, k, $V_{1,ss}$, $V_{2,ss}$, and $V_{3,ss}$:

$$\frac{1}{c} - \beta V_{1,ss} = 0$$

$$V_{1,ss} = \beta V_{1,ss} \alpha k^{\alpha-1}$$

$$V_{2,ss} = \beta \left[V_{1,ss} k^{\theta} + \rho V_{2,ss} \right]$$

$$V_{3,ss} = \beta V_{3,ss}$$

- Then:
  1. $V_{1,ss} = \frac{1}{\beta c} > 0$.
  2. $V_{2,ss} = \frac{\beta}{1-\beta\rho} \frac{k^{\alpha}}{c} = \frac{\beta}{(1-\alpha\beta)(1-\beta\rho)} > 0$.
  3. $V_{3,ss} = 0$.

Quadratic components of the value function

From the previous derivations, we have:

$$
\begin{aligned}
c_t^{-1} - \beta \mathbb{E}_t V_{1,t+1} &= 0 \\
V_{1,t} &= \beta \mathbb{E}_t V_{1,t+1} \alpha e^{z_t} k_t^{\alpha-1} \\
V_{2,t} &= \beta \mathbb{E}_t \left[V_{1,t+1} e^{z_t} k_t^{\alpha} + \rho V_{2,t+1} \right] \\
V_{3,t} &= \beta \mathbb{E}_t \left[V_{2,t+1} \sigma \varepsilon_{t+1} + V_{3,t+1} \right]
\end{aligned}
$$

- We will now take derivatives of each of the four equations with respect to k_t, z_t, and λ.
- We will take advantage of the equality of cross derivatives.
- The envelope theorem does not hold anymore (we are taking derivatives of the derivatives of the value function).

The welfare cost of the business cycle

- An advantage of performing the perturbation on the value function is that we have evaluation of welfare readily available.

- Note that at the deterministic steady state, we have:

$$V(k, 0; \chi) \simeq V_{ss} + \frac{1}{2} V_{33,ss}$$

- Hence $\frac{1}{2} V_{33,ss}$ is a measure of the welfare cost of the business cycle.

- Note that this quantity is not necessarily negative. Indeed, it may well be positive in many models, like in a RBC with leisure choice. See Cho and Cooley (2000).

Our example

- We know that $V_{ss} = \frac{\log c}{1-\beta}$.
- Then, we can compute the decrease in consumption τ that will make the household indifferent between consuming $(1-\tau)\,c$ units per period with certainty or c_t units with uncertainty.
- To do so, note that:

$$\frac{\log c}{1-\beta} + \frac{1}{2} V_{33,ss} = \frac{\log c}{1-\beta} + \frac{\log(1-\tau)}{1-\beta} \Rightarrow$$
$$\tau = 1 - \exp\left(\frac{1-\beta}{2} V_{33,ss}\right)$$

A numerical example I

- A more realistic example

$$V\left(k_t, z_t\right) = \max_{c_t} \left[\left(1-\beta\right) \frac{c_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t V\left(k_{t+1}, z_{t+1}\right) \right]$$

$$\text{s.t. } c_t + k_{t+1} = e^{z_t} k_t^{\theta} + \left(1-\delta\right) k_t$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t,\ \varepsilon_t \sim \mathcal{N}\left(0,1\right)$$

- We pick standard parameter values by setting

$$\beta = 0.99, \gamma = 2, \delta = 0.0294, \theta = 0.3, \text{ and } \rho = 0.95.$$

A numerical example II

- Then, we get:

$$\begin{aligned}
V\left(k_t, z_t; 1\right) &\simeq -0.54000 + 0.00295\left(k_t - k_{ss}\right) + 0.11684 z_t \\
&\quad -0.00007\left(k_t - k_{ss}\right)^2 - 0.00985 z_t^2 \\
&\quad -0.97508 - 0.00225\left(k_t - k_{ss}\right) z_t \\
c\left(k_t, z_t; 1\right) &\simeq 1.85193 + 0.04220\left(k_t - k_{ss}\right) + 0.74318 z_t
\end{aligned}$$

- Also, the consumption equivalent of the welfare cost of the business cycle is 8.8475e-005, even lower than Lucas' (1987) original computation because of the smoothing possibilities implied by the presence of capital.

- Use as an initial guess for VFI.