Perturbation Methods II: General Case

(Lectures on Solution Methods for Economists VI)

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The General case

- Most of arguments in the previous set of lecture notes are easy to generalize.
- The set of equilibrium conditions of many DSGE models can be written using recursive notation as:

$$\mathbb{E}_t \mathcal{H}(y, y', x, x') = 0,$$

  where y_t is a $n_y \times 1$ vector of controls and x_t is a $n_x \times 1$ vector of states.
- $n = n_x + n_y$.
- $\mathcal{H}$ maps $R^{n_y} \times R^{n_y} \times R^{n_x} \times R^{n_x}$ into R^n.

Partitioning the state vector

- The state vector x_t can be partitioned as $x = [x_1; x_2]^t$.
- x_1 is a $(n_x - n_\epsilon) \times 1$ vector of endogenous state variables.
- x_2 is a $n_\epsilon \times 1$ vector of exogenous state variables.
- Why do we want to partition the state vector?

Exogenous stochastic process I

$$x_2' = Ax_2 + \lambda \eta_\epsilon \epsilon'$$

- Process with 3 parts:

  1. The deterministic component Ax_2, where A is a $n_\epsilon \times n_\epsilon$ matrix, with all eigenvalues with modulus less than one.
  2. The scaled innovation $\eta_\epsilon \epsilon'$, where:

     2.1 η_ϵ is a known $n_\epsilon \times n_\epsilon$ matrix.

     2.2 ϵ is a $n_\epsilon \times 1$ i.i.d. innovation with bounded support, zero mean, and variance/covariance matrix I.
  3. The perturbation parameter λ.

Exogenous stochastic process II

- We can accommodate very general structures of x_2 through changes in the definition of the state space: i.e., stochastic volatility.
- More general structure:

$$x_2' = \Gamma(x_2) + \lambda \eta_\epsilon \epsilon'$$

  where Γ is a non-linear function satisfying that all eigenvalues of its first derivative evaluated at the non-stochastic steady state lie within the unit circle.

- Note we do not impose Gaussanity.

The perturbation parameter

- The scalar $\lambda \geq 0$ is the perturbation parameter.
- If we set $\lambda = 0$, we have a deterministic model.
- Important: there is only ONE perturbation parameter. The matrix η_ϵ takes account of relative sizes of different shocks.
- Why bounded support? Samuelson (1970), Jin and Judd (2002).

Solution of the model

- The solution to the model is of the form:

$$y = g(x; \lambda)$$
$$x' = h(x; \lambda) + \lambda \eta \epsilon'$$

  where g maps $R^{n_x} \times R^{+}$ into R^{n_y} and h maps $R^{n_x} \times R^{+}$ into R^{n_x}.

- The matrix η is of order $n_x \times n_\epsilon$ and is given by:

$$\eta = \begin{bmatrix} \emptyset \\ \eta_\epsilon \end{bmatrix}$$

Perturbation

- We wish to find a perturbation approximation of the functions g and h around the non-stochastic steady state, $x_t = \bar{x}$ and $\lambda = 0$.
- We define the non-stochastic steady state as vectors $(\bar{x}, \bar{y})$ such that:

$$\mathcal{H}(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0.$$

- Note that $\bar{y} = g(\bar{x}; 0)$ and $\bar{x} = h(\bar{x}; 0)$.
- This is because, if $\lambda = 0$, $\mathbb{E}_t \mathcal{H} = \mathcal{H}$.

Plugging-in the proposed solution

- Substituting the proposed solution, we define:

$$F(x;\lambda) \equiv \mathbb{E}_t \mathcal{H}(g(x;\lambda), g(h(x;\lambda) + \eta\lambda\epsilon', \lambda), x, h(x;\lambda) + \eta\lambda\epsilon') = 0$$

- Since $F(x;\lambda) = 0$ for any values of x and λ, the derivatives of any order of F must also be equal to zero.

- Formally:

$$F_{x^k \lambda^j}(x;\lambda) = 0 \quad \forall x, \lambda, j, k,$$

where $F_{x^k \lambda^j}(x,\lambda)$ denotes the derivative of F with respect to x taken k times and with respect to λ taken j times.

First-order approximation

- We are looking for approximations to g and h around $(x, \lambda) = (\bar{x}, 0)$ of the form:

$$\begin{aligned} g(x;\lambda) &= g(\bar{x};0) + g_x(\bar{x};0)(x-\bar{x}) + g_\lambda(\bar{x};0)\lambda \\ h(x;\lambda) &= h(\bar{x};0) + h_x(\bar{x};0)(x-\bar{x}) + h_\lambda(\bar{x};0)\lambda \end{aligned}$$

- As explained earlier, $g(\bar{x};0) = \bar{y}$ and $h(\bar{x};0) = \bar{x}$.
- The remaining four unknown coefficients of the first-order approximation to g and h are found by using the fact that:

$$F_x(\bar{x};0) = 0$$

and

$$F_\lambda(\bar{x};0) = 0$$

- Before doing so, we need to introduce the tensor notation.

Tensors

- General trick from physics.
- An n^{th}-rank tensor in a m-dimensional space is an operator that has n indices and m^n components and obeys certain transformation rules.
- $[\mathcal{H}_y]^i_\alpha$ is the (i, α) element of the derivative of $\mathcal{H}$ with respect to y:
  1. The derivative of $\mathcal{H}$ with respect to y is an $n \times n_y$ matrix.
  2. Thus, $[\mathcal{H}_y]^i_\alpha$ is the element of this matrix located at the intersection of the i-th row and α-th column.
  3. Thus, $[\mathcal{H}_y]^i_\alpha [g_x]^\alpha_\beta [h_x]^\beta_j = \sum_{\alpha=1}^{n_y} \sum_{\beta=1}^{n_x} \frac{\partial \mathcal{H}^i}{\partial y^\alpha} \frac{\partial g^\alpha}{\partial x^\beta} \frac{\partial h^\beta}{\partial x^j}$.
- $[\mathcal{H}_{y'y'}]^i_{\alpha\gamma}$:
  1. $\mathcal{H}_{y'y'}$ is a three dimensional array with n rows, n_y columns, and n_y pages.
  2. Then $[\mathcal{H}_{y'y'}]^i_{\alpha\gamma}$ denotes the element of $\mathcal{H}_{y'y'}$ located at the intersection of row i, column α and page γ.

Solving the system I

- g_x and h_x can be found as the solution to the system:

$$
\begin{aligned}
[F_x(\bar{x};0)]^i_j &= [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_x]^\beta_j + [\mathcal{H}_y]^i_\alpha [g_x]^\alpha_j + [\mathcal{H}_{x'}]^i_\beta [h_x]^\beta_j + [\mathcal{H}_x]^i_j = 0; \\
i &= 1, \ldots, n; \quad j, \beta = 1, \ldots, n_x; \quad \alpha = 1, \ldots, n_y
\end{aligned}
$$

- Note that the derivatives of $\mathcal{H}$ evaluated at $(y, y', x, x') = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$ are known.
- Then, we have a system of $n \times n_x$ quadratic equations in the $n \times n_x$ unknowns given by the elements of g_x and h_x.
- We can solve with a standard quadratic matrix equation solver.

Solving the system II

- g_λ and h_λ are the solution to the n equations:

$$
\begin{gathered}
[F_\lambda(\bar{x};0)]^i = \\
\mathbb{E}_t\{[\mathcal{H}_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_\lambda]^\beta + [\mathcal{H}_{y'}]^i_\alpha[g_x]^\alpha_\beta[\eta]^\beta_\phi[\epsilon']^\phi + [\mathcal{H}_{y'}]^i_\alpha[g_\lambda]^\alpha \\
+[\mathcal{H}_y]^i_\alpha[g_\lambda]^\alpha + [\mathcal{H}_{x'}]^i_\beta[h_\lambda]^\beta + [\mathcal{H}_{x'}]^i_\beta[\eta]^\beta_\phi[\epsilon']^\phi\} \\
i = 1,\ldots,n; \quad \alpha = 1,\ldots,n_y; \quad \beta = 1,\ldots,n_x; \quad \phi = 1,\ldots,n_\epsilon.
\end{gathered}
$$

- Then:

$$
\begin{gathered}
[F_\lambda(\bar{x};0)]^i \\
= [\mathcal{H}_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_\lambda]^\beta + [\mathcal{H}_{y'}]^i_\alpha[g_\lambda]^\alpha + [\mathcal{H}_y]^i_\alpha[g_\lambda]^\alpha + [f_{x'}]^i_\beta[h_\lambda]^\beta = 0; \\
i = 1,\ldots,n; \quad \alpha = 1,\ldots,n_y; \quad \beta = 1,\ldots,n_x; \quad \phi = 1,\ldots,n_\epsilon.
\end{gathered}
$$

- Certainty equivalence: linear and homogeneous equation in g_λ and h_λ. Thus, if a unique solution exists, it satisfies:

$$
\begin{aligned}
h_\lambda &= 0 \\
g_\lambda &= 0
\end{aligned}
$$

Second-order approximation I

The second-order approximations to g around $(x;\lambda) = (\bar{x};0)$ is

$$
\begin{aligned}
[g(x;\lambda)]^i \quad = \quad & [g(\bar{x};0)]^i + [g_x(\bar{x};0)]^i_a[(x-\bar{x})]_a + [g_\lambda(\bar{x};0)]^i[\lambda] \\
& + \frac{1}{2}[g_{xx}(\bar{x};0)]^i_{ab}[(x-\bar{x})]_a[(x-\bar{x})]_b \\
& + \frac{1}{2}[g_{x\lambda}(\bar{x};0)]^i_a[(x-\bar{x})]_a[\lambda] \\
& + \frac{1}{2}[g_{\lambda x}(\bar{x};0)]^i_a[(x-\bar{x})]_a[\lambda] \\
& + \frac{1}{2}[g_{\lambda\lambda}(\bar{x};0)]^i[\lambda][\lambda]
\end{aligned}
$$

where $i = 1, \ldots, n_y$, $a, b = 1, \ldots, n_x$, and $j = 1, \ldots, n_x$.

Second-order approximation II

The second-order approximations to h around $(x;\lambda) = (\bar{x};0)$ is

$$\begin{aligned}
[h(x;\lambda)]^j &= [h(\bar{x};0)]^j + [h_x(\bar{x};0)]^j_a[(x-\bar{x})]_a + [h_\lambda(\bar{x};0)]^j[\lambda] \\
&+ \frac{1}{2}[h_{xx}(\bar{x};0)]^j_{ab}[(x-\bar{x})]_a[(x-\bar{x})]_b \\
&+ \frac{1}{2}[h_{x\lambda}(\bar{x};0)]^j_a[(x-\bar{x})]_a[\lambda] \\
&+ \frac{1}{2}[h_{\lambda x}(\bar{x};0)]^j_a[(x-\bar{x})]_a[\lambda] \\
&+ \frac{1}{2}[h_{\lambda\lambda}(\bar{x};0)]^j[\lambda][\lambda],
\end{aligned}$$

where $i = 1,\ldots,n_y$, $a, b = 1,\ldots,n_x$, and $j = 1,\ldots,n_x$.

Second-order approximation III

- The unknowns of these expansions are $[g_{xx}]^i_{ab}$, $[g_{x\lambda}]^i_a$, $[g_{\lambda x}]^i_a$, $[g_{\lambda\lambda}]^i$, $[h_{xx}]^j_{ab}$, $[h_{x\lambda}]^j_a$, $[h_{\lambda x}]^j_a$, $[h_{\lambda\lambda}]^j$.
- These coefficients can be identified by taking the derivative of $F(x;\lambda)$ with respect to x and λ twice and evaluating them at $(x;\lambda) = (\bar{x};0)$.
- By the arguments provided earlier, these derivatives must be zero.

Solving the system I

We use $F_{xx}(\bar{x};0)$ to identify $g_{xx}(\bar{x};0)$ and $h_{xx}(\bar{x};0)$:

$$
\begin{gathered}
[F_{xx}(\bar{x};0)]^i_{jk} = \\
\left([\mathcal{H}_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k + [\mathcal{H}_{y'x'}]^i_{\alpha\delta}[h_x]^\delta_k + [\mathcal{H}_{y'x}]^i_{\alpha k}\right)[g_x]^\alpha_\beta[h_x]^\beta_j \\
+[\mathcal{H}_{y'}]^i_\alpha[g_{xx}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j + [\mathcal{H}_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_{xx}]^\beta_{jk} \\
+\left([\mathcal{H}_{yy'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{yy}]^i_{\alpha\gamma}[g_x]^\gamma_k + [\mathcal{H}_{yx'}]^i_{\alpha\delta}[h_x]^\delta_k + [\mathcal{H}_{yx}]^i_{\alpha k}\right)[g_x]^\alpha_j \\
+[\mathcal{H}_y]^i_\alpha[g_{xx}]^\alpha_{jk} \\
+\left([\mathcal{H}_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{x'y}]^i_{\beta\gamma}[g_x]^\gamma_k + [\mathcal{H}_{x'x'}]^i_{\beta\delta}[h_x]^\delta_k + [\mathcal{H}_{x'x}]^i_{\beta k}\right)[h_x]^\beta_j \\
+[\mathcal{H}_{x'}]^i_\beta[h_{xx}]^\beta_{jk} \\
+[\mathcal{H}_{xy'}]^i_{j\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{xy}]^i_{j\gamma}[g_x]^\gamma_k + [\mathcal{H}_{xx'}]^i_{j\delta}[h_x]^\delta_k + [\mathcal{H}_{xx}]^i_{jk} = 0; \\
i = 1,\dots n, \quad j,k,\beta,\delta = 1,\dots n_x; \quad \alpha,\gamma = 1,\dots n_y.
\end{gathered}
$$

Solving the system II

- We know the derivatives of $\mathcal{H}$.
- We also know the first derivatives of g and h evaluated at $(y, y', x, x') = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$.
- Hence, the above expression represents a system of $n \times n_x \times n_x$ linear equations in then $n \times n_x \times n_x$ unknowns elements of g_{xx} and h_{xx}.

Solving the system III

Similarly, $g_{\lambda\lambda}$ and $h_{\lambda\lambda}$ can be obtained by solving:

$$
\begin{aligned}
[F_{\lambda\lambda}(\bar{x};0)]^i \quad = \quad & [\mathcal{H}_{y'}]^i_{\alpha}[g_x]^{\alpha}_{\beta}[h_{\lambda\lambda}]^{\beta} \\
& +[\mathcal{H}_{y'y'}]^i_{\alpha\gamma}[g_x]^{\gamma}_{\delta}[\eta]^{\delta}_{\xi}[g_x]^{\alpha}_{\beta}[\eta]^{\beta}_{\phi}[I]^{\phi}_{\xi} \\
& +[\mathcal{H}_{y'x'}]^i_{\alpha\delta}[\eta]^{\delta}_{\xi}[g_x]^{\alpha}_{\beta}[\eta]^{\beta}_{\phi}[I]^{\phi}_{\xi} \\
& +[\mathcal{H}_{y'}]^i_{\alpha}[g_{xx}]^{\alpha}_{\beta\delta}[\eta]^{\delta}_{\xi}[\eta]^{\beta}_{\phi}[I]^{\phi}_{\xi} + [\mathcal{H}_{y'}]^i_{\alpha}[g_{\lambda\lambda}]^{\alpha} \\
& +[\mathcal{H}_{y}]^i_{\alpha}[g_{\lambda\lambda}]^{\alpha} + [\mathcal{H}_{x'}]^i_{\beta}[h_{\lambda\lambda}]^{\beta} \\
& +[\mathcal{H}_{x'y'}]^i_{\beta\gamma}[g_x]^{\gamma}_{\delta}[\eta]^{\delta}_{\xi}[\eta]^{\beta}_{\phi}[I]^{\phi}_{\xi} \\
& +[\mathcal{H}_{x'x'}]^i_{\beta\delta}[\eta]^{\delta}_{\xi}[\eta]^{\beta}_{\phi}[I]^{\phi}_{\xi} = 0; \\
i \quad = \quad & 1,\ldots,n;\alpha,\gamma = 1,\ldots,n_y;\beta,\delta = 1,\ldots,n_x;\phi,\xi = 1,\ldots,n_{\epsilon}
\end{aligned}
$$

a system of n linear equations in the n unknowns given by the elements of $g_{\lambda\lambda}$ and $h_{\lambda\lambda}$.

Cross-derivatives

- The cross derivatives $g_{x\lambda}$ and $h_{x\lambda}$ are zero when evaluated at $(\bar{x}, 0)$.
- Why? Write the system $F_{\lambda x}(\bar{x}; 0) = 0$ taking into account that all terms containing either g_λ or h_λ are zero at $(\bar{x}, 0)$.
- Then:

$$[F_{\lambda x}(\bar{x}; 0)]^i_j = [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\lambda x}]^\beta_j + [\mathcal{H}_{y'}]^i_\alpha [g_{\lambda x}]^\alpha_\gamma [h_x]^\gamma_j +$$
$$[\mathcal{H}_y]^i_\alpha [g_{\lambda x}]^\alpha_j + [\mathcal{H}_{x'}]^i_\beta [h_{\lambda x}]^\beta_j = 0;$$
$$i = 1, \ldots n; \quad \alpha = 1, \ldots, n_y; \quad \beta, \gamma, j = 1, \ldots, n_x.$$

- This is a system of $n \times n_x$ equations in the $n \times n_x$ unknowns given by the elements of $g_{\lambda x}$ and $h_{\lambda x}$.
- The system is homogeneous in the unknowns.
- Thus, if a unique solution exists, it is given by:

$$\begin{aligned} g_{\lambda x} &= 0 \\ h_{\lambda x} &= 0 \end{aligned}$$

Structure of the solution

- The perturbation solution of the model satisfies:

$$
\begin{aligned}
g_{\lambda}(\bar{x};0) &= 0 \\
h_{\lambda}(\bar{x};0) &= 0 \\
g_{x\lambda}(\bar{x};0) &= 0 \\
h_{x\lambda}(\bar{x};0) &= 0
\end{aligned}
$$

- Standard deviation only appears in:

  1. A constant term given by $\frac{1}{2}g_{\lambda\lambda}\lambda^2$ for the control vector y_t.

  2. The first $n_x - n_\epsilon$ elements of $\frac{1}{2}h_{\lambda\lambda}\lambda^2$.

- Correction for risk.

- Quadratic terms in endogenous state vector x_1.

- Those terms capture non-linear behavior.

Higher-order approximations

- We can iterate this procedure as many times as we want.
- We can obtain n-th order approximations.
- Problems:

  1. Existence of higher order derivatives (Santos, 1992).
  2. Numerical instabilities.
  3. Computational costs.