Linearization

(Lectures on Solution Methods for Economists V: Appendix)

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Basic RBC

- Benchmark set up:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t + \psi \log \left(1 - l_t\right) \right\}$$

$$\begin{aligned} c_t + k_{t+1} &= k_t^{\alpha} \left(e^{z_t} l_t\right)^{1-\alpha} + \left(1 - \delta\right) k_t, \ \forall \ t > 0 \\ z_t &= \rho z_{t-1} + \varepsilon_t, \ \varepsilon_t \sim \mathcal{N}(0, \sigma) \end{aligned}$$

- This is a dynamic optimization problem.
- The previous problem does not have a "paper and pencil" solution.
- Traditional solution: linearization.

Equilibrium conditions

- From the household problem+firms's problem+aggregate conditions:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left\{ \frac{1}{c_{t+1}} \left(1 + \alpha k_{t+1}^{\alpha-1} \left(e^{z_{t+1}} l_{t+1} \right)^{1-\alpha} - \delta \right) \right\}$$

$$\psi \frac{c_t}{1 - l_t} = (1-\alpha) k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} l_t^{-1}$$

$$c_t + k_{t+1} = k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

- Do we substitute first?

Steady state I

- If $\sigma = 0$, the equilibrium conditions are:

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} \left(1 + \alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} - \delta\right)$$
$$\psi \frac{c_t}{1 - l_t} = (1 - \alpha) k_t^{\alpha} l_t^{-\alpha}$$
$$c_t + k_{t+1} = k_t^{\alpha} l_t^{1-\alpha} + (1 - \delta) k_t$$

- The equilibrium conditions imply a steady state:

$$\frac{1}{c} = \beta \frac{1}{c} \left(1 + \alpha k^{\alpha-1} l^{1-\alpha} - \delta\right)$$
$$\psi \frac{c}{1 - l} = (1 - \alpha) k^{\alpha} l^{-\alpha}$$
$$c + \delta k = k^{\alpha} l^{1-\alpha}$$

Steady state II

Solution:

$$
\begin{aligned}
k &= \frac{\mu}{\Omega + \varphi\mu} \\
l &= \varphi k \\
c &= \Omega k \\
y &= k^{\alpha} l^{1-\alpha}
\end{aligned}
$$

where $\varphi = \left(\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right)^{\frac{1}{1-\alpha}}$, $\Omega = \varphi^{1-\alpha} - \delta$, and $\mu = \frac{1}{\psi}\left(1-\alpha\right)\varphi^{-\alpha}$.

Linearization I

- Loglinearization or linearization?
- Loglinearization:
  1. Take variable x_t and substitute by $xe^{\widehat{x}_t}$ where:
  $$\widehat{x}_t = \log \frac{x_t}{x}$$
  2. A variable $\widehat{x}_t$ represents the log-deviation with respect to the steady state.
  3. Linearize with respect to $\widehat{x}_t$.
- Advantages and disadvantages.
- We can linearize and perform later a change of variables.

Linearization II

We linearize:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left\{ \frac{1}{c_{t+1}} \left(1 + \alpha k_{t+1}^{\alpha-1} \left(e^{z_{t+1}} l_{t+1} \right)^{1-\alpha} - \delta \right) \right\}$$

$$\psi \frac{c_t}{1 - l_t} = (1-\alpha) k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} l_t^{-1}$$

$$c_t + k_{t+1} = k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

around l, k, and c with a First-order Taylor Expansion.

Linearization III

We get:

$$-\frac{1}{c}(c_t - c) = \mathbb{E}_t \left\{ \begin{array}{c} -\frac{1}{c}(c_{t+1} - c) + \alpha(1-\alpha)\beta\frac{y}{k}z_{t+1} + \\ \alpha(\alpha-1)\beta\frac{y}{k^2}(k_{t+1} - k) + \alpha(1-\alpha)\beta\frac{y}{kl}(l_{t+1} - l) \end{array} \right\}$$

$$\frac{1}{c}(c_t - c) + \frac{1}{(1-l)}(l_t - l) = (1-\alpha)z_t + \frac{\alpha}{k}(k_t - k) - \frac{\alpha}{l}(l_t - l)$$

$$(c_t - c) + (k_{t+1} - k) = \left\{ \begin{array}{c} y\left((1-\alpha)z_t + \frac{\alpha}{k}(k_t - k) + \frac{(1-\alpha)}{l}(l_t - l)\right) \\ +(1-\delta)(k_t - k) \end{array} \right\}$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

Rewriting the system I

$$\alpha_1 (c_t - c) = \mathbb{E}_t \{\alpha_1 (c_{t+1} - c) + \alpha_2 z_{t+1} + \alpha_3 (k_{t+1} - k) + \alpha_4 (l_{t+1} - l)\}$$

$$(c_t - c) = \alpha_5 z_t + \frac{\alpha}{k} c (k_t - k) + \alpha_6 (l_t - l)$$

$$(c_t - c) + (k_{t+1} - k) = \alpha_7 z_t + \alpha_8 (k_t - k) + \alpha_9 (l_t - l)$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

$$\alpha_1 = -\tfrac{1}{c} \qquad \alpha_2 = \alpha (1 - \alpha) \beta \tfrac{y}{k}$$

$$\alpha_3 = \alpha (\alpha - 1) \beta \tfrac{y}{k^2} \qquad \alpha_4 = \alpha (1 - \alpha) \beta \tfrac{y}{kl}$$

$$\alpha_5 = (1 - \alpha) c \qquad \alpha_6 = -\left(\tfrac{\alpha}{l} + \tfrac{1}{(1-l)}\right) c$$

$$\alpha_7 = (1 - \alpha) y \qquad \alpha_8 = y \tfrac{\alpha}{k} + (1 - \delta)$$

$$\alpha_9 = y \tfrac{(1-\alpha)}{l} \qquad y = k^\alpha l^{1-\alpha}$$

Rewriting the system II

- After some algebra the system is reduced to:

$$A\left(k_{t+1}-k\right)+B\left(k_t-k\right)+C\left(l_t-l\right)+Dz_t=0$$

$$\mathbb{E}_t\left(\begin{array}{c} G\left(k_{t+1}-k\right)+H\left(k_t-k\right)+J\left(l_{t+1}-l\right) \\ +K\left(l_t-l\right)+Lz_{t+1}+Mz_t \end{array}\right)=0$$

$$\mathbb{E}_t z_{t+1}=\rho z_t$$

- We have eliminated one control: c_t. This is not necessary in general:

  1. Policy functions that we find.
  2. Numerical differences.

- How do we solve this system of equations? Different yet equivalent approaches.

Undetermined coefficients

- We guess policy functions of the form

$$
\begin{aligned}
(k_{t+1} - k) &= P_1 (k_t - k) + P_2 z_t \\
(l_t - l) &= R_1 (k_t - k) + R_2 z_t
\end{aligned}
$$

- Plug them in, use linearity of expectation and

$$\mathbb{E}_t z_{t+1} = \rho z_t$$

to get:

$$A (P_1 (k_t - k) + P_2 z_t) + B (k_t - k) + C (R_1 (k_t - k) + R_2 z_t) + D z_t = 0$$

$$G (P_1 (k_t - k) + P_2 z_t) + H (k_t - k) + J (R_1 (P_1 (k_t - k) + P_2 z_t) + R_2 N z_t)$$

$$+ K (R_1 (k_t - k) + R_2 z_t) + (LN + M) z_t = 0$$

Solving the system I

- Since these equations need to hold for any value $(k_{t+1} - k)$ or z_t, we need to equate each coefficient to zero.
- Coefficients on $(k_t - k)$:

$$AP_1 + B + CR_1 = 0$$

$$GP_1 + H + JR_1 P_1 + KR_1 = 0$$

- Coefficients on z_t:

$$AP_2 + CR_2 + D = 0$$

$$(G + JR_1)\,P_2 + JR_2 N + KR_2 + LN + M = 0$$

Solving the system II

- We have a system of four equations on four unknowns.
- To solve it, first note that $R_1 = -\frac{1}{C}(AP_1 + B) = -\frac{1}{C}AP_1 - \frac{1}{C}B$
- Then:

$$P_1^2 + \left(\frac{B}{A} + \frac{K}{J} - \frac{GC}{JA}\right)P_1 + \frac{KB - HC}{JA} = 0$$

  a quadratic equation on P_1.

Solving the system III

- We have two solutions:

$$P_1 = -\frac{1}{2}\left(-\frac{B}{A} - \frac{K}{J} + \frac{GC}{JA} \pm \left(\left(\frac{B}{A} + \frac{K}{J} - \frac{GC}{JA}\right)^2 - 4\frac{KB - HC}{JA}\right)^{0.5}\right)$$

  one stable and another unstable.

- If we pick the stable root and find $R_1 = -\frac{1}{C}(AP_1 + B)$, we have to a system of two linear equations on two unknowns with solution:

$$P_2 = \frac{-D(JN + K) + CLN + CM}{AJN + AK - CG - CJR_1}$$
$$R_2 = \frac{-ALN - AM + DG + DJR_1}{AJN + AK - CG - CJR_1}$$

Practical implementation

- How do we do this in practice?
- Solving quadratic equations: “A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily” by Harald Uhlig.
- Using `dynare`.

General structure of linearized system

Given m states s_t, n controls y_t, and k exogenous stochastic processes z_{t+1}, we have:

$$
\begin{aligned}
As_t + Bs_{t-1} + Cy_t + Dz_t = 0 \\
\mathbb{E}_t \left(Fs_{t+1} + Gs_t + Hs_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t \right) = 0 \\
\mathbb{E}_t z_{t+1} = Nz_t
\end{aligned}
$$

where C is of size $l \times n$, $l \geq n$ and of rank n, F is of size $(m+n-l) \times n$, and that N has only stable eigenvalues.

Policy functions I

We guess policy functions of the form:

$$s_t = Ps_{t-1} + Qz_t$$
$$y_t = Rs_{t-1} + Uz_t$$

where P, Q, R, and U are matrices such that the computed equilibrium is stable.

Policy functions II

For simplicity, suppose $l = n$ (standard case, see Uhlig's chapter for the general case). Then:

1. P satisfies the matrix quadratic equation:

$$\left(F - JC^{-1}A\right)P^2 - \left(JC^{-1}B - G + KC^{-1}A\right)P - KC^{-1}B + H = 0$$

   The equilibrium is stable iff $\max\left(abs\left(eig\left(P\right)\right)\right) < 1$.

2. R is given by:

$$R = -C^{-1}\left(AP + B\right)$$

3. Q satisfies:

$$N' \otimes \left(F - JC^{-1}A\right) + I_k \otimes \left(JR + FP + G - KC^{-1}A\right) vec\left(Q\right) = vec\left(\left(JC^{-1}D - L\right)N + KC^{-1}D - M\right)$$

4. U satisfies:

$$U = -C^{-1}\left(AQ + D\right)$$

How to solve quadratic equations

To solve for the $m \times m$ matrix P in

$$\Psi P^2 - \Gamma P - \Theta = 0$$

1. Define the $2m \times 2m$ matrices:

$$\Xi = \begin{bmatrix} \Gamma & \Theta \\ I_m & 0_m \end{bmatrix}, \text{ and } \Delta = \begin{bmatrix} \Psi & 0_m \\ 0_m & I_m \end{bmatrix}$$

2. Let s be the generalized eigenvector and λ be the corresponding generalized eigenvalue of Ξ w.r.t. Δ. Then, we can write $s' = [\lambda x', x']$ for some $x \in \mathbb{R}^m$.
3. If $\exists$ m generalized eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$ with generalized eigenvectors $s_1, ..., s_m$ of Ξ w.r.t. Δ, written as $s' = [\lambda x_i', x_i']$ for some $x_i \in \mathbb{R}^m$ and if $(x_1, ..., x_m)$ is linearly independent, then:

$$P = \Omega \Lambda \Omega^{-1}$$

is a solution to the matrix quadratic equation where $\Omega = [x_1, ..., x_m]$ and $\Lambda = [\lambda_1, ..., \lambda_m]$. The solution of P is stable if $\max |\lambda_i| < 1$. Conversely, any diagonalizable solution P can be written in this way.