HA Models in Continuous Time

Jesús Fernández-Villaverde[1]

February 11, 2022

[1]University of Pennsylvania

Why continuous time?

Solving HA Models = Solving systems of PDEs

- A system of two PDEs:
  1. Hamilton-Jacobi-Bellman equation for individual choices.
  2. Kolmogorov forward equation for evolution of distribution.
- Many well-developed methods in applied math for analyzing and solving these equations. Furthermore, active area of new research.
- Apparatus is very general: applies to any heterogeneous agent model with continuum of atomistic agents:
  1. Heterogeneous households (Aiyagari, Bewley, Huggett,...).
  2. Heterogeneous producers (Hopenhayn,...).
- Can be extended to handle aggregate shocks (Krusell-Smith).

Computational advantages of continuous time

1. Borrowing constraints only show up in boundary conditions:
   - FOCs always hold with "$=$".
2. "Tomorrow is today":
   - FOCs are "static," compute by hand: $c^{-\sigma} = v_a'(a, y)$.
3. Sparsity:
   - Solutions require inverting matrices: very sparse ("tridiagonal").
   - Reason: continuous time $\Rightarrow$ one step left or one step right.
4. Two birds with one stone:
   - Tight link between solving (HJB) and (KF) for distribution.
   - Matrix in discrete (KF) is transpose of matrix in discrete (HJB).
   - Reason: diff. operator in (KF) is adjoint of operator in (HJB).

Extensions to more general models

- Non-convexities.
- Stopping time problems (e.g., search and matching, Ss, ...).
- Multiple assets.
- Financial frictions.
- Aggregate shocks through linearization.

Optimal control

Optimal control

- Standard deterministic optimal control problem in continuous time:

$$v(x_0) = \max_{\{\alpha(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} h(x(t), \alpha(t))\, dt$$

  subject to the law of motion for the state:

$$\dot{x}(t) = f(x(t), \alpha(t)) \quad \text{and} \quad \alpha(t) \in A$$

  for $t \geq 0$, $x(0) = x_0$ given.

- $\rho \geq 0$: discount rate.
- $x \in X \subseteq \mathbb{R}^m$: state vector.
- $\alpha \in A \subseteq \mathbb{R}^n$: control vector.
- $h : X \times A \to \mathbb{R}$: instantaneous return function.

Consumption-savings problem

- Sequence formulation of consumption savings problem

$$\max_{c_t} \int_{t=0}^{\infty} e^{-\rho t} u(c(t))$$

subject to

$$\dot{a}(t) = ra(t) + y(t) - c(t)$$

$$a(t) \geq \underline{a}$$

$$a(0) \text{ given}$$

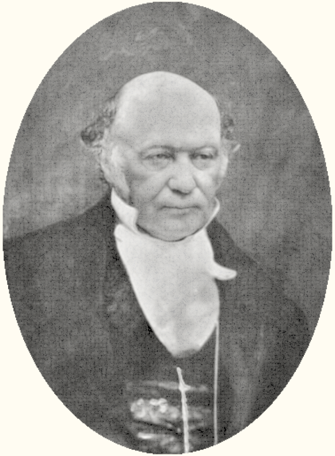
$$\lim_{t\to\infty} a(t) \geq 0$$

- Solution consists of time paths for consumption $c(t)$ and assets $a(t)$.
- Note that FOCs are not the solution of the problem, just a property of the solution. Thus, we need to work beyond finding the FOCs.

Optimization in continuous time

- We are interested in optimization in continuous time, both in deterministic and stochastic environments.
- Elegant and powerful math (differential equations, stochastic processes...).
- Three approaches:
    1. Calculus of Variations.
    2. Hamiltonians.
    3. Dynamic Programming.
- We will focus on dynamic programming.
    1. It can do everything economists need from calculus of variations.
    2. It is better than Hamiltonians for the stochastic case.

Hamilton-Jacobi-Bellman



William Hamilton

Carl Jacobi

Richard Bellman

Recursive formulation

- Recursive formulation known as Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho V(a) = \max_c u(c) + V'(a)\dot{a}$$

$$\text{subject to}$$

$$\dot{a} = ra + y - c$$

$$a \geq \underline{a}$$

- Solution to the HJB is a value function $V(a)$, a policy function $c(a)$, and a policy function for savings $\dot{a} = s(a)$ (not a path for optimal choices, as in a Hamiltonian).
- Note time invariance of these functions.
- When will the functions be time-dependent?

Viscosity solutions, I

- Relevant notion of "solutions" to HJB introduced by Pierre-Louis Lions and Michael G. Crandall in 1983 in the context of PDEs.
- Classical solution of a PDE:

$$F(x, u, Du, D^2u) = 0$$

  is a function u in Ω that is continuous and differentiable that satisfies the PDE above.
- We want a weaker class of solutions than classical solutions.
- More concretely, we want to allow for points of non-differentiability of u (in this case, $V(a)$).
- Similarly, we want to allow for convex kinks in the value function $V(a)$.
- Different classes of "weaker solutions."

Viscosity solutions, II

- Subsolution: An upper semicontinuous function u in Ω is a "subsolution" of a PDE in the "viscosity sense" if for any point $x_0 \in \Omega$ and any C^2 function ϕ such that $\phi(x_0) = u(x_0)$ and $\phi \geq u$ in a neighborhood of x_0, we have:

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0$$

- Supersolution: A lower semicontinuous function u in Ω is defined to be a "supersolution" of a PDE in the "viscosity sense" if for any point $x_0 \in \Omega$ and any C^2 function ϕ such that $\phi(x_0) = u(x_0)$ and $\phi \leq u$ in a neighborhood of x_0, we have:

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$$

- Viscosity solution: A continuous function "u" is a "viscosity solution" of the PDE if it is both a supersolution and a subsolution.

Viscosity solutions, III

- Viscosity solution is unique.
- A baby example: consider the boundary value problem $F(u') = |u'| - 1 = 0$, on $(-1, 1)$ with boundary conditions $u(-1) = u(1) = 0$. The unique viscosity solution is the function $u(x) = 1 - |x|$.
- Coincides with solution to sequence problem.
- Numerical methods designed to find viscosity solutions.
- Check, for more background, *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations* by Michael G. Crandall, Hitoshi Ishii, and Pierre-louis Lions.
- Also, *Controlled Markov Processes and Viscosity Solutions* by Wendell H. Fleming and Halil Mete Soner.

Derivation of the HJB

- Discrete time BE with period length Δ, $\beta(\Delta) = e^{-\rho\Delta}$:

$$V(a_t) = \max_{c_t} u(c_t)\Delta + e^{-\rho\Delta} V(a_{t+\Delta})$$

subject to

$$c_t\Delta + a_{t+\Delta} \leq (1 + r\Delta)a_t + y_t\Delta$$

- Subtract $(1 - \rho\Delta)V(a_t)$ from both sides and since for $\Delta \approx 0$, $e^{-\rho\Delta} \approx 1 - \rho\Delta$.

$$\rho\Delta V(a_t) = \max_{c_t} u(c_t)\Delta + (1 - \rho\Delta)\left[V(a_{t+\Delta}) - V(a_t)\right]$$

- Divide both sides by Δ and re-arrange:

$$\rho V(a_t) = \max_{c_t} u(c_t) + (1 - \rho\Delta)\frac{V(a_{t+\Delta}) - V(a_t)}{a_{t+\Delta} - a_t}\frac{a_{t+\Delta} - a_t}{\Delta}$$

- Take $\Delta \to 0$:

$$\rho V(a_t) = \max_{c_t} u(c_t) + V'(a_t)\,\dot{a}_t$$

Optimality conditions

- HJB is:

$$\rho V(a) = \max_c u(c) + V'(a)[ra + y - c]$$

- FOC for c is:

$$u'(c) = V'(a)$$

- We can also get a continuous-time Euler Equation:

$$\frac{\dot{c}}{c} = \sigma(c)(r - \rho)$$

where $\sigma(c) \equiv -\frac{u'(c)}{cu''(c)}$ is EIS.

- And a borrowing constraint imposed via state constraint:

$$V'(\underline{a}) \geq u'(r\underline{a} + y)$$

Derivation of Euler Equation

- Envelope condition: differentiate HJB with respect to assets a:

$$\rho V'(a) = V'(a)r + V''(a)\dot{a}$$

- Differentiate FOC for c with respect to t:

$$u''(c)\dot{c} = V''(a)\dot{a}$$

- Substitute FOC and $\frac{d}{dt}$FOC into envelope condition:

$$(\rho - r)u'(c) = u''(c)\dot{c}$$

- Divide by c and rearrange:

$$\frac{\dot{c}}{c} = -\frac{u'(c)}{cu''(c)}(r - \rho) = \sigma(r - \rho)$$

Derivation of borrowing constraint

- At $a = \underline{a}$, savings must be non-negative:

$$\dot{a} \geq 0$$

- From the budget constraint, this implies:

$$c \leq r\underline{a} + y$$

- Applying u' to both sides:

$$u'(c) \geq u'(r\underline{a} + y)$$

- And using the FOC for consumption at equality:

$$V'(\underline{a}) \geq u'(r\underline{a} + y)$$

Poisson process for income

- Focus on Poisson process for income $y_j \in \{y_1, \ldots, y_J\}$.
- Hazard of switching from state j to j' is $\lambda_{jj'}$.
- General Markov transition matrix for J states is:

$$\Lambda = \begin{bmatrix} -\sum_{i \neq 1} \lambda_{1i} & \lambda_{12} & \ldots & \lambda_{1J} \\ \lambda_{21} & -\sum_{i \neq 2} \lambda_{2i} & \ldots & \lambda_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{J1} & \lambda_{J2} & \ldots & -\sum_{i \neq J} \lambda_{Ji} \end{bmatrix}$$

- With $J = 3$ transition matrix is:

$$\Lambda = \begin{bmatrix} -\lambda_{12} - \lambda_{13} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & -\lambda_{21} - \lambda_{23} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & -\lambda_{31} - \lambda_{32} \end{bmatrix}$$

- Alternatives?

Income distribution for Poisson process

- Measure of individuals in each income state is a $J \times 1$ vector $g(t)$.
- Distribution $g(t)$ evolves according to:

$$\dot{g}(t) = \Lambda' g(t)$$

- Stationary distribution therefore satisfies:

$$\Lambda' g = 0$$

HJB with Poisson income

- HJB with Poisson income process:

$$\rho V(a, y_j) = \max_c u(c) + V_a(a, y_j)\,\dot{a} + \sum_{j' \neq j} \underbrace{\lambda_{jj'}}_{Pr(y_{j'}|y_j)} \underbrace{[V(a, y_{j'}) - V(a, y_j)]}_{\Delta utility\ from\ switching\ states}$$

subject to

$$\dot{a} = ra + y_j - c$$

- Associated stochastic continuous-time Euler equation:

$$\frac{\dot{c}}{c} = \sigma(c)\left(r - \rho + \sum_{j' \neq j} \lambda_{jj'}\left[\frac{u'(c; y_{j'})}{u'(c)} - 1\right]\right)$$

where $\sigma(c) \equiv \frac{-u'(c)}{cu''(c)}$.

Finite difference methods

Discretized HJB

- Define assets grid $\mathcal{A} = \{a_1, \ldots, a_N\}$ with $a_1 = \underline{a}$. How do we discretize?
- Denote grid spacing between point $i-1$ and i as Δa_i.
- HJB at each grid point a_i is:

$$\rho V_i = \max_c u(c) + V_i^{'} [ra_i + y - c]$$

- Substitute FOC for consumption, $c = u'^{-1}(V_i')$:

$$\rho V_i = u(u'^{-1}(V_i')) + V_i^{'} \left[ra_i + y - u'^{-1}(V_i')\right]$$

- System of non-linear equations in (V_i, V_i').
- At borrowing constraint $\underline{a} = a_1$, also require:

$$V_1' \geq u'(ra_1 + y)$$

Power-spaced grids

- Policy functions are typically very non-linear close to the borrowing constraint, yet very linear when away from it.
- Thus with linear interpolation, we need more grid points close to the constraint for accuracy, which can be achieved with power-space grids.
- Let $[\underline{a}, \bar{a}]$ be the possible range of asset holdings. Let $\mathcal{Z}$ be an equi-spaced grid on $[0, 1]$.
- For each grid point $z \in \mathcal{Z}$, define $x = z^{\alpha}$ for some $\alpha \in (1, \infty)$ to create a non-linear spaced grid $\mathcal{X}$ on $[0, 1]$. Notice that as $\alpha \to \infty$, $\mathcal{X}$ has more and more points closer to 0.
- We can create the asset grid $\mathcal{A}$ by rescaling each $x \in \mathcal{X}$

$$a = \underline{a} + (\bar{a} - \underline{a})x$$

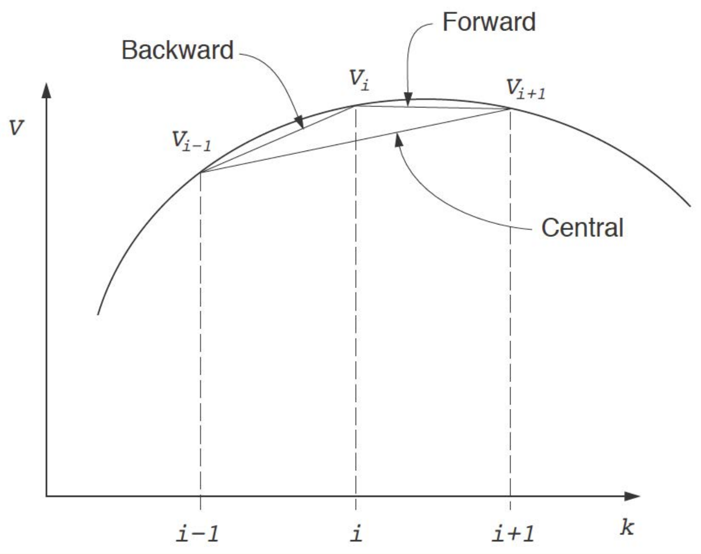
Finite difference approximations of V'

- FD approximation converts to system of non-linear equations in V_i only.
- Three possible FD approximations of V_i':

$$V_i' \approx \frac{V_i - V_{i-1}}{\Delta a_i} = V_{iB}' \qquad \text{backward difference}$$
$$V_i' \approx \frac{V_{i+1} - V_i}{\Delta a_{i+1}} = V_{iF}' \qquad \text{forward difference}$$
$$V_i' \approx \frac{V_{i+1} - V_{i-1}}{\Delta a_i + \Delta a_{i+1}} = V_{iC}' \qquad \text{central difference}$$

- Alternative: complex-step differentiation.

Finite difference approximation of V'



Upwinding

- Optimal savings with forward difference approximation:

$$s_{iF} \equiv ra_i + y - u'^{-1}(V'_{iF})$$

- Optimal savings with backward difference approximation:

$$s_{iB} \equiv ra_i + y - u'^{-1}(V'_{iB})$$

- If $s_{iF} > 0$, $s_{iB} \geq 0$:

$$V'_i = V'_{iF}, \quad s_i = s_{iF} \implies \dot{a} > 0, \quad c_{iF} = u'^{-1}(V'_{iF})$$

- If $s_{iF} \leq 0$, $s_{iB} < 0$:

$$V'_i = V'_{iB}, \quad s_i = s_{iB} \implies \dot{a} < 0, \quad c_{iB} = u'^{-1}(V'_{iB})$$

- If $s_{iF} \leq 0$, $s_{iB} \geq 0$:

$$s_i = 0 \implies \dot{a} = 0, \quad c_{i0} = ra_i + y, \quad V'_i = u'(ra_i + y)$$

Convex points

- What if $s_{iF} > 0$, $s_{iB} < 0$?
- Implies $V(a)$ is convex at $a = a_i$:

$$
\begin{aligned}
u'^{-1}\left(V_{iF}^{'}\right) < ra_i + y < u'^{-1}\left(V_{iB}^{'}\right) \\
\implies u'^{-1}\left(V_{iF}^{'}\right) < u'^{-1}\left(V_{iB}^{'}\right) \\
\implies V_{iF}^{'} > V_{iB}^{'}
\end{aligned}
$$

- Choose direction with highest Hamiltonian:

$$H_i \equiv u\left(c_i\right) + V_i^{'}\left[ra_i + y - c_i\right]$$

- Should also check whether $\dot{a} = 0$ gives a higher Hamiltonian.

FD approximation to HJB

- Define indicators variables for directions implied by upwinding.
- At non-convex points:
  - $I_{iF} = 1 \iff s_{iF} > 0 \implies s_i = s_{iF}$
  - $I_{iB} = 1 \iff s_{iB} < 0 \implies s_i = s_{iB}$
  - $I_{i0} = 1 \iff s_{iB} \geq 0, \; s_{iF} \leq 0 \implies s_i = 0$
- Discretized HJB as:

$$\begin{aligned}\rho V_i &= u(c_i) + \frac{V_{i+1} - V_i}{\Delta a_{i+1}} I_{iF} s_i + \frac{V_i - V_{i-1}}{\Delta a_i} I_{iB} s_i \\ &= u(c_i) - \frac{I_{iB} s_i}{\Delta a_i} V_{i-1} - \left(\frac{I_{iF} s_i}{\Delta a_{i+1}} - \frac{I_{iB} s_i}{\Delta a_i} \right) V_i + \frac{I_{iF} s_i}{\Delta a_{i+1}} V_{i+1}\end{aligned}$$

- In matrix notation:

$$\rho V = u + \mathbf{A} V$$

where $\mathbf{A}$ is $N \times N$ matrix.

The A matrix

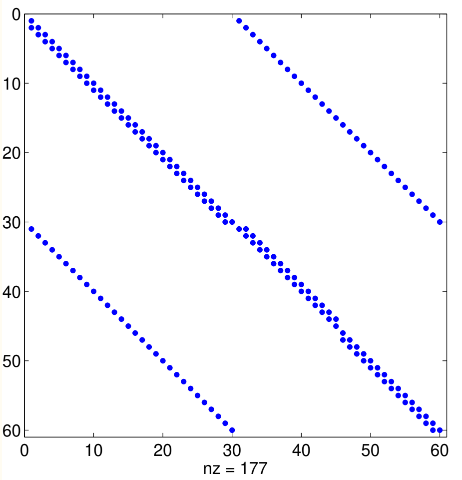
- FD method approximates process for k with discrete Poisson process, $\mathbf{A}$ summarizes Poisson intensities:
  - entries in row i:

$$\begin{bmatrix} \underbrace{-\frac{l_{iB} s_i}{\Delta k}}_{\text{inflow}_{i-1} \geq 0} & \underbrace{\frac{l_{iB} s_i}{\Delta k} - \frac{l_{iF} s_i}{\Delta k}}_{\text{outflow}_i \leq 0} & \underbrace{\frac{l_{iF} s_i}{\Delta k}}_{\text{inflow}_{i+1} \geq 0} \end{bmatrix} \begin{bmatrix} v_{i-1} \\ v_i \\ v_{i+1} \end{bmatrix}$$

  - negative diagonals, positive off-diagonals, rows sum to zero.
  - tridiagonal matrix, very sparse.
- $\mathbf{A}$ depends on v (nonlinear problem).
- Two iterative methods for solving $\rho V = u(V) + \mathbf{A}(\mathbf{V})V$.

A matrix structure

A is very sparse (only tridiagonal).



Explicit updating: Basic idea

- Start with guess V_0 for V.
- Use shorthand notation:

$$\begin{aligned} u_l &= u(V_l) \\ \mathbf{A}_l &= \mathbf{A}(V_l) \end{aligned}$$

- Update V_{l+1} from V_l:

$$\rho V_{l+1} = u_l + \mathbf{A}_l V_l \Rightarrow$$
$$V_{l+1} = \frac{u_l}{\rho} + \frac{1}{\rho}\mathbf{A}_l V_l$$

- Problem: this is not a contraction since $\frac{1}{\rho}$ is typically above 1.

Explicit updating

- Use partial updating to ensure convergence, for $\omega \in (0, 1)$:

$$V_{l+1} = (1 - \omega) V_l + \omega \left[\frac{u_l}{\rho} + \frac{1}{\rho} \mathbf{A}_l V_l \right]$$

- Usually, the step size is $\Delta \equiv \frac{\omega}{\rho}$ and updating rule becomes:

$$V_{l+1} = (1 - \Delta\rho) V_l + \Delta [u_l + \mathbf{A}_l V_l] \Rightarrow$$
$$V_{l+1} = \Delta u_l + [\mathbf{I} + (\mathbf{A}_l - \rho\mathbf{I}) \Delta] V_l$$

which can be arranged as:

$$V_{l+1} = \Delta u_l + \left[\left(\frac{1}{\Delta} - \rho \right) \mathbf{I} + \mathbf{A}_l \right] \Delta V_l$$

which is a simple matrix multiplication operation.

Explicit updating as forward iteration

- Explicit updating is sometimes referred to as forward time iteration.
- To see this, re-write updating rule as:

$$\frac{V_{l+1} - V_l}{\Delta} + \rho V_l = u_l + \mathbf{A}_l V_l$$

  which is a discretized version of:

$$\dot{V} + \rho V = u + AV$$

  if we re-define the steps l as time t.

- The iterative rule starts at $t = l$ and moves forward to $t = l + 1$.
- Conditional stability: only converges for sufficiently low Δ.

Implicit updating

- Start with guess V_0 for V.
- Use V_{l+1} rather than V_l in HJB wherever possible:

$$\frac{V_{l+1} - V_l}{\Delta} + \rho V_{l+1} = u_l - \mathbf{A}_l V_{l+1}$$

$$\left[\left(\rho + \frac{1}{\Delta}\right)\mathbf{I} - \mathbf{A}_l\right] V_{l+1} = u_l + \frac{V_l}{\Delta},$$

- Update V_{l+1} from V_l by solving linear system:

$$V_{l+1} = \left[\left(\rho + \frac{1}{\Delta}\right)\mathbf{I} + \mathbf{A}_l\right] \backslash \left(u_l + \frac{V_l}{\Delta}\right)$$

- Also known as backward-time iteration.
- Unconditional stability: converges for any $\Delta > 0$, so very fast.

Intuition for performance of implicit updating

- Consider linear ODE

$$\dot{y}(t) = -\alpha y(t) \text{ with } \alpha > 0$$

  and initial condition $y(0) = 1$ and solution $y(t) = e^{-\alpha t}$.

- Approximating $y(\Delta)$ with explicit method:

$$\frac{y(\Delta) - y(0)}{\Delta} = -\alpha y(0)$$

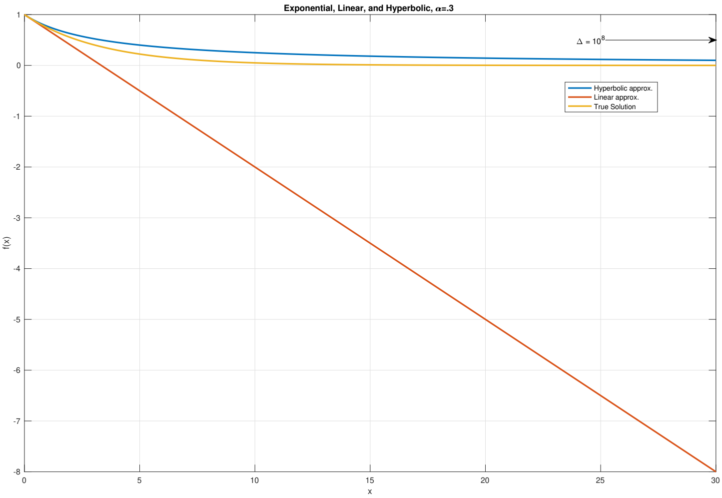
$$y(\Delta) = 1 - \alpha\Delta \quad \textbf{linear approximation}$$

- Approximating $y(\Delta)$ with implicit method:

$$\frac{y(\Delta) - y(0)}{\Delta} = -\alpha y(\Delta)$$

$$y(\Delta) = \frac{1}{1 + \alpha\Delta} \quad \textbf{hyperbolic approximation}$$

Linear vs. hyperbolic approximations to exponential



Implicit updating with Poisson income

- Discretized value function is matrix with entries corresponding to (a_i, y_j):

$$\rho V_{ij} = u(c_{ij}) - \frac{I_{iB} s_i}{\Delta a_i} V_{i-1} - \left(\frac{I_{iF} s_i}{\Delta a_{i+1}} - \frac{I_{iB} s_i}{\Delta a_i} \right) V_i + \frac{I_{iF} s_i}{\Delta a_{i+1}} V_{i+1}$$
$$+ \sum_{j' \neq j} \lambda_{jj'} \left[V_{ij'} - V_{ij} \right]$$

- V can be vectorized into $NJ \times 1$ vector.
- Combine $\mathbf{A}$ and Λ matrices to create $NJ \times NJ$ matrix.

Barles-Souganidis

Why does the method work?

- Well-developed theory for numerical solution of HJB equation using finite difference methods.
- Barles and Souganidis (1991), "Convergence of approximation schemes for fully nonlinear second order equations."
- Result: finite difference scheme converges to unique viscosity solution under three conditions
  1. Monotonicity.
  2. Consistency.
  3. Stability.
- Good reference: Tourin (2013), *An Introduction to Finite Difference Methods for PDEs in Finance*.

Barles-Souganidis conditions

1. Monotonicity: the numerical scheme is monotone, that is S is non-increasing in both V_{i-1} and V_{i+1}.

2. Consistency: the numerical scheme is consistent, that is for every smooth function v with bounded derivatives:

$$S\left(\Delta a, a_i, V(k_i); V(a_{i-1}), v(a_{i+1})\right) \to G(V(a), V'(a), v''(a))$$

   as $\Delta a \to 0$ and $a_i \to a$.

3. Stability: the numerical scheme is stable, that is for every $\Delta a > 0$, it has a solution $V_i, i = 1, .., I$, which is uniformly bounded independently of Δa.

Barles-Souganidis theorem

Theorem

If the scheme (S) satisfies the monotonicity, consistency, and stability conditions 1 to 3, then as $\Delta a \to 0$ its solution $V_i, i = 1, ..., I$ converges locally uniformly to the unique viscosity solution of (G).

- Convergence here has nothing to do with iterative algorithm converging to fixed point.
- Instead: convergence of $V_i \to V$ as $\Delta ka \to 0$. More momentarily.

Intuition for monotonicity condition

- Write (S) as:

$$\rho V_i = \tilde{S}(\Delta a, a_i, V_i; V_{i-1}, V_{i+1})$$

- For example, in consumption-savings model:

$$\tilde{S}(\Delta a, a_i, a_i; V_{i-1}, V_{i+1}) = u(c_i) + \frac{V_{i+1} - V_i}{\Delta a}(ra_i + y - c_i)^+ + \frac{V_i - V_{i-1}}{\Delta a}(ra_i + y - c_i)^-$$

- Monotonicity: $\tilde{S} \uparrow$ in V_{i-1}, V_{i+1} ($\Leftrightarrow S \downarrow$ in V_{i-1}, V_{i+1}).
- Intuition: if my continuation value at $i-1$ or $i+1$ is larger, I must be at least as well off (i.e., V_i on LHS must be at least as high).

Checking monotonicity

- Recall upwind scheme:

$$S\left(\Delta a, a_i, V_i; V_{i-1}, V_{i+1}\right) = \rho V_i - u(c_i) - \frac{V_{i+1} - V_i}{\Delta a}(ra_i + y - c_i)^+ - \frac{V_i - V_{i-1}}{\Delta a}(ra_i + y - c_i)^-$$

- Can check that it satisfies monotonicity: S is indeed non-increasing in both V_{i-1} and V_{i+1}.
- c_i depends on V_i's but doesn't affect monotonicity due to envelope condition.

Meaning of "convergence"

- Convergence is about $\Delta a \to 0$.
- So what is content of theorem?
  1. System of I non-linear equations $S(\Delta a, a, V_i; V_{i-1}, V_{i+1}) = 0$.
  2. Theorem guarantees that as $\Delta a \to 0$, the solutions of (S) converge to solution the HJB equation (G).
     Theorem does not guarantee that (S) has solution for fixed Δa: stability assumption.

Why does iterative scheme work?

Two interpretations:

1. Newton method for solving system of non-linear equations (S).
2. Iterative scheme $\Leftrightarrow$ solve (HJB) backward in time.

$$\frac{V_i^{n+1} - V_i^n}{\Delta} + \rho V_i^n = u(c_i^n) + (V^n)'(a_i)(ra_i + y - c_i^n)$$

In effect, it sets $V(k, T) =$ initial guess and solves

$$\rho V(k, t) = \max_c \; u(c) + \partial_a V(a, t)(ra + y - c) + \partial_t V(a, t)$$

backwards in time. $V(a) = \lim_{t \to -\infty} V(a, t)$.

Stationary distribution

Kolmogorov forward equation

- Also known as the Fokker-Planck equation.
- Quite important in physics and population genetics.
- Stationary distribution $g(a, y)$ solves KFE:

$$0 = -\partial_a \left[s\left(a, y_j\right) g\left(a, y_j\right) \right] - g\left(a, y_j\right) \sum_{j'=1}^{J} \lambda_{jj'} + \sum_{j'=1}^{J} \lambda_{j'j} g\left(a, y_{j'}\right)$$

- In the deterministic version:

$$0 = -\partial_a \left[s\left(a, y_j\right) g\left(a, y_j\right) \right]$$

Derivation of the deterministic KFE, I

- CDF: fraction of people with wealth below a at time t:

$$G(a, t) = \Pr(\tilde{a}_t \leq a)$$

- Over time period of length Δ, wealth evolves as:

$$\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta s(\tilde{a}_t)$$

- Fraction of people with wealth below a evolves as:

$$\begin{aligned} G(a, t+\Delta) &= \Pr(\tilde{a}_{t+\Delta} \leq a) \\ &= \Pr(\tilde{a}_t \leq a - \Delta s(a)) \\ &= G(a - \Delta s(a), t) \end{aligned}$$

- Intuition: The individuals with wealth $< a - \Delta s(a)$ at t, are the individuals who have wealth $< a$ at $t + \Delta$.

Derivation of the deterministic KFE, II

- Subtract $G(a,t)$ from both sides and divide by Δ:

$$\frac{G(a,t+\Delta)-G(a,t)}{\Delta} = \frac{G(a-\Delta s(a),t)-G(a,t)}{\Delta}$$

- Take the limit as $\Delta \to 0$:

$$\partial_t G(a,t) = -s(a)\partial_a G(a,t)$$

where we have used that:

$$\lim_{\Delta\to 0}\frac{G(a-\Delta s(a),t)-G(a,t)}{\Delta} = \lim_{x\to 0}\frac{G(a-x,t)-G(a,t)}{x}s(a) = -s(a)\partial_a G(a,t)$$

- Differentiate with respect to a, use $g(a,t)=\partial_a G(a,t)$ and set $\partial_t G(a,t)=0$:

$$0 = -\partial_a\left[s\left(a,y_j\right)g\left(a,y_j\right)\right]$$

Intuition for Poisson KFE

- Dynamics of marginal CDF $G(a, y_j)$ outside stationary distribtion

$$\frac{d}{dt} G_t\left(a, y_j\right) = -s\left(a, y_j\right) g\left(a, y_j\right) - G\left(a, y_j\right) \sum_{j'=1}^{J} \lambda_{jj'} + \sum_{j'=1}^{J} \lambda_{j'j} G\left(a, y_{j'}\right)$$

- Changes over time on LHS is due to:

  1. Agents with a assets might save (dis-save).
  2. Agents with y_j income hit by shocks and leave y_j.
  3. Agents with income y_j' hit by shocks that bring them to y_j.

- Differentiate with respect to a and set change to zero yields stationary KFE.

Solving the KFE

- Operator implied by KFE is adjoint of operator implied by HJB:

$$0 = \mathbf{A}' g$$

- Eigenvalue problem subject to normalization:

$$\sum_{a \in \mathcal{A}} \sum_{j=1}^{J} g_0(a_i, y_j) = 1$$

- In practice, iterate with implicit updating starting from g_0:

$$\dot{g} = \mathbf{A}' g$$

$$\frac{g_n - g_{n-1}}{\Delta} = \mathbf{A}' g_n$$

$$(I - \Delta \mathbf{A}') g_n = g_{n-1}$$

- Converges in handful of iterations for large Δ.

KFE for non-uniform grids

- Adjust discretized version of KFE to preserves mass.
- Use trapezoidal rule

$$\int_{\underline{a}}^{\overline{a}} f(a)\, g(a, y_j) \approx \sum_{i=1}^{N} f(a_i)\, g(a_i, y_j)\, \tilde{\Delta} a_i$$

$$\text{with} \quad \tilde{\Delta} a_i = \begin{cases} \frac{1}{2}\Delta a_2 & \text{if } i = 1 \\ \frac{1}{2}\left(\Delta a_i + \Delta a_{i+1}\right) & \text{if } i = 2, \ldots, N-1 \\ \frac{1}{2}\Delta a_N & \text{if } i = N \end{cases}$$

- Discretized KFE becomes:

$$\left(I - \Delta \mathbf{A}'\right) \tilde{g}_n = \tilde{g}_{n-1}$$

where $\tilde{g}_i \equiv g_i \tilde{\Delta} a_i$

Aiyagari Models

Aggregate savings for ergodic distribution

- For a given interest rate r, we can compute stationary distribution $g(a, y; r)$. Since g is a measure, it satisfies:

$$g(a, y) > 0, \qquad \sum_j \int_a g(a, y_j; r) da = 1$$

- Compute aggregate savings in stationary distribution:

$$A(r) = \sum_j \int_a a g(a, y_j; r) da$$

- When $r = -1$, no-one saves and $A(-1) = \bar{a}$.
- When $r = \beta^{-1} - 1$ or $r = \rho$, assets explode: $A(r) \to \infty$.

Super-martingale convergence, I

To see that asset holdings diverge when $r = \beta^{-1} - 1$.

- Super-martingale: sequence of random variables X_t such that $X_t \geq \mathbb{E}_t[X_{t+1}]$.
- Super-Martingale convergence theorem: If $X_t \geq 0$ is a non-negative super-martingale, then X_t converges *almost surely* to a random variable X with $\mathbb{E}(X) < \infty$.

Super-martingale convergence, II

- Suppose we have *IID* y_{it} and state variable x_{it}.
- From HJB, envelope theorem, and $\beta(1+r)=1$

$$V'(x_t) \geq \mathbb{E}_t[V'(x_{t+1})]$$

  and $V'(x_t) > 0$, so we have a non-negative super martingale.
- On any infinite path y^∞, $V'(x_t)(y^\infty)$ settles down to constant, possibly $\infty \implies x_t(y^\infty)$ settles down to a constant, possibly ∞.
- But from BC: $x_{t+1} = (1+r)(x_t - c_t) + y_{t+1}$ with y_{t+1} *IID* and random means $\forall t$ never settles down to finite value $\implies x_t(y^\infty) \to \infty$.
- True for all income histories, so $x_t \to \infty$ *almost surely*.

Precautionary savings

- Intuition for why savings diverge when $R = \beta^{-1}$ is precautionary savings.
- Households have three motives for saving in this model:
  1. Inter-temporal motive: difference between $1 + r$ and β.
  2. Smoothing motive: concavity of utility function.
  3. Precautionary motive: either (i) presence of occasionally binding borrowing constraint; or (ii) convexity of marginal utility of consumption.
- Precautionary motive: agents continue wanting to save even when inter-temporal motive is shutdown, i.e., $1 + r = \beta^{-1}$.
- Thus, for total assets to remain bounded, we require $r < \beta^{-1} - 1$.

Shape of aggregate savings function

- $A(r)$ is continuous if no discontinuity in underlying consumption-savings problem when varying r.
- If EIS> 1, then $A(r)$ is strictly increasing. But this is not a necessary condition.
- In general $A(r)$ need not be strictly increasing, but in almost all applications it is.
- While we need to numerically check that $A(r)$ is strictly increasing, knowing that most likely it is can help to build intuition.

Stationary equilibrium interest rate

- Stationary equilibrium interest rate r determined by equating demand and supply in the market for assets in the ergodic distribution of households.
- Since $A(r) \in [0, \infty)$ and continuous, an equilibrium will exist if the demand for assets is either constant or decreasing in the interest rate.
- The supply of assets depend on the type of model we are dealing with:
    1. Huggett model: private IOUs in zero net supply.
    2. Bewley model: money or bonds in positive net supply.
    3. Aiyagari model: capital in positive net supply.

Huggett model: Assets in zero net supply

- Aggregate savings is a vertical line.
- Equilibrium interest rate determined by market clearing condition $A(r) = 0$.
- Important that households are allowed to borrow, i.e., $\underline{a} < 0$.
- Compute by iterating on interest rate until convergence or using a one-dimensional equation solver.

Huggett Model: Equilibrium

A stationary recursive competitive equilibrium (RCE) is

1. Policy functions: $c(a, y), s(a, y), V(a, y)$.
2. Interest rate: r.
3. Distribution of households: $g(a, y)$.

such that:

1. Given r, the functions $c(a, y), s(a, y), V(a, y)$ solve the household problem, i.e., satisfies the HJB:

$$\rho V\left(a, y_j\right) = u\left(c(a, y_j)\right) + V_a\left(a, y_j\right)\left[ra + y_j - c(a, y_j\right] + \sum_{j'} \lambda_{jj'}\left[V\left(a, y_{j'}\right) - V\left(a, y_j\right)\right]$$

2. Given the savings policy function $s(a, y)$, the distribution $g(a, y)$ is stationary. i.e., satisfies the KFE:

$$0 = -\partial_a\left[s(a, y_j)g(a, y_j)\right] - g(a, y_j)\sum_{j'} \lambda_{jj'} + \sum_{j'} \lambda_{j'j} g(a, y_{j'})$$

3. Given the distribution $g(a, y)$, the market for asset clears:

$$\sum_j \int_a a g(a, y_j) da = 0$$

Bewley model: Assets in positive supply

- Government issues real bonds B, finances interest payments and govt spending G by collecting taxes according to tax function $\tau(a, y)$.
- Total tax revenues are:

$$T(r) = \sum_j \int_a \tau(a, y_j) g(a, y_j; r) da$$

- Government budget constraint: $G + rB = T(r)$.
- Market clearing condition $A(r) = B$.
- Computation with exogenous B: As in Huggett economy,determine $G(r) = T(r) - rB$ as residual, provided $G(r) \geq 0$.
- Computation with exogenous G: Solve $A(r) = \frac{T(r)-G}{r}$ and determined equilibrium B endogenously.

Monetary interpretation of Bewley model, I

- Replace government with monetary authority who issues an exogenous, possibly time-varying quantity of nominal assets M_t, i.e., money.
- Denote the price level at time t by P_t, then inflation rate is:

$$\pi_t = \frac{P_{t+1}}{P_t} - 1$$

  and real return on holding money $-\pi_t$.
- Stationary equilibrium must have a constant real interest rate, all stationary equilibria must have a constant inflation rate π.
- Market clearing condition in stationary equilibrium:

$$A(-\pi) = \frac{M_t}{P_t}$$

Monetary interpretation of Bewley model, II

- Since $A(\pi)$ is constant in a stationary equilibrium, real money supply $\frac{M_t}{P_t}$ must also be constant.
- So if money grows at exogenous rate μ_t, the price level must grow at the same rate, i.e. $\pi_t = \mu_t$.
- But since inflation must be constant, only constant money growth rules are consistent with a stationary equilibrium and, hence, $\pi = \mu$.
- Plugging into market clearing conditions yields:

$$A(-\mu) = \frac{M_0}{P_0}$$

which determines the initial price level as a function of the level and growth rate of money.

- Thus, (M_0, μ) uniquely pin down (P_0, π).

Aiyagari model: Capital

- Representative firm with CRS production technology $Y = K^{\alpha} L^{1-\alpha}$.
- Firm rents capital from households at rate r and hires efficiency units of labor at wage rate w:

$$r + \delta = \alpha \left(\frac{K}{L} \right)^{\alpha - 1}$$

$$w = (1 - \alpha) \left(\frac{K}{L} \right)^{\alpha}$$

which implies a one-to-one mapping between w and r:

$$w = (1 - \alpha) \left(\frac{r + \delta}{\alpha} \right)^{\frac{\alpha}{\alpha - 1}}$$

Market clearing

- Labor market clearing with exogenous labor supply:

$$L = \sum_j \int_a y_j g\left(a, y_j; r\right) da = \sum_j y_j \pi_j$$

- Capital market clearing:

$$\begin{aligned} A(r) &= K(r) \\ &= L\left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} \end{aligned}$$

Aiyagari Model: Equilibrium

A stationary Recursive Competitive Equilibrium (RCE) is

1. Policy functions: $c(a,y), s(a,y), V(a,y)$
2. Factor Demands: K, L
3. Prices: r,w
4. Distribution of households: $g(a,y)$

such that:

1. Given r, w, the functions $c(a,y), s(a,y), V(a,y)$ solve the household problem, i.e., satisfies the HJB:

$$\rho V(a,y_j) = u(c(a,y_j)) + V_a(a,y_j)[ra + y_j - c(a,wy_j] + \sum_{j'} \lambda_{jj'}\left[V(a,y_{j'}) - V(a,y_j)\right]$$

2. Given r, w, the factor demands K, L solve the firm FOC.
3. Given the savings policy function $s(a,y)$, the distribution $g(a,y)$ is stationary. i.e, satisfies the KFE:

$$0 = -\partial_a[s(a,y_j)g(a,y_j)] - g(a,y_j)\sum_{j'}\lambda_{jj'} + \sum_{j'}\lambda_{j'j}g(a,y_{j'})$$

4. Given the distribution $g(a,y)$, the markets for capital and labor clear:

$$\sum_j \int_a a g(a,y_j)da = K \qquad \sum_j y_j \pi_j = L$$

Computation of equilibrium

- Any non-linear equation solver can be used to solve: $A(r) = K(r)$
- Often useful to iterate on $\kappa \equiv \frac{K}{L}$:

$$\frac{A\left(\alpha\kappa^{\alpha-1} - \delta\right)}{L} = \kappa$$

  suggests updating rule:

$$\kappa_{l+1} = \omega \frac{A\left(\alpha\kappa_l^{\alpha-1} - \delta\right)}{L} + (1-\omega)\,\kappa_l$$

  where $\omega \in [0,1]$ is dampening parameter.

- Useful to normalize average labor efficiency so $Y = 1$:

$$1 = K^{\alpha} L^{1-\alpha}$$

$$1 = \kappa^{\alpha} L$$

$$\implies L = \mathbb{E}[y] = \kappa^{-\alpha}$$

Endogenous labor supply

- Aggregate labor supply:

$$H(r) = \sum_j \int_a y_j h(a, y_j; r)\, g(a, y_j; r)\, da$$

where $h(a, y_j; r)$ is the optimal hours policy function.

- Market clearing condition for capital-labor ratio:

$$\frac{A(r)}{H(r)} = \kappa(r)$$

$$\frac{A\left(\alpha\kappa^{\alpha-1} - \delta\right)}{H\left(\alpha\kappa^{\alpha-1} - \delta\right)} = \kappa$$

- Iterate on κ as previously:

$$\kappa_{l+1} = \omega \frac{A\left(\alpha\kappa_l^{\alpha-1} - \delta\right)}{H\left(\alpha\kappa_l^{\alpha-1} - \delta\right)} + (1-\omega)\kappa_l$$

MPCs and hand-to-mouth households

Marginal propensity to consume

- Discrete time: define MPC $\mathfrak{m}$ as:

$$\mathfrak{m}(a,y) = \frac{\partial c(a,y)}{\partial a}$$

For discrete change Δ:

$$\mathfrak{m}_{\Delta}(a,y) = \frac{c(a+\Delta,y) - c(a,y)}{\Delta}$$

- Continuous time: define consuption over period τ, $\tilde{c}_\tau$

$$\tilde{c}_\tau(a,y) = E\left[\int_0^\tau c(a_t,y_t)\,dt \,\middle|\, a_t = a, y_t = y\right]$$

Define MPC $\mathfrak{m}_\tau$ as:

$$\mathfrak{m}_\tau(a,y) = \frac{\partial \tilde{c}_\tau(a,y)}{\partial a}$$

For discrete change Δ:

$$\mathfrak{m}_{\tau,\Delta}(a,y) = \frac{\tilde{c}_\tau(a+\Delta,y) - \tilde{c}_\tau(a,y)}{\Delta}$$

Computation of continuous-time MPC

- Feynman-Kac formula: $\tilde{c}_\tau(a,y) = \Gamma(a,y,0)$ where

$$0 = c(a,y_j) + \Gamma_a(a,y_j,o)\,s(a,y_j) + \sum_{j'\neq j}\lambda_{jj'}\left[\Gamma(a,y_{j'},t) - \Gamma(a,y_j,t)\right] + \partial_t\Gamma(a,y_j,t)$$

  with terminal condition is $\Gamma(a,y_j,\tau) = 0$.

- Discretized version has same transition matrix as HJB and satisfies:

$$0 = c + \mathbf{A}\Gamma + \dot{\Gamma}$$

- Backward iteration with implicit updating yields:

$$0 = c + \mathbf{A}\Gamma_l + \frac{\Gamma_{l+1} - \Gamma_l}{\Delta t}$$

$$\Gamma_l = \left(\frac{\mathbf{I}}{\Delta t} - \mathbf{A}\right)^{-1}\left(c + \frac{\Gamma_{l+1}}{\Delta t}\right)$$

- Start iterations at $t = \tau$ with $\Gamma_{\frac{\tau}{\Delta t}} = 0$ and update.

Measuring MPCs

1. Revealed preference:
    - Natural experiments: fiscal stimulus payments, tax rebates, lottery winnings, mortgage modifications . . .
    - Transitory income shocks: statistical model and theory to extract unexpected component of regular income fluctuations.
2. Stated preference.

- Two ways measure consumption:
    - Survey data on consumption.
    - Back out from household budget constraint.

Stylized Facts on MPCs

1. Average MPC $>> r$. Av. quarterly MPC out of unexpected $\$200 - \$2,000$ windfall is between $15\% - 30\%$. Av. annual MPC is a bit larger: $20\% - 40\%$.
2. Heterogeneity and bi-modality. Two groups of households:
    - Group of responders: high MPCs around $\sim 50\%$ or more.
    - Group of non-responders: MPC ≈ 0.
3. Excess sensitivity. MPCs out of anticipated windfalls are very similar to MPCs out of actual windfalls.
4. Sign and size asymmetry.
    - People respond more to gains than to losses.
    - Some evidence that larger windfalls generate larger responses.

MPC for high-wealth households

- Discrete time: with CRRA utility MPC approaches:

$$\lim_{a \to \infty} \mathfrak{m}(a, y) \approx R\left(\beta R\right)^{-\frac{1}{\gamma}} - 1$$

- Continuous time: with CRRA utility MPC approaches:

$$\lim_{a \to \infty} \mathfrak{m}_\tau(a, y) \approx \tau\left(\frac{\rho - r}{\gamma} + r\right)$$

  where $\frac{1}{\gamma}$ is EIS.

- Special case: $\beta R = 1$ or $\rho = r \Rightarrow$ MPC$= r$.

- Annual calibration with log utility ($\gamma = 1$), $\beta = 0.96$ and $R = 1.03$ gives MPC of 4.2%.

MPC for low-wealth households

- Discrete time: Household at $a = \underline{a}$ has MPC $\mathfrak{m} = 1$ if $a_{t+1} = \underline{a}$, i.e.

$$u'(c) > \beta E[V_a(a', y') | y]$$

  For discrete change, whether $\mathfrak{m}_\Delta = 1$ depends on whether household is also constrained at $\underline{a} + \Delta$.

- Continuous time: Household at $a = \underline{a}$ has MPC $\mathfrak{m} = 1$ if $\dot{a} = 0$.
  Whether $\mathfrak{m}_{\tau,\Delta} = 1$ depends on $\dot{a}$ at $\underline{a} + \Delta$.

- Hand-to-mouth (HtM) households: consume all disposable income in a given period and so have high MPC.

- Because of concavity c, MPCs can be large also for non-HtM households who are close to a kink or constraint, depending on nature of income risk.

MPCs in consumption-savings model

- Generating average MPC above interest rate boils down to generating wealth distribution with substantial fraction of (close to) HtM households.
- Challenging because of precautionary motives: holding little wealth exposes households to consumption fluctuations, which they dislike, so they tend to save themselves away from high MPC region.
- When we choose discount factor, interest rate, risk aversion to generate realistic amount of average wealth, i.e. $\approx 2.5 - 3.5\times$ average income, there are typically very few HtM households ($< 5\%$), so average MPC $< 10\%$.

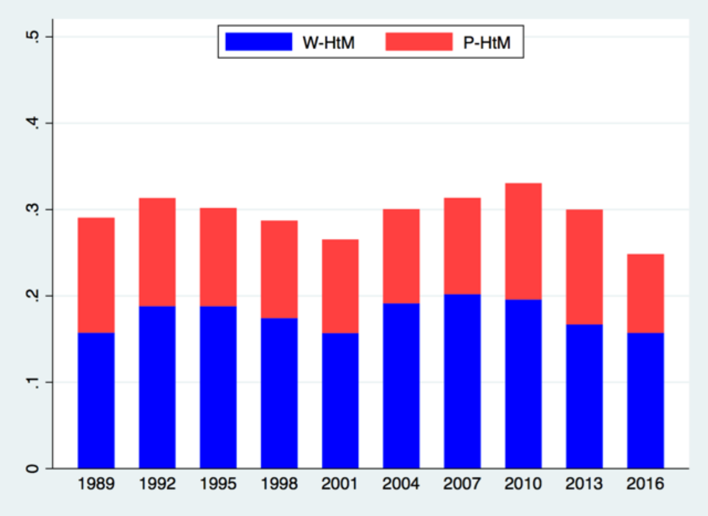
Generating HtM in the consumption-savings model

1. Liquid wealth calibration, i.e., give up on matching mean assets:
    - Overstates fraction of households with low wealth.
    - Problematic in GE models and models with investment.
    - Miss potentially important wealth effects.
2. Discount rate heterogeneity
    - Extreme form: spender-saver model, i.e., fraction with $\rho = \infty$.
    - Less extreme from: stochastic transitions.
3. Effective interest rate heterogeneity
    - High implicit tax rates from phasing out of means-tested benefits.
    - Luxury warm-glow bequest motive.
4. Illiquid assets
    - Generates wealthy hand-to-mouth households.
    - Matches both MPC and wealth distributions.

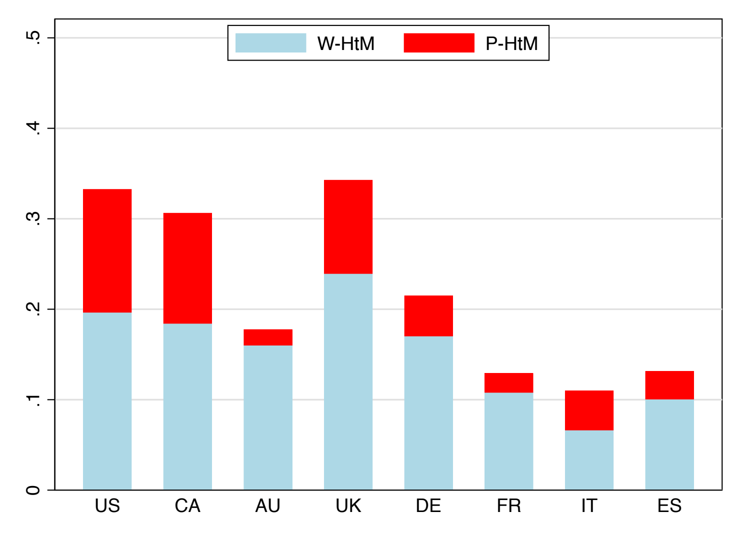
Wealthy hand-to-mouth

- Few HtM in terms of net worth.
- Many HtM in terms of liquid wealth: exclude housing wealth, tax-deferred retirement accounts, term deposits, and business equity.
- Three types of households:
  - Poor HtM : zero net worth.
  - Wealthy HtM : zero liquid assets but positive illiquid assets.
  - Non HtM : positive liquid assets.
- Around 10% of US households are P-HtM.
- Around 20% of US households are W-HtM.

Wealthy hand-to-mouth: U.S.



Wealthy hand-to-mouth: International



Why so many W-HtM households?

- Why live hand-to-mouth, rather than use wealth to smooth shocks?
- High-return illiquid assets generate trade-off:
  - Better consumption smoothing (short-run) vs
  - Higher lifetime consumption (long-run).
- Smoothing requires either:
  1. Opportunity cost of holding large cash balances.
  2. Borrowing at expensive rates.
  3. Paying transaction cost to adjust illiquid asset.
- Intuition: Welfare losses from not smoothing are second order.
- Aside: EIS vs. risk-aversion.

Simple model of wealthy HtM households

- Three periods $t \in \{0, 1, 2\}$, no uncertainty.
- $t = 0$: portfolio choice for endowment of 1 unit.
  - Liquid asset with return 1, $m_1 \geq 0$.
  - Illiquid asset with return $R^{\frac{1}{2}} > 1$, cannot be accessed at $t = 1$.
- $t = 1$: receive y_1 and chooses c_1 and m_2.
- $t = 2$: receive y_2 and consume c_2.
- Household problem:

$$\max u(c_1) + u(c_2)$$

$$\text{s.t.}$$

[t=0]: $$m_1 + a = 1$$

[t=1]: $$c_1 + m_2 = y_1 + m$$

[t=2]: $$c_2 = y_2 + m_2 + Ra$$

Wealthy HtM at $t = 1$

- Interior solution at $t = 0$:
$$u'(c_1)\left[1 - \frac{\partial m_2}{\partial m_1}\right] = u'(c_2)\left[R - \frac{\partial m_2}{\partial m_1}\right]$$

- Optimality condition at $t = 1$: $u'(c_1) \geq u'(c_2)$.

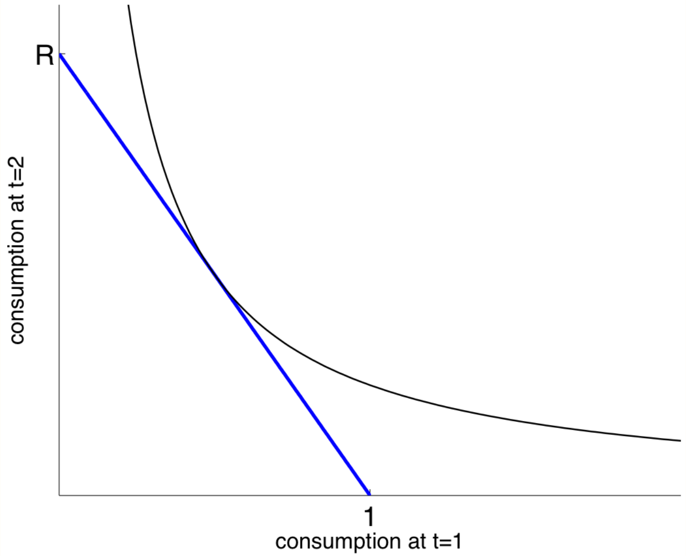
- Suppose borrowing constraint not binding at $t = 1$ (i.e. constraint holds with equality). This would imply $\frac{\partial m_2}{\partial m_1} = \frac{1}{2}$.

- Substituting into FOC at $t = 0$:
$$u'(c_1) = (2R - 1)\, u'(c_2) > u'(c_2)$$
contradicting assumption that constraint not binding at $t = 1$.

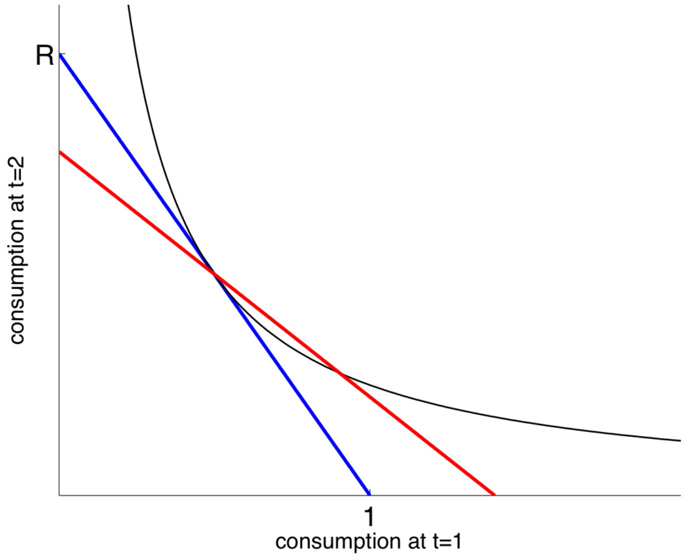
- Hence, household is constrained at $t = 1$ and sets $m_2 = 0$ and $\frac{\partial m_2}{\partial m_1} = 0$.

- Even though household is not constrained at $t = 0$, it chooses a portfolio so that it will be wealthy hand-to-mouth in period $t = 1$ and have large MPC.

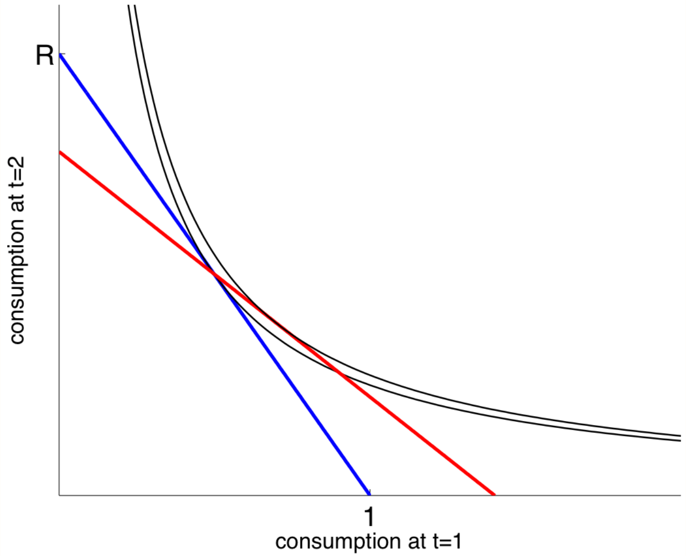
Optimal savings with illiquid asset



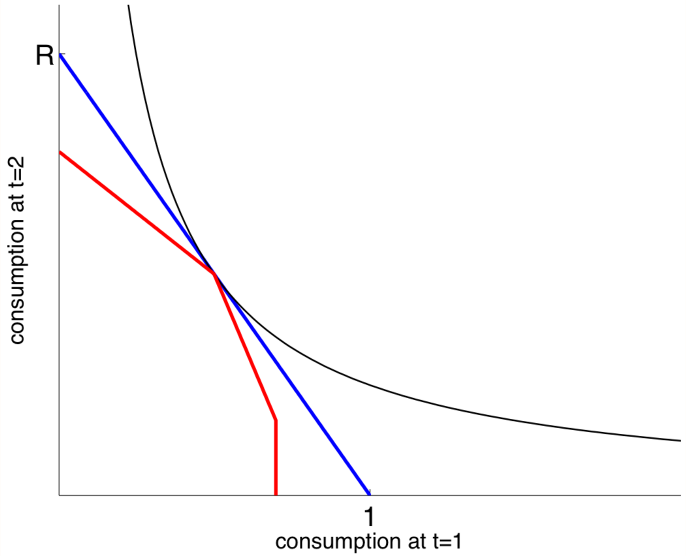
Optimal savings with illiquid asset



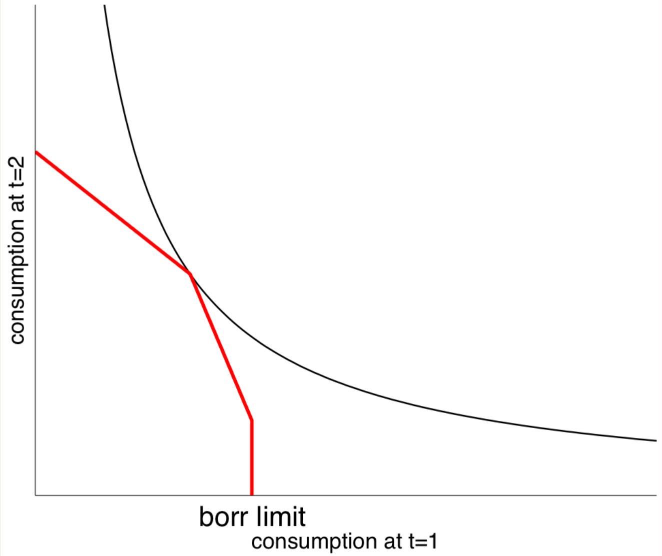
Optimal savings with illiquid asset



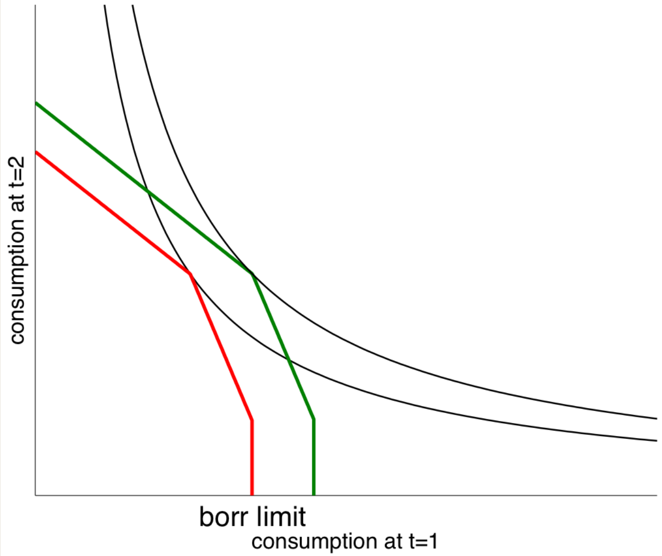
Optimal savings with illiquid asset



Optimal savings with illiquid asset



Optimal savings with illiquid asset



HANK

Building blocks

Households

- Uninsured idiosyncratic labor income risk: Consume, supply labor.
- Hold two assets: liquid and illiquid.

Firms

- Monopolistically competitive intermediate-good producers.
- Quadratic price adjustment costs à la Rotemberg (1982).

Fiscal Authority

- Issues liquid debt, spends, taxes.

Monetary Authority

- Sets nominal rate on liquid assets based on a Taylor rule.

Assets

- Liquid assets: nominal return set by monetary policy.
- Illiquid assets: real return determined by profitability of capital.

Households

$$\max_{\{c_t, \ell_t, \}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-(\rho+\lambda)t} u(c_t, \ell_t) dt \quad \text{s.t.}$$

$$\dot{b}_t = r^b(b_t) b_t + w z_t \ell_t - c_t$$

$$z_t = \text{some Markov process}$$

$$b_t \geq -\underline{b}$$

- c_t: non-durable consumption
- b_t: liquid assets
- z_t: individual productivity
- ℓ_t: hours worked
- a_t
- d_t
- χ
- T
- Γ

Households

$$\max_{\{c_t,\ell_t,d_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-(\rho+\lambda)t} u(c_t,\ell_t)dt \quad \text{s.t.}$$

$$\dot{b}_t = r^b(b_t)b_t + wz_t\ell_t - d_t - \chi(d_t,a_t) - c_t + \Gamma - T(wz_t\ell_t + \Gamma)$$

$$\dot{a}_t = r^a a_t + d_t$$

$$z_t = \text{some Markov process}$$

$$b_t \geq -\underline{b}, \quad a_t \geq 0$$

- c_t: non-durable consumption
- b_t: liquid assets
- z_t: individual productivity
- ℓ_t: hours worked
- a_t: illiquid assets
- d_t: illiquid deposits ($\gtrless 0$)
- χ: transaction cost function
- T: labor income tax/transfer
- Γ: income from firm ownership

no housing – see working paper

Households

$$\max_{\{c_t, \ell_t, d_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-(\rho+\lambda)t} u(c_t, \ell_t) dt \quad \text{s.t.}$$

$$\dot{b}_t = r^b(b_t) b_t + w z_t \ell_t - d_t - \chi(d_t, a_t) - c_t - \tilde{T}\left(w z_t \ell_t + \Gamma\right) + \Gamma$$

$$\dot{a}_t = r^a a_t + d_t$$

$$z_t = \text{some Markov process}$$

$$b_t \geq -\underline{b}, \quad a_t \geq 0$$

- c_t: non-durable consumption
- b_t: liquid assets
- z_t: individual productivity
- ℓ_t: hours worked
- a_t: illiquid assets
- d_t: illiquid deposits ($\gtrless 0$)
- χ: transaction cost function
- $\tilde{T}$: income tax/transfer
- Γ: income from firm ownership

no housing – see working paper

Households

$$\max_{\{c_t,\ell_t,d_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-(\rho+\lambda)t} u(c_t,\ell_t)dt \quad \text{s.t.}$$

$$\dot{b}_t = r^b(b_t)b_t + wz_t\ell_t - d_t - \chi(d_t, a_t) - c_t - \tilde{T}\left(wz_t\ell_t + \Gamma\right) + \Gamma$$

$$\dot{a}_t = r^a a_t + d_t$$

$$z_t = \text{some Markov process}$$

$$b_t \geq -\underline{b}, \quad a_t \geq 0$$

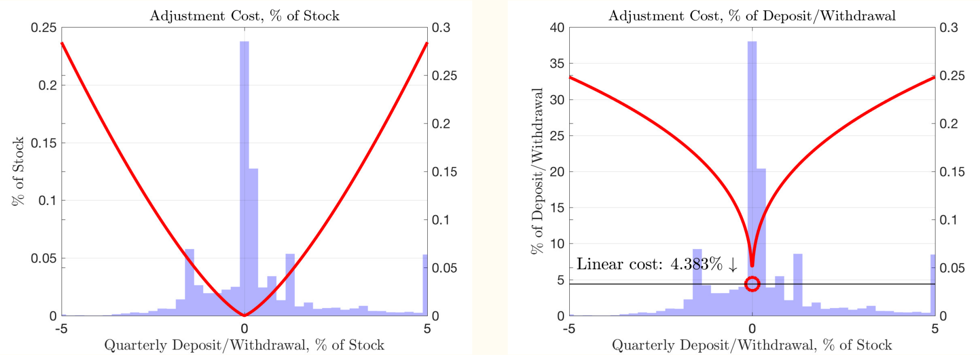
- c_t: non-durable consumption
- b_t: liquid assets
- z_t: individual productivity
- ℓ_t: hours worked
- a_t: illiquid assets
- d_t: illiquid deposits ($\gtrless 0$)
- χ: transaction cost function
- $\tilde{T}$: income tax/transfer
- Γ: income from firm ownership

no housing – see working paper

Adjustment cost function

$$\chi(d,a) = \chi_0 |d| + \chi_1 \left|\frac{d}{\tilde{a}}\right|^{\chi_2} \tilde{a} \qquad \text{where} \qquad \tilde{a} \equiv \max\{a, \underline{a}\}$$

- Linear component implies: inaction region (Bertola-Caballero, Abel-Eberly,...).
- Convex component implies finite deposit rates.



Elements of household solution

- Recursive solution of household problem consists of:

  1. Consumption policy function $c(a, b, z; w, r^a, r^b)$.
  2. Deposit policy function $d(a, b, z; w, r^a, r^b)$.
  3. Labor supply policy function $\ell(a, b, z; w, r^a, r^b)$.
  4. Joint distribution of households $\mu(da, db, dz; w, r^a, r^b)$.

Firms

- Representative competitive final goods producer:

$$Y = \left(\int_0^1 y_j^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad \Rightarrow \quad y_j = \left(\frac{p_j}{P} \right)^{-\varepsilon} Y$$

- Monopolistically competitive intermediate goods producers:
  - Technology: $y_j = Zk_j^{\alpha} n_j^{1-\alpha} \quad \Rightarrow \quad m = \frac{1}{Z} \left(\frac{r}{\alpha} \right)^{\alpha} \left(\frac{w}{1-\alpha} \right)^{1-\alpha}$.
  - Set prices subject to quadratic adjustment costs:

$$\Theta \left(\frac{\dot{p}}{p} \right) = \frac{\theta}{2} \left(\frac{\dot{p}}{p} \right)^2 PY$$

- Exact NK Phillips curve:

$$\left(r^a - \frac{\dot{Y}}{Y} \right) \pi = \frac{\varepsilon}{\theta} (m - \bar{m}) + \dot{\pi}, \quad \bar{m} = \frac{\varepsilon-1}{\varepsilon}$$

Illiquid return and profits

- Illiquid assets = part capital, part equity:

$$a = k + qs$$

  - k: capital, pays return $r - \delta$.
  - s: shares, price q, pay dividends $\omega\Pi = \omega(1-m)Y$.

- Arbitrage:

$$\frac{\omega\Pi + \dot{q}}{q} = r - \delta := r^a$$

- Remaining $(1-\omega)\Pi$? Scaled lump-sum transfer to households:

$$\Gamma = (1-\omega)\frac{z}{\bar{z}}\Pi$$

- Set $\omega = \alpha$ (capital share) $\Rightarrow$ neutralize countercyclical markups:

$$\text{total illiquid flow} = rK + \omega\Pi = \alpha m Y + \omega(1-m)Y = \alpha Y$$

$$\text{total liquid flow} = wL + (1-\omega)\Pi = (1-\alpha)Y$$

Monetary authority and government

- Taylor rule:
$$i = \bar{r}^b + \phi\pi + \epsilon, \quad \phi > 1$$
with $r^b := i - \pi$ (Fisher equation), $\epsilon =$ innovation ("MIT shock")

- Progressive tax on labor income:
$$\tilde{T}(wz\ell + \Gamma) = -T + \tau \times (wz\ell + \Gamma)$$

- Government budget constraint (in steady state):
$$G - r^b B^g = \int \tilde{T} d\mu$$

- Transition? Ricardian equivalence fails $\Rightarrow$ this matters!

Summary of market clearing conditions

- Liquid asset market:

$$B^h + B^g = 0$$

- Illiquid asset market:

$$A = K + q$$

- Labor market:

$$N = \int z\ell(a, b, z)d\mu$$

- Goods market:

$$Y = C + I + G + \chi + \Theta + \text{borrowing costs}$$

HANK: Devil is in the details

- Modeling choices that are inconsequential in RANK can matter tremendously in HANK.

1. Fiscal policy adjustment:
    - Timing matters: failure of Ricardian equivalence.
    - Distribution matters: progressivity of available tax instrument.
2. Distribution of profits:
    - Equity market vs. exogenous claims.
3. Discount rate used by firms:
    - No unique stochastic discount factor.
4. Incidence of fluctuations in labor demand:
    - Concentration of labor shortfalls, heterogeneity in exposure.

Three key aspects of parameterization

1. Measurement and partition of asset categories into:
    - Liquid (cash, bank accounts + government/corporate bonds).
    - Illiquid (equity, housing).
2. Income process with leptokurtic income changes:
    - Nature of earnings risk affects household portfolio.
3. Adjustment cost function and discount rate:
    - Match mean liquid/illiquid wealth and fraction HtM.

- Production side: standard calibration of NK models.
- Standard separable preferences: $u(c, \ell) = \log c - \frac{1}{2}\ell^2$.

Continuous-time earnings dynamics

- Key challenge: inferring within-year dynamics from annual data.
- Higher order moments of annual changes are informative.
- Target moments of 1-year and 5-year labor earnings growth from SSA data.
- Model generates a thick right tail for earnings levels.

Two-component jump-drift process

- Flow earnings $(y = wz\ell)$ modeled as sum of two components:

$$\log y_t = y_{1t} + y_{2t}$$

- Each component is a jump-drift with:
    - mean-reverting drift: $-\beta y_{it} dt$.
    - jumps with arrival rate: λ_i, drawn from $\mathcal{N}(0, \sigma_i)$.
- Estimate using SMM aggregated to annual frequency.
- Choose six parameters to match eight moments.

Earnings process estimates

Parameter		Component $j = 1$	Component $j = 2$
Arrival rate	λ_j	0.080	0.007
Mean reversion	β_j	0.761	0.009
St. Deviation of innovations	σ_j	1.74	1.53

Table 4: Earnings Process Parameter Estimates. Rates expressed as quarterly values.

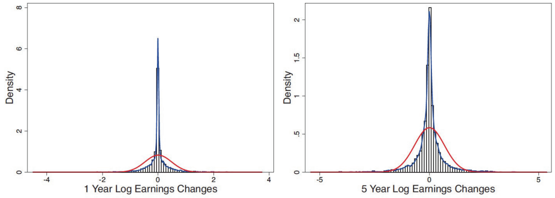
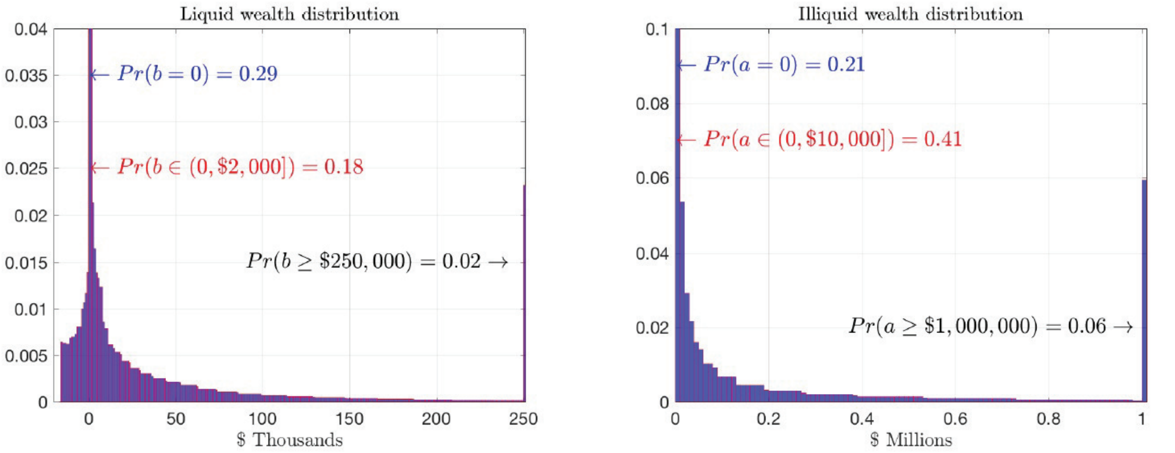


Figure D.1: Growth Rate Distribution of Estimated Earnings Process

Wealth distributions



(a) Liquid wealth distribution

(b) Illiquid wealth distribution

Figure 1: Distributions of liquid and illiquid wealth.

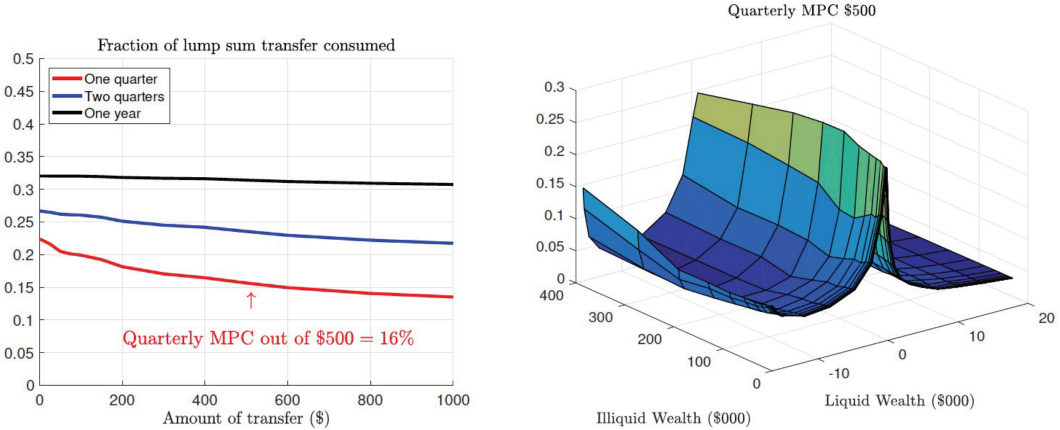
Wealth distributions

	Data	Model
Mean illiquid assets	2.92	2.92
Mean liquid assets	0.26	0.23
Frac. with $b = 0$ and $a = 0$	0.10	0.10
Frac. with $b = 0$ and $a > 0$	0.20	0.19
Frac. with $b < 0$	0.15	0.15

	Liquid Wealth		Illiquid Wealth	
Moment	Data	Model	Data	Model
Top 0.1% share	17%	2.3%	12%	7%
Top 1% share	47%	18%	33%	40%
Top 10% share	86%	75%	70%	88%
Bottom 50% share	-4%	-3%	3%	0.1%
Bottom 25% share	-5%	-3%	0%	0%
Gini coefficient	0.98	0.86	0.81	0.82

Table 5: Left panel: Moments targeted in calibration and reproduced by the model. Means are expressed as ratios to annual output. Right panel: Statistics for the top and bottom of the wealth distribution not targeted in the calibration. Source: SCF 2004.

Large and heterogeneous MPCs



(a) $\int \mathrm{MPC}_\tau^x(a,b,z)d\mu$ by τ, x

(b) $\mathrm{MPC}_1^{\$500}(a,b,z)$

Figure 2: MPC Heterogeneity

Equivalence between HA and RA models, I

- IRF of C to a shock η in model m:
$$dC_t^m = \int_i dc_{it}^m di$$

- Non-equivalence $\Rightarrow$ different IRF:
$$dC_t^{HA} \neq dC_t^{RA} \quad \forall\ t \geq 0$$

- Weak equivalence $\Rightarrow$ same IRF:
$$dC_t^{HA} = dC_t^{RA} \quad \forall\ t \geq 0$$

- Strong equivalence $\Rightarrow$ same IRF $+$ same mechanism.

Equivalence between HA and RA models, II

- Propose three criteria for assessing similarity of mechanism.
- IRF depends on vector of J equilibrium objects Θ^m that include:
    - Fiscal policy variables $(T, \tau)^m$.
    - Prices $(w, r^a, r^b, q)^m$.
    - Shock itself η (same across models):

$$dC_t^m = \sum_{j=1}^{J} \int_{\tau=0}^{\infty} \frac{\partial C_t^m}{\partial \Theta_{j\tau}} d\Theta_{j\tau}^m d\tau \quad \text{for} \quad t = 0, ..., \infty$$

Criteria to assess strong equivalence

1. Same IRF decomposition into response to w, r^a, T ...

$$\int_{\tau=0}^{\infty} \frac{\partial C_t^{HA}}{\partial \Theta_{j\tau}} d\Theta_{j\tau}^{HA} d\tau = \int_{\tau=0}^{\infty} \frac{\partial C_t^{RA}}{\partial \Theta_{j\tau}} d\Theta_{j\tau}^{RA} d\tau \quad \forall\ t \geq 0, \quad \forall j = 1 \dots J$$

Criteria to assess strong equivalence

1. Same IRF decomposition into response to w, r^a, T ...

$$\int_{\tau=0}^{\infty} \frac{\partial C_t^{HA}}{\partial \Theta_{j\tau}} d\Theta_{j\tau}^{HA} d\tau = \int_{\tau=0}^{\infty} \frac{\partial C_t^{RA}}{\partial \Theta_{j\tau}} d\Theta_{j\tau}^{RA} d\tau \quad \forall\ t \geq 0, \quad \forall j = 1 \ldots J$$

2. Both PE and GE discrepancies are zero:

$$dC_t^{HA} - dC_t^{RA} = \underbrace{\sum_{j=1}^{J} \int_0^{\infty} \frac{\partial C_t^{HA}}{\partial \Theta_{j\tau}} \left(d\Theta_{j\tau}^{HA} - d\Theta_{j\tau}^{RA} \right) d\tau}_{\text{GE discrepancy}}$$

$$+ \underbrace{\sum_{j=1}^{J} \int_0^{\infty} \left(\frac{\partial C_t^{HA}}{\partial \Theta_{j\tau}} - \frac{\partial C_t^{RA}}{\partial \Theta_{j\tau}} \right) d\Theta_{j\tau}^{RA} d\tau}_{\text{PE discrepancy}}$$

Criteria to assess strong equivalence

1. Same IRF decomposition into response to w, r^a, T ...

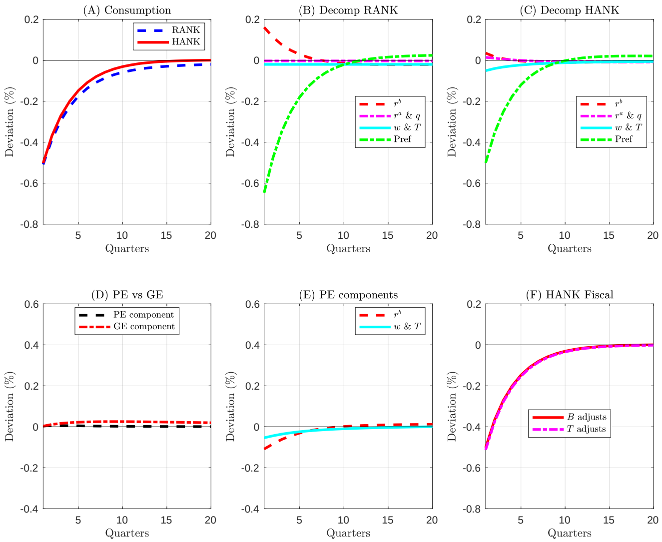
$$\int_{\tau=0}^{\infty} \frac{\partial C_t^{HA}}{\partial \Theta_{j\tau}} d\Theta_{j\tau}^{HA} d\tau = \int_{\tau=0}^{\infty} \frac{\partial C_t^{RA}}{\partial \Theta_{j\tau}} d\Theta_{j\tau}^{RA} d\tau \quad \forall\ t \geq 0, \quad \forall j = 1 \dots J$$

2. Both PE and GE discrepancies are zero:

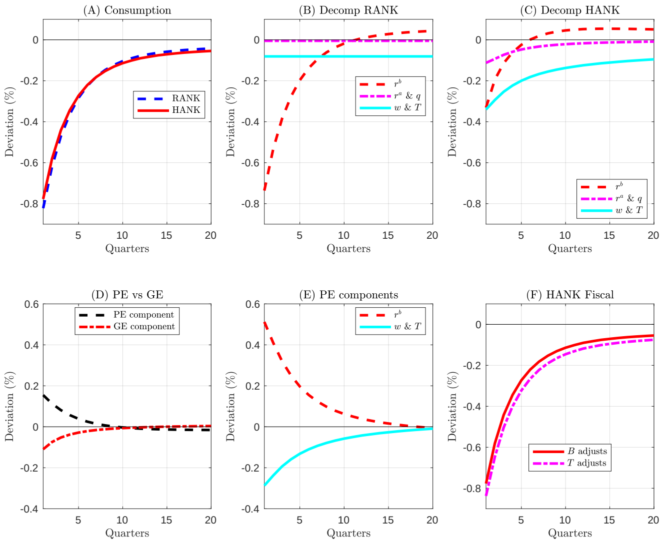
$$dC_t^{HA} - dC_t^{RA} = \underbrace{\sum_{j=1}^{J} \int_0^{\infty} \frac{\partial C_t^{HA}}{\partial \Theta_{j\tau}} \left(d\Theta_{j\tau}^{HA} - d\Theta_{j\tau}^{RA} \right) d\tau}_{\text{GE discrepancy}}$$
$$+ \underbrace{\sum_{j=1}^{J} \int_0^{\infty} \left(\frac{\partial C_t^{HA}}{\partial \Theta_{j\tau}} - \frac{\partial C_t^{RA}}{\partial \Theta_{j\tau}} \right) d\Theta_{j\tau}^{RA} d\tau}_{\text{PE discrepancy}}$$

3. Sensitivity to fiscal rule.

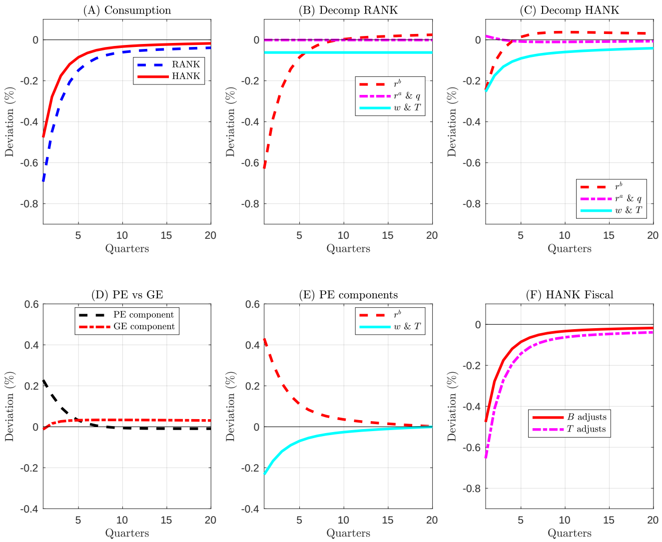
Preference shock: Strong equivalence



TFP shock: Weak equivalence

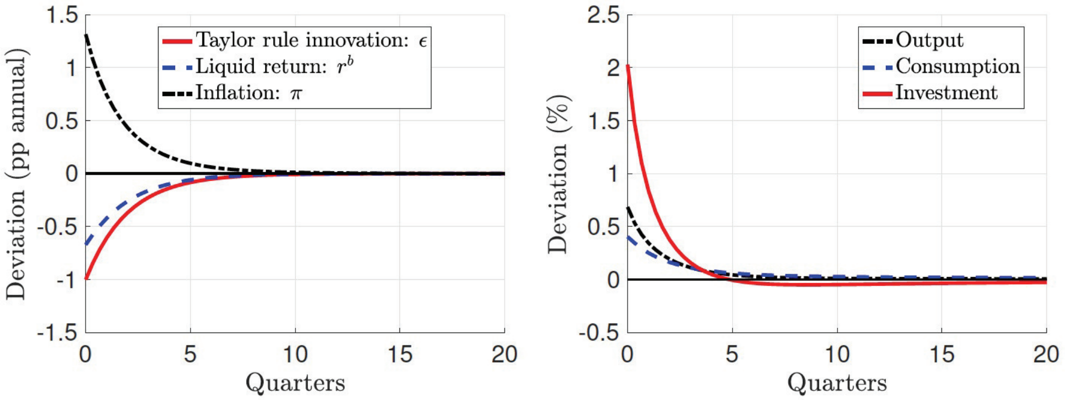


Monetary shock: Non-equivalence



Aggregate effect of MP shock

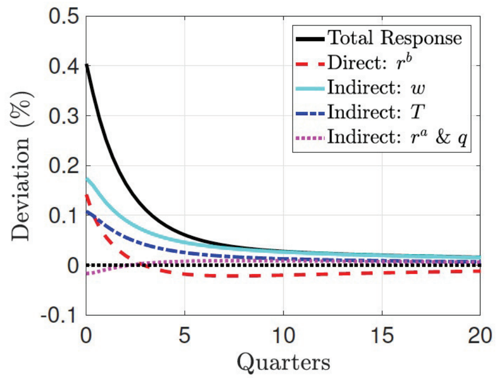
- Innovation $\epsilon < 0$ to the Taylor rule: $i = \bar{r}^b + \phi\pi + \epsilon$.
- All experiments: $\epsilon_0 = -0.0025$, i.e. -1% annualized.



(a) Monetary Shock, Interest Rate, Inflation (b) Aggregate Quantities

Decomposition of MP shock, I

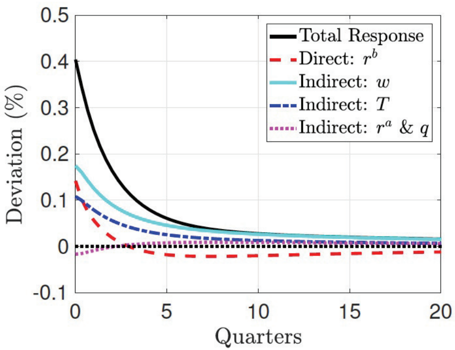
$$dC_0 = \underbrace{\int_0^\infty \frac{\partial C_0}{\partial r_t^b} dr_t^b dt}_{\text{direct}} + \underbrace{\int_0^\infty \left[\frac{\partial C_0}{\partial r_t^a} dr_t^a + \frac{\partial C_0}{\partial w_t} dw_t + \frac{\partial C_0}{\partial T_t} dT_t \right] dt}_{\text{indirect}}$$



(b) Consumption Decomposition

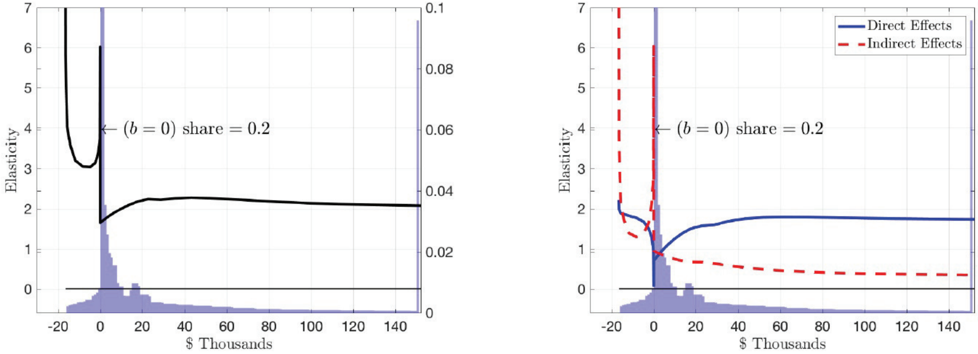
Decomposition of MP Shock, II

$$dC_0 = \underbrace{\int_0^\infty \frac{\partial C_0}{\partial r_t^b} dr_t^b dt}_{19\%} + \underbrace{\int_0^\infty \left[\frac{\partial C_0}{\partial r_t^a} dr_t^a + \frac{\partial C_0}{\partial w_t} dw_t + \frac{\partial C_0}{\partial T_t} dT_t\right] dt}_{81\%}$$



(b) Consumption Decomposition

Monetary transmission by liquid wealth



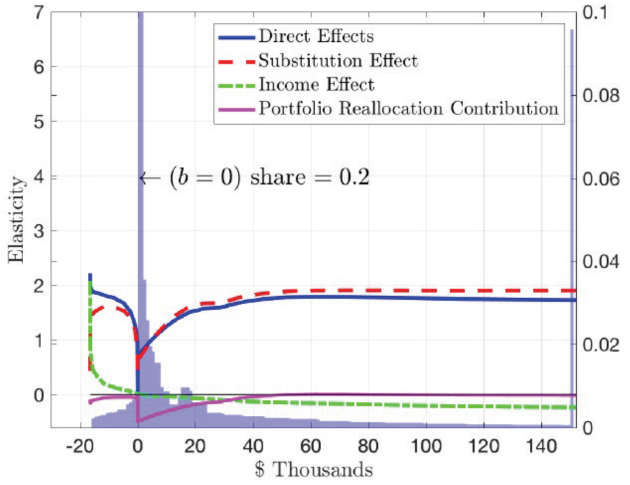
(a) Elasticity with respect to r^b

(b) Consumption Change: Indirect and direct

Figure 5: Consumption Responses by Liquid Wealth Position

- Total change = c-weighted sum of (direct + indirect) at each b.

Why small direct effects?



(a) Breakdown of direct effect

Fiscal response and total effect

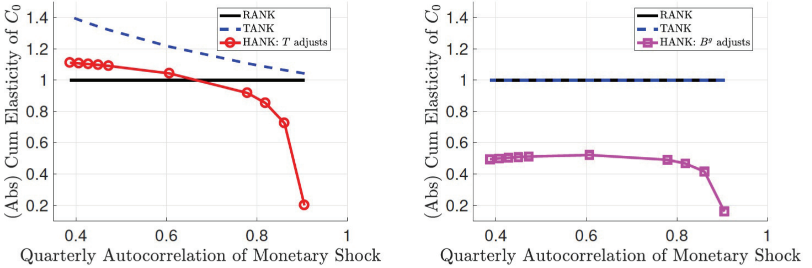
	T adjusts (1)	G adjusts (2)	B^g adjusts (3)
Elasticity of C_0 to r^b	-2.21	-2.07	-1.48
Share of Direct effects:	19%	22%	46%

- Fiscal response to lower interest payments on debt:
  - T adjusts: stimulates AD through MPC of HtM households.
  - G adjusts: translates 1-1 into AD.
  - B^g adjusts: no initial stimulus to AD from fiscal side.

When is HANK $\neq$ RANK? Persistence

- RANK: $\frac{\dot{C}_t}{C_t} = \frac{1}{\gamma}(r_t - \rho) \Rightarrow C_0 = \bar{C} \exp\left(-\frac{1}{\gamma}\int_0^\infty (r_s - \rho)ds\right)$.
- Cumulative r-deviation $R_0 := \int_0^\infty (r_s - \rho)ds$ is sufficient statistic.
- Persistence η only matters insofar as it affects R_0:

$$-\frac{d \log C_0}{dR_0} = \frac{1}{\gamma} = 1 \quad \text{for all } \eta$$



(a) T adjusts (b) B^g adjusts

One-asset HANK vs. two-asset HANK

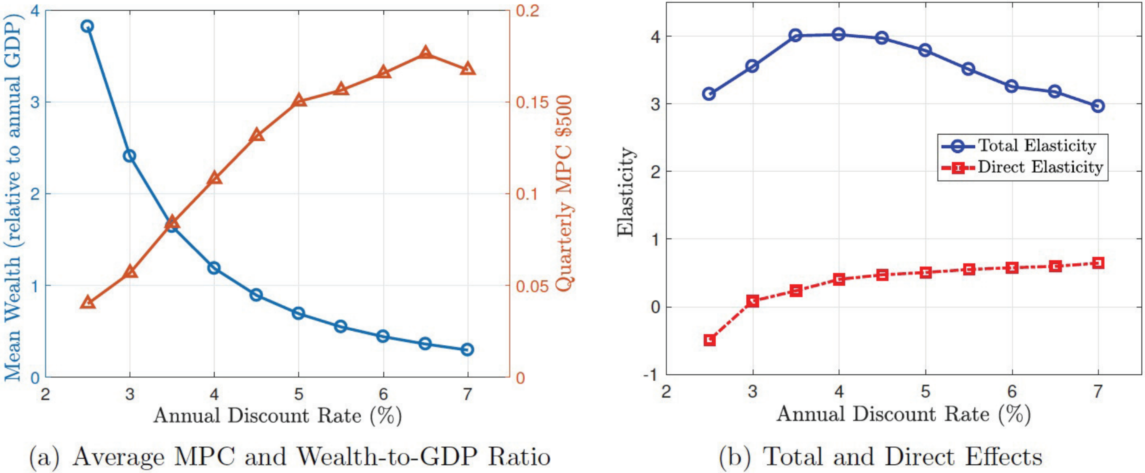


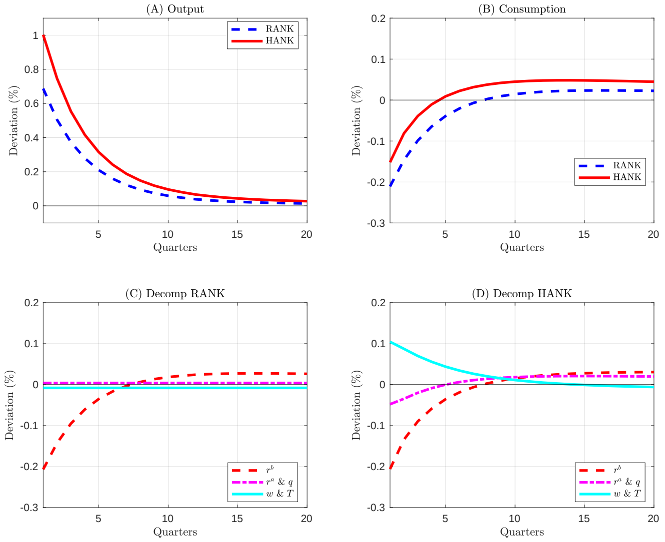
Figure 7: Key Features of One-Asset Model for Different Calibrations

Fiscal stimulus

Stark examples of non-equivalence between RANK and HANK.

1. Temporary expansion of G expenditures:
    - Larger output multiplier in HANK.
    - Weaker crowding out of consumption.

Fiscal stimulus: Government Spending

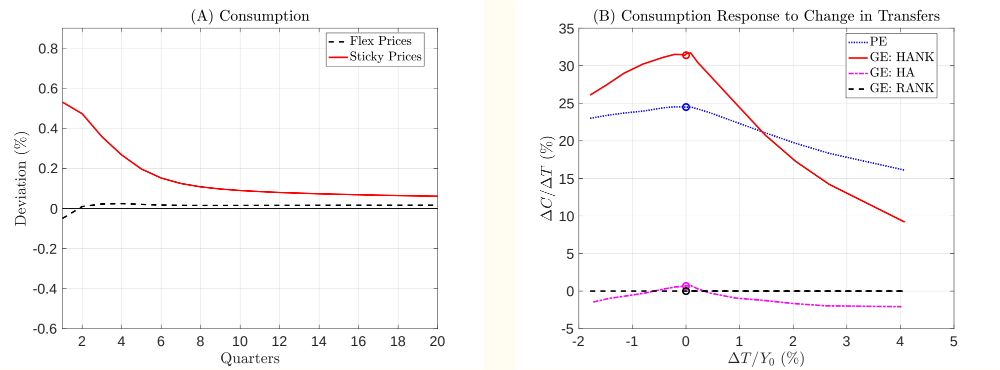


Fiscal stimulus

Stark examples of non-equivalence between RANK and HANK.

1. Temporary expansion of G expenditures:
   - Larger output multiplier in HANK.
   - Weaker crowding out of consumption.
2. Temporary expansion of lump-sum transfers T:
   - RANK: No impact due to Ricardian equivalence.
   - HANK: Positive impact, sign/size asymmetries.

Fiscal stimulus: Transfers



- Size effect: $|\Delta C|$ falls with $|\Delta T|$.
- Sign asymmetry: $|\Delta C|$ larger for negative ΔT.
- GE amplifies stimulus for small ΔT, but for large ΔT inflationary pressure leading to $r^b \uparrow$ dominates.

Appendices

HJB with two endogenous states

- Ignoring income risk, HJB becomes:

$$\rho V(a,b) = \max_c \; u(c) + V_b(a,b)(w + r^b b - d - \chi(d,a) - c) \\ + V_a(a,b)(d + r^a a)$$

- For simplicity, assume $\chi(d,a) = \left(\frac{d}{a}\right)^2 a$.
- FOC for deposits d:

$$(1 + \chi_d(d,a))V_b(a,b) = V_a(a,b)$$

$$d = \left(\frac{V_a(a,b)}{V_b(a,b)} - 1\right) a$$

- Intuition: optimal deposit rate depends on difference in marginal values.

Upwinding with two endogenous states

- Standard upwind scheme at point (a_i, b_j):

$$\rho V_{i,j} = u(c_{i,j}) + \frac{V_{i+1,j} - V_{i,j}}{\Delta b} I_F^b s_{i,j}^b + \frac{V_{i,j} - V_{i-1,j}}{\Delta b} I_B^b s_{i,j}^b$$
$$+ \frac{V_{i,j+1} - V_{i,j}}{\Delta a} I_F^a s_{i,j}^a + \frac{V_{i,j} - V_{i,j-1}}{\Delta a} I_B^a s_{i,j}^a$$

  where:

$$s_{i,j}^b = w + r^b b_i - d_{i,j} - \chi(d_{i,j}, a_j) - c_{i,j}$$
$$s_{i,j}^a = r^a a_i + d_{i,j}$$

- Difficulty: $d_{i,j}$ depends on forward/backward choice for V_b and V_a.

- Could end up using different $d_{i,j}$ in each term.

Splitting the drift

- Convenient trick: split the drift

$$\begin{aligned}\rho V(a,b) = \max_c \ u(c) &+ V_b(a,b)(w + r^b b - c) \\ &+ V_b(a,b)(-d - \chi(d,a)) \\ &+ V_a(a,b)d \\ &+ V_a(a,b)r^a a\end{aligned}$$

  and upwind each term separately.

- Satisfies Barles-Souganidis monotonicity condition.

- Important: **A** matrix that goes into KFE must be based on actual upwinding, i.e. based on the actual directions of $\dot{a}$ and $\dot{b}$, because Markov transition matrix in KFE must describe actual dynamics of the system.

Fixed transaction cost version

$$\rho V(a,b) = \max_c u(c) + V_a \dot{a} + V_b \dot{b}$$

subject to

$$\dot{a} = r^a a$$

$$\dot{b} = r^b b + y - c$$

$$V(a,b) \geq W(a,b)$$

where:

$$W(a,b) \equiv \max_{a',b'} V(a',b')$$

subject to

$$a' + b' \leq a + b - \kappa$$

Optimal adjustment decision

- At each point (a, b) in state space there are 2 cases:

  1. $\rho V(a,b) = u(c(a,b)) + V_a\dot{a} + V_b\dot{b}$ and $V(a,b) > W(a,b)$.
  2. $\rho V(a,b) < u(c(a,b)) + V_a\dot{a} + V_b\dot{b}$ and $V(a,b) = W(a,b)$.

  where $c(a,b) \equiv u\left[u'^{-1}(V_b)\right]$.

HJBVI: Variational inequality

- Write two conditions compactly as:

$$\rho V(a,b) = \max\left\{u(c(a,b)) + V_a\dot{a} + V_b\dot{b},\ W(a,b;V)\right\} = 0$$

  which is equivalent to:

$$\min\left\{\rho V(a,b) - u(c(a,b)) - V_a\dot{a} - V_b\dot{b},\ V(a,b) - W(a,b;V)\right\} = 0$$

  $\Longrightarrow$ called a HJB variational inequality (HJBVI).

- HJBVI is equivalent to

$$\begin{aligned} [V(a,b) - W(a,b;V)]\ \left[\rho V(a,b) - u(c(a,b)) - V_a\dot{a} - V_b\dot{b}\right] &= 0 \\ \rho V(a,b) - u(c(a,b)) - V_a\dot{a} - V_b\dot{b} &\geq 0 \\ V(a,b) - W(a,b;v) &\geq 0 \end{aligned}$$

  for all (a,b).

Linear complementarity problems

- Prototypical LCP: given matrix $\mathbf{B}$ and vector q, find z such that:

$$
\begin{aligned}
z'(\mathbf{B}z + q) &= 0 \\
z &\geq 0 \\
\mathbf{B}z + q &\geq 0
\end{aligned}
$$

- There are many good LCP solvers in Julia, Matlab, and other languages.

- Special case of quadratic programming problem.

- A good one for $\mathbf{B}$ large but sparse (Newton-based):
  `http://www.mathworks.com/matlabcentral/fileexchange/20952`

Solving discretized HJBVI through LCP

- Discretized HJBVI is:

$$\begin{aligned} [V - W(V)]' [\rho V - u(V) - \mathbf{A}(V) V] &= 0 \\ \rho V - \mathbf{A}(V) V &\geq 0 \\ V - W(V) &\geq 0 \end{aligned}$$

- Non-linear complementarity problem since u, A and W depend on V. But implicit update steps are exactly an LCP:

$$\begin{aligned} \left(V^{n+1} - W^n\right)' \left(\frac{V^{n+1} - V^n}{\Delta} + \rho V^{n+1} - u^n - \mathbf{A^n} V^{n+1}\right) &= 0 \\ \frac{V^{n+1} - V^n}{\Delta} + \rho V^{n+1} - \mathbf{A^n} V^{n+1} &\geq 0 \\ V^{n+1} - W^n &\geq 0 \end{aligned}$$

- LCP with:

$$\begin{aligned} z &= V^{n+1} - W^n \\ \mathbf{B} &= (1 + \rho\Delta)\mathbf{I} - \Delta \mathbf{A} \end{aligned}$$

Solution algorithm

- Follow same steps as for HJB with implicit updating.
- Replace linear solver with LCP solver.
- Update vector of adjustment values w^n after each update:

$$W_i^n \equiv \max_j V_j^n$$

$$\text{subject to}$$

$$a_i + b_i \leq a_j + b_j - \kappa$$

(Very) simple NK models

Goal:

- Introduce decomposition of C response to r change.

Setup:

- Prices and wages perfectly rigid $= 1$, GDP=labor $= Y_t$.
- Households: CRRA(γ), income Y_t, interest rate r_t

$$\Rightarrow \quad C_t(\{r_s, Y_s\}_{s\geq 0})$$

- Monetary policy: sets time path $\{r_t\}_{t\geq 0}$, special case:

$$r_t = \rho + e^{-\eta t}(r_0 - \rho), \quad \eta > 0 \tag{1}$$

- Equilibrium: $C_t(\{r_s, Y_s\}_{s\geq 0}) = Y_t$.
- Overall effect of monetary policy:

$$C_t = \bar{C} \exp\left(-\frac{1}{\gamma}\int_t^\infty (r_s - \rho)ds\right) \quad \Rightarrow \quad \frac{d \log C_0}{dr_0} = -\frac{1}{\gamma\eta}$$

Decomposition of consumption response

- Decompose C response by totally differentiating $C_0(\{r_t, Y_t\}_{t\geq 0})$:

$$dC_0 = \underbrace{\int_0^\infty \frac{\partial C_0}{\partial r_t} dr_t dt}_{\text{direct response to } r} + \underbrace{\int_0^\infty \frac{\partial C_0}{\partial Y_t} dY_t dt}_{\text{indirect effects due to } Y}$$

- With exponentially decaying interest rate path:

$$-\frac{d \log C_0}{dr_0} = \frac{1}{\gamma\eta} \left[\underbrace{\frac{\eta}{\rho+\eta}}_{\text{direct response to } r} + \underbrace{\frac{\rho}{\rho+\eta}}_{\text{indirect effects due to } Y} \right]$$

- Reasonable parameterizations $\Rightarrow$ very small indirect effects, e.g.,
  - $\rho = 0.5\%$ quarterly.
  - $\eta = 0.5$, i.e. quarterly autocorr $e^{-\eta} = 0.61$.

$$\Rightarrow \quad \frac{\eta}{\rho+\eta} = 99\%, \qquad \frac{\rho}{\rho+\eta} = 1\%$$

RANK with government debt

- Assume households hold assets B issued by government.
- Govt levies lump-sum taxes T_t to finance interest payments on debt.
- Changes in interest rates necessarily require a fiscal response in order to maintain budget balance $\rightarrow$ additional source of indirect effects:

$$dC_0 = \underbrace{\int_0^\infty \frac{\partial C_0}{\partial r_t} dr_t dt}_{\text{direct response to } r} + \underbrace{\int_0^\infty \left(\frac{\partial C_0}{\partial Y_t} dY_t + \frac{\partial C_0}{\partial T_t} dT_t \right) dt}_{\text{indirect effects}}$$

- Decomposition becomes:

$$-\frac{d \log C_0}{dr_0} = \frac{1}{\gamma\eta}\left[\underbrace{\frac{\eta}{\rho+\eta}\left(1 - \rho\gamma\frac{B_0}{\bar{Y}}\right)}_{\text{direct}} + \underbrace{\frac{\rho}{\rho+\eta}}_{\text{indirect: } Y} + \underbrace{\frac{\eta}{\rho+\eta}\rho\gamma\frac{B_0}{\bar{Y}}}_{\text{indirect: } T} \right]$$

- Overall effect of monetary policy not affected: Ricardian Equivalence.
- Direct effect smaller.

HtM households

- Spender-saver or two-agent New Keynesian (TANK) model.
- Fraction Λ are HtM "spenders": $C_t^{sp} = Y_t$.
- Decomposition becomes:

$$-\frac{d\log C_0}{dr_0} = \frac{1}{\gamma\eta}\left[\underbrace{(1-\Lambda)\frac{\eta}{\rho+\eta}}_{\text{direct response to } r} + \underbrace{(1-\Lambda)\frac{\rho}{\rho+\eta}+\Lambda}_{\text{indirect effects due to } Y}\right].$$

- Overall effect of monetary policy not affected.
- Indirect effects larger $\approx \Lambda = 20-30\%$.

HtM households with government debt

- Fall in r_t implies a fall in interest payments of $(r_t - \rho) B$.
- Fraction Λ^T of income gains transferred to spenders.
- Overall consumption response:

$$-\frac{d \log C_0}{dr_0} = \frac{1}{\gamma\eta} \quad + \quad \underbrace{\frac{\Lambda^T}{1-\Lambda} \frac{B}{\bar{Y}}}_{\text{fiscal redistribution channel}} \quad .$$

- Interaction between non-Ricardian households and debt in positive net supply matters for overall effect of monetary policy.
- Specifics of fiscal policy (Λ^T) determine strength of this channel.

Richer RANK and TANK models

	RANK				TANK		
	$B=0$	$B>0$	S-W	$B,K>0$	$B=0$	$B>0$	$B,K>0$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Elasticity of C	−2.00	−2.00	−0.74	−2.07	−2.00	−2.43	−2.77
P.E. elast. of C	−1.98	−1.96	−0.73	−1.95	−1.38	−1.39	−1.39
Direct effects	99%	98%	99%	94%	69%	57%	50%

Table 1: Elasticity of aggregate consumption and share of direct effects in several versions of the RANK and TANK models.

Notes: $B=0$ denotes the simple models of Section 2 with wealth in zero net supply. $B>0$ denotes the extension of these models with government bonds in positive net supply. In RANK, we set $\gamma=1, \eta=0.5, \rho=0.005$, and $B_0/Y=1$. In addition, in TANK we set $\Lambda=\Lambda^T=0.3$. 'S-W' is the medium-scale version of the RANK model described in Appendix A.4 based on Smets-Wouters. '$B,K>0$' denotes the richer version of the representative-agent and spender-saver New Keynesian model featuring a two-asset structure, as in HANK. See Appendix A.5 for a detailed description of this model and its calibration. In all cases lump-sum transfers adjust to balance the government budget constraint in the economies with bonds in positive supply. 'P.E. elast of C' is the partial equilibrium (or direct) elasticity computed as total elasticity times the share of direct effects.

Fiscal stimulus payments (FSP)

- Direct cash transfers to households:
  - Small (relative to household budget).
  - Lump-sum.
  - Temporary (i.e one-off).
  - Open anticipated by time received.
- Used either to alleviate economic hardship during recessions or as as a source of fiscal stimulus, justified by fiscal multiplier.
- Recent examples:
  - 2009 ARRA: up to $400 per adult.
  - 2008 ESA: $300-$600 per adult. Total payout of $79b, equivalent to 2.2% quarterly GDP.
  - 2001 EGTRRA: up to $300 per adult. Total payout of $38b, equivalent to 1.7% of quarterly GDP.

Bush tax cuts (EGTRRA 2001)

- Large scale reduction in federal taxes $\sim 5\%$.
- Enacted in May 2001, but first mentioned in second half of 2000.
- Lowest tax rate applied to \$6,000 of individual (\$12,000 of joint married) income: reduced from 15% to 10%.
- Part of tax reform paid as 'rebate' in July-September 2001.
- In total around 92m taxpayers received checks totaling \$38b. 80% of households who received checks, received \$600.
- Random timing of checks received, based on last 2 digits of social-security.

Consumption response to tax rebate

- Johnson, Parker and Souleles (2006) (JPS) added question to 2001 Consumer Expenditure Survey (CEX) to ask whether a tax rebate was received by each household and, if so, how much.

- Regression specification:

$$\Delta c_{it} = \sum_s \beta_{0,s} m_s + \beta_1' X_{i,t-1} + \beta_2 R_{it} + \varepsilon_{it},$$

  where m is month dummies, $X_{i,t-1}$ are controls, R_{it} is the dollar amount of the tax rebate received. β_2 is rebate coefficient.

- ε_{it} may be correlated with R_{it}:
  - Eligibility based on tax filing status and income in 2000.
  - Rebate amount depends on number of earners and marital status.
  - R_{it} depends on actual income in 2000 if less than full amount.

Rebate receipt as instrument

- Measure effect of *receipt* of rebate check.
- Exploit randomization of timing as an instrument.
- Estimate β_2 with 2SLS using indicator $D_{it} = \mathbf{1}\{R_{it} > 0\}$ for R_{it}.

	Nondurables
JPS 2006, 2SLS ($N = 13,066$)	0.375 (0.136)
Trim top & bottom 0.5%, 2SLS ($N = 12,935$)	0.237 (0.093)
Trim top & bottom 1.5%, 2SLS ($N = 12,679$)	0.219 (0.079)
MS 2011, IVQR ($N = 13,066$)	0.244 (0.057)

Table 1: Estimates of the 2001 rebate coefficient ($\hat{\beta}_2$). Nondurables include food (at home and away), utilities, household operations, public transportation and gas, personal care, alcohol and tobacco, miscellaneous goods, apparel good and services, reading materials, and out-of-pocket health care expenditures. JPS 2006: Johnson, Parker and Souleles (2006); MS 2011: Misra and Surico (2011). 2SLS: Two-Stage Least Squares; IVQR: Instrumental Variable Quantile Regression.

Heterogeneity in rebate coefficients

- Splitting Sample: JPS find low income and low wealth households have high rebate coefficients.
- Quantile IV Regression: Misra and Surico (2014)
  1. Lots of heterogeneity. Implied average rebate coefficient lower than JPS (about 0.24) and more precisely estimated.
  2. Around half of households have rebate coefficients of zero: for about 45% point estimate is zero, for about 60% cannot reject zero.
  3. Around 30% of households have high rebate coefficients: for about 15% can't reject that rebate coefficient is 1.
  4. High income households are found at both ends of the distribution of rebate coefficients.

Interpretation of rebate coefficient

- Rebate coefficients may not be same as MPC: even if R_{it} is random, OLS interpretation of β_2 is complicated.
- Treatment group: households who received the rebate at time t.
- Control group: mix of two types of households:
  1. Households who will receive rebate in the future.
  2. Households who have received rebate in the past.
- Consumption growth of control group is mix of:
  1. MPC out of news about a future receipt of a check: $= 0$?
  2. Lagged MPC out of the receipt of the check: $\neq 0$.

Rebate coefficient versus MPC: Example

- Group A: early recipients who receive the rebate check in 2001:Q2.
- Group B: late recipients who receive the rebate check in 2001:Q3.
- OLS estimate of β_2 is:

$$\hat{\beta}_2 = \frac{1}{2}\left(\Delta c_{Q2}^{A} - \Delta c_{Q2}^{B}\right) + \frac{1}{2}\left(\Delta c_{Q3}^{B} - \Delta c_{Q3}^{A}\right)$$

- To interpret $\hat{\beta}_2$ as an MPC we need:
  - Δc_{Q2}^{A} and Δc_{Q3}^{B} (the treatments) to reflect consumption responses to surprise rebate checks.
  - Δc_{Q2}^{B} and Δc_{Q3}^{A} (the controls) to be zero.
- These depend on assumed information structure.

Rebate coefficient versus MPC: Example

	Quarter 2 (Q2)		Quarter 3 (Q3)	
	Group A	Group B	Group A	Group B
Surprise for group A	Δc to surprise check	Δc to news	Lagged Δc to surprise check	Δc to anticipated check
Anticipated by all	Δc to anticipated check	0	Lagged Δc to anticipated check	Δc to anticipated check
Surprise for all	Δc to surprise check	0	Lagged Δc to surprise check	Δc to surprise check

Table 2: Economic interpretation of the components of the rebate coefficient β_2 in equation (2) under the three alternative information structures.

Requirements for large rebate coefficient

- Consider information structure: surprise for group A.
- Large rebate coefficient ($\approx 25\%$) requires:
  1. Large average MPC out of a surprise check (so Δc^A_{Q2} is large).
  2. Small MPC out of news about a future check (so Δc^B_{Q2} is small).
  3. Large MPC out of anticipated check (so Δc^B_{Q3} is large).
  4. Small MPC to a lagged surprise check (so Δc^A_{Q3} is small).
- Under PIH, or non-HtM households in consumption-savings model:
  - (1) is small.
  - (2) $\approx$ (1): Q2 rebate coefficient ≈ 0.
  - (3) is zero.
  - (4) is often negative.

Solution: HtM households?

- Qualitatively: HtM households could satisfy all these conditions:
  1. MPCs are large for HtM households so (1) satisfied.
  2. HtM households are not able to increase consumption in response to news about a future payment satisfying (2).
  3. Implies (3) can be large.
- Quantitatively: Plausible calibrations of standard consumption-savings model fail because too few HtM households.
- Can increase fraction of HtM in model via aforementioned tweaks.
- One asset consumption-savings model disciplined by data on net worth: $< 10\%$ of households are HtM.