Krusell-Smith Models

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A basic model with aggregate uncertainty, I

- We want to deal with models with aggregate uncertainty.
- Why?
- Issues of interpretation (forecasting vs. policy analysis, etc.).
- Internal vs. external propagation.
- Exogenous vs. endogenous risk.

A basic model with aggregate uncertainty, I

- Aggregate production function
$$Y_t = s_t F(K_t, L_t)$$
where $\{s_t\}$ is a sequence of random variables.

- Let
$$s_t \in \{s_b, s_g\} = S$$
with $s_b < s_g$ and conditional probabilities $\pi(s'|s)$.

- s_b is an economic recession and s_g is an expansion.

- Easy to extend to richer specifications.

A basic model with aggregate uncertainty, II

- Idiosyncratic labor productivity y_t

$$y_t \in Y = \{y_u, y_e\}$$

  with $y_u < y_e$.

- y_u stands for the household being unemployed and y_e stands for the household being employed.

- The distribution of y_t is correlated with aggregate productivity s_t.

- Probability of being unemployed is higher during recessions than during expansions.

- Let π be a 4×4 matrix with entry

$$\pi(y', s'|y, s) > 0$$

  that gives the conditional probability of individual productivity y', aggregate state s' tomorrow, conditional on (y, s) today.

Cross-sectional distributions

- Consistency requires that:

$$\sum_{y' \in Y} \pi(y', s'|y, s) = \pi(s'|s) \text{ all } y \in Y, \text{ all } s, s' \in S$$

- Law of large numbers: idiosyncratic risk averages out, only aggregate risk determines number of agents in states $y \in Y$.

- *Assume* that, cross-sectionally, the fraction of the population in idiosyncratic state $y = y_u$ is only a function of the aggregate state s. Denote the cross-sectional distribution by $\Pi_s(y)$.

- This assumption imposes additional restrictions on $\pi(y', s'|y, s)$:

$$\Pi_{s'}(y') = \sum_{y \in Y} \frac{\pi(y', s'|y, s)}{\pi(s'|s)} \Pi_s(y) \text{ for all } s, s' \in S$$

Recursive formulation

- Individual state variables (a, y).
- Aggregate state variables (s, Φ).
- Recursive formulation of household problem:

$$v(a, y, s, \Phi) = \max_{c, a' \geq 0} \{U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi(y', s' | y, s) v(a', y', s', \Phi')\}$$

$$\begin{aligned} \text{s.t. } c + a' &= w(s, \Phi) y + (1 + r(s, \Phi)) a \\ \Phi' &= H(s, \Phi, s') \end{aligned}$$

Recursive competitive equilibrium, I

A RCE is value function $v : Z \times S \times \mathcal{M} \to R$, household policy functions $c, a' : Z \times S \times \mathcal{M} \to R$, firm policy functions $K, L : S \times \mathcal{M} \to R$, pricing functions $r, w : S \times \mathcal{M} \to R$, aggregate law of motion $H : S \times \mathcal{M} \times S \to \mathcal{M}$ s.t.

1. v, a', c are measurable wrt $\mathcal{B}(S)$, v satisfies the household's Bellman equation and a', c are the associated policy functions, given $r()$ and $w()$
2. K, L satisfy, given $r()$ and $w()$

$$\begin{aligned} r(s, \Phi) &= F_K(K(s, \Phi), L(s, \Phi)) - \delta \\ w(s, \Phi) &= F_L(K(s, \Phi), L(s, \Phi)) \end{aligned}$$

Recursive competitive equilibrium, II

3. For all $\Phi \in \mathcal{M}$ and all $s \in S$

$$\begin{aligned} K(H(s,\Phi)) &= \int a'(a,y,s,\Phi)d\Phi \\ L(s,\Phi) &= \int y d\Phi \end{aligned}$$

$$\int c(a,y,s,\Phi)d\Phi + \int a'(a,y,s,\Phi)d\Phi = F(K(s,\Phi),L(s,\Phi)) + (1-\delta)K(s,\Phi)$$

4. Aggregate law of motion H is generated by exogenous Markov chain π and policy function a'.

Transition functions

- Define $Q_{\Phi,s,s'} : Z \times \mathcal{B}(Z) \to [0,1]$ by:

$$Q_{\Phi,s,s'}((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y', s'|y, s) \text{ if } a'(a, y, s, \Phi) \in \mathcal{A} \\ 0 \text{ else} \end{cases}$$

- Aggregate law of motion:

$$\Phi'(\mathcal{A},\mathcal{Y}) = (H(s,\Phi,s'))\,(\mathcal{A},\mathcal{Y}) = \int Q_{\Phi,s,s'}((a,y),(\mathcal{A},\mathcal{Y}))\Phi(da \times dy)$$

Lack of theoretical results

- We do not know about the existence of a recursive equilibrium in which the aggregate state only contains the current shock and the current wealth distribution (in fact, some counterexamples in Kubler and Schmedders, 2002).
- Miao (2006): existence of recursive equilibrium when we also track the cross-sectional distribution of expected payoffs.
- Similarly, we do not know about uniqueness.
- We will compute approximate equilibrium with boundedly rational agents (where approximation is not just due to numerical error).
- No sense as to whether this equilibrium is close to a true recursive equilibrium.
- Recently, idea of self-justified equilibria by Kubler and Scheidegger (2019).

Keeping track of wealth distribution

- Key challenge: wealth distribution Φ is an infinite-dimensional object.
- Why do agents need to keep track of Φ? In order to forecast future capital stock and, with it, future prices.
- But for K' need entire Φ since:

$$K' = \int a'(a, y, s, \Phi) d\Phi$$

- If a' were linear in a, with same slope for all $y \in Y$, exact aggregation obtained and average capital stock today *is* sufficient statistic for the average capital stock tomorrow.
- Krusell and Smith's proposal: approximate distribution Φ with a finite set of moments.

Computation, I

- Let n-dimensional vector m denote first n moments of asset distribution.
- Agents use an approximate law of motion:

$$m' = H_n(s, m)$$

- Agents are boundedly rational in the sense that moments of higher order than n of the current wealth distribution may help to more accurately forecast the first n moments tomorrow.
- Choose the number of moments and the functional form of the function H_n.

Computation, II

- Krusell and Smith pick $n = 1$ and pose

$$\log(K') = a_s + b_s \log(K)$$

  for $s \in \{s_b, s_g\}$. Here (a_s, b_s) are parameters that need to be determined.

- Household problem

$$v(a, y, s, K) = \max_{c, a' \geq 0} \{U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi(y', s' | y, s) v(a', y', s', K')\}$$

$$\begin{aligned} \text{s.t. } c + a' &= w(s, K)y + (1 + r(s, K))a \\ \log(K') &= a_s + b_s \log(K) \end{aligned}$$

- Reduction of the state space to a four dimensional space $(a, y, s, K) \in \mathbf{R} \times Y \times S \times \mathbf{R}$.

Algorithm, I

1. Guess (a_s, b_s).

2. Solve households problem to obtain $a'(a, y, s, K)$.

3. Simulate for large number of T periods, large number N of households:

   - Initial conditions for economy (s_0, K_0), for each household (a_0^i, y_0^i).

   - Draw random sequences $\{s_t\}_{t=1}^{T}$ and $\{y_t^i\}_{t=1,i=1}^{T,N}$, use decision rule $a'(a, y, s, K)$, perceived law of motion for K to generate $\{a_t^i\}_{t=1,i=1}^{T,N}$.

   - Aggregate:

$$K_t = \frac{1}{N} \sum_{i=1}^{N} a_t^i$$

Algorithm, II

4. Run the regressions

$$\log(K') = \alpha_s + \beta_s \log(K)$$

   to estimate (α_s, β_s) for $s \in S$.

5. If the R^2 for this regression is high and $(\alpha_s, \beta_s) \approx (a_s, b_s)$ stop. An approximate equilibrium is found. Otherwise update guess for (a_s, b_s).

6. If guesses for (a_s, b_s) converge, but R^2 remains low, add higher moments to the aggregate law of motion and/or use different functional form.

Parallelization

- As with Aiyagari Models, there are gains to parallelization.
- Computation of value function and simulation can be parallelized.
- Update of law of motion for moments of distribution is harder to parallelize.

A quantitative example

- Model period 1 quarter (business cycle model).
- *CRRA* utility with $\sigma = 1$ (i.e. log-utility).
- The time discount factor $\beta = 0.99^4 = 0.96$, i.e. $\rho = 4.1\%$.
- Capital share $\alpha = 0.36$.
- Annual depreciation rate $\delta = (1 - 0.025)^4 - 1 = 9.6\%$.

Aggregate shocks

- Two state process: expansion and recession: $S = \{0.99, 1.01\}$.
- St. dev. of technology shock is $\sigma_s = 0.01$.
- Transition matrix symmetric:

$$\pi(s_g|s_g) = \pi(s_b|s_b)$$

- Expected time in good state and bad state 8 quarters, hence

$$\begin{aligned} 8 &= [1 - \pi(s_g|s_g)] \left[1 + 2\pi(s_g|s_g) + 3\pi(s_g|s_g)^2 + \ldots\right] \\ \pi(s_g|s_g) &= \frac{7}{8} \end{aligned}$$

- Thus

$$\pi(s'|s) = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix}$$

Idiosyncratic shocks

- Two state process: employment and unemployment

$$Y = \{0.25, 1\}$$

- Unemployed person makes 25% of the labor income of an employed person.
- Transition probabilities:

$$\pi(y'|s', y, s) = \frac{\pi(y', s'|y, s)}{\pi(s'|s)}$$

  or

$$\pi(y', s'|y, s) = \pi(y'|s', y, s) * \pi(s'|s)$$

- Specify the four 2×2 matrices $\pi(y'|s', y, s)$ indicating, conditional on an aggregate transition from s to s', what the individual's probabilities of transition from employment to unemployment are.

Income process, I

- Expansion: average time of unemployment equal to 1.5 quarters:

$$\begin{aligned} 1.5 &= [1-\pi(y'=y_u|s'=s_g, y=y_u, s=s_g)] * \\ \sum_{i=1}^{\infty} i * \pi(y' &= y_u|s'=s_g, y=y_u, s=s_g)^{i-1} \\ \pi(y' &= y_u|s'=s_g, y=y_u, s=s_g) = \frac{1}{3} \end{aligned}$$

  Hence $\pi(y'=y_e|s'=s_g, y=y_u, s=s_g)=\frac{2}{3}$.

- Recession: average time of unemployment equal to 2.5 quarters:

$$\begin{aligned} \pi(y' &= y_u|s'=s_b, y=y_u, s=s_b) = 0.6 \\ \pi(y' &= y_e|s'=s_b, y=y_u, s=s_b) = 0.4 \end{aligned}$$

Income process, II

- Probability of remaining unemployed after switch from expansion to recession is 1.25 times the same probability when the economy was already in a recession

$$
\begin{aligned}
\pi(y' &= y_u | s' = s_b, y = y_u, s = s_g) = 0.75 \\
\pi(y' &= y_e | s' = s_b, y = y_u, s = s_g) = 0.25
\end{aligned}
$$

- Probability of remaining unemployed after switch from recession to expansion is 0.75 times the same probability when times were already good.

$$
\begin{aligned}
\pi(y' &= y_u | s' = s_g, y = y_u, s = s_b) = 0.25 \\
\pi(y' &= y_e | s' = s_g, y = y_u, s = s_b) = 0.75
\end{aligned}
$$

Income process, III

- Unemployment rate:

  1. Recessions: $\Pi_{s_b}(y_u) = 10\%$.
  2. Expansions: $\Pi_{s_g}(y_u) = 4\%$.

- Consistency with aggregate transition probabilities requires:

$$\Pi_{s'}(y') = \sum_{y \in Y} \frac{\pi(y', s'|y, s)}{\pi(s'|s)} \Pi_s(y) \text{ for all } s, s' \in S$$

Income process, IV

- Then:

$$
\begin{aligned}
\pi(y' &= y_u|s' = s_g, y = y_e, s = s_g) = 0.028 \\
\pi(y' &= y_e|s' = s_g, y = y_e, s = s_g) = 0.972 \\
\pi(y' &= y_u|s' = s_b, y = y_e, s = s_b) = 0.04 \\
\pi(y' &= y_e|s' = s_b, y = y_e, s = s_b) = 0.96 \\
\pi(y' &= y_u|s' = s_b, y = y_e, s = s_g) = 0.079 \\
\pi(y' &= y_e|s' = s_b, y = y_e, s = s_g) = 0.921 \\
\pi(y' &= y_u|s' = s_g, y = y_e, s = s_b) = 0.02 \\
\pi(y' &= y_e|s' = s_g, y = y_e, s = s_b) = 0.98
\end{aligned}
$$

- Best times for finding job when economy moves from recession to boom, worst chances when economy moves from boom into recession.

Income process, V

$$\pi = \begin{pmatrix} 0.525 & 0.035 & 0.09375 & 0.0099 \\ 0.35 & 0.84 & 0.03125 & 0.1151 \\ 0.03125 & 0.0025 & 0.292 & 0.0245 \\ 0.09375 & 0.1225 & 0.583 & 0.8505 \end{pmatrix}$$

Results, I

- Only thing to forecast is K'. Hence, we try $n = 1$.
- Converged law of motion:

$$\begin{aligned}\log(K') &= 0.095 + 0.962\log(K) \text{ for } s = s_g \\ \log(K') &= 0.085 + 0.965\log(K) \text{ for } s = s_b\end{aligned}$$

- Use simulated time series for aggregate capital stock with sequence of aggregate shocks $\{(s_t, K_t\}_{t=0}^{T}$. Divide sample into periods with $s_t = s_b$ and $s_t = s_g$ and run:

$$\log(K_{t+1}) = \alpha_j + \beta_j \log(K_t) + \varepsilon_{t+1}^{j}$$

- Define

$$\hat{\varepsilon}_{t+1}^{j} = \log(K_{t+1}) - \hat{\alpha}_j - \hat{\beta}_j \log(K_t) \text{ for } j = g, b$$

Results, II

- Define

$$
\begin{aligned}
\sigma_j &= \left(\frac{1}{T_j} \sum_{t \in \tau_j} \left(\hat{\tilde{\varepsilon}}_t^j \right)^2 \right)^{0.5} \\
R_j^2 &= 1 - \frac{\sum_{t \in \tau_j} \left(\hat{\tilde{\varepsilon}}_t^j \right)^2}{\sum_{t \in \tau_j} \left(\log K_{t+1} - \log \bar{K} \right)^2}
\end{aligned}
$$

- If $\sigma_j = 0$ for $j = g, b$ (if $R_j^2 = 1$ for $j = g, b$) then agents do not make forecasting errors.
- Estimates:

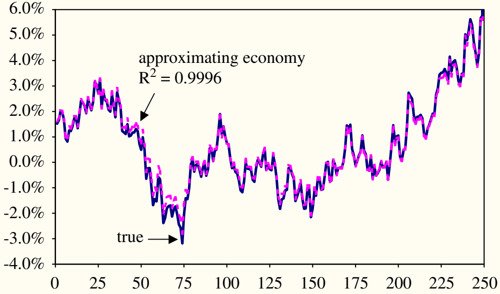
$$
\begin{aligned}
R_j^2 &= 0.999998 \text{ for } j = b, g \\
\sigma_g &= 0.0028 \\
\sigma_b &= 0.0036
\end{aligned}
$$

- Maximal forecasting errors for interest rates 25 years into the future is 0.1%. Corresponding utility losses?

Caution about accuracy

- Den Haan (2010) shows that one must be careful interpreting accuracy measures.
- R^2 measures comovement.
- Alternative measures (Euler equation errors,...).
- Interaction among errors.

A



approximating economy
$R^2 = 0.9996$
true
6.0%
5.0%
4.0%
3.0%
2.0%
1.0%
0.0%
-1.0%
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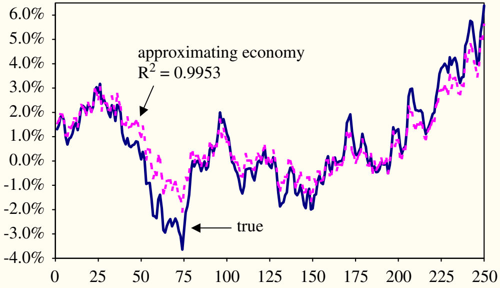
-3.0%
-4.0%
0 25 50 75 100 125 150 175 200 225 250

B



approximating economy
$R^2 = 0.9953$
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6.0%
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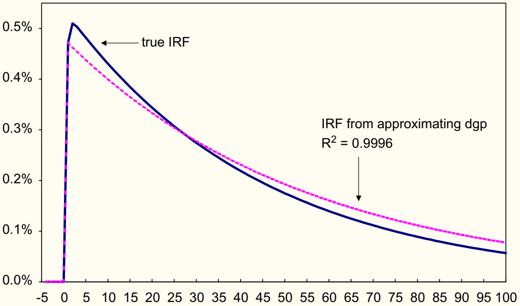
A



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0.4%
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0.2%
0.1%
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true IRF
IRF from approximating dgp
R2 = 0.9996
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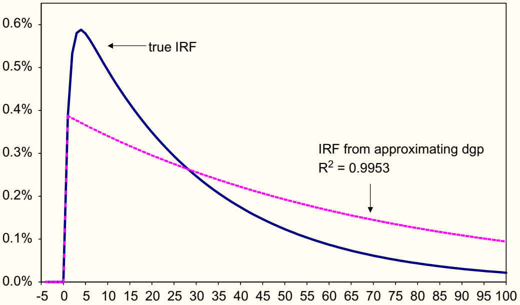
B



0.6%
0.5%
0.4%
0.3%
0.2%
0.1%
0.0%
true IRF
IRF from approximating dgp
R2 = 0.9953
-5 0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100

Quasi-aggregation

- Suppose all agents have linear savings functions with same marginal propensity to save

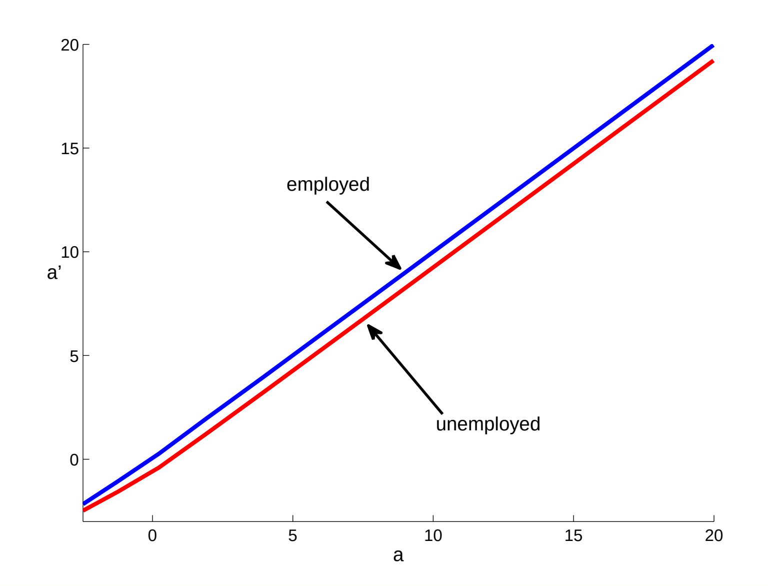
$$a'(a, y, s, K) = a_s + b_s a + c_s y$$

- Then

$$K' = \int a'(a, y, s, K) d\Phi = a_s + b_s \int a d\Phi + c_s \bar{L} = \tilde{a}_s + b_s K$$

- Exact aggregation obtains: first moment of the current wealth distribution is a sufficient statistic for Φ for forecasting aggregate capital stock tomorrow.
- In this economy: savings functions almost linear with same slope for $y = y_u$ and $y = y_e$.
- Only exceptions: unlucky ($y = y_u$) liquidity constrained agents. But these agents hold negligible fraction of aggregate wealth and do not matter for aggregate capital dynamics.
- Hence quasi-aggregation!

Policy functions



Why is marginal propensity to consume close to 1?

- Consumption under certainty equivalence and $r = \frac{1}{\beta} - 1$

$$c_t = \frac{r}{1+r}\left(\mathbb{E}_t \sum_{s=0}^{T-t} \frac{y_{t+s}}{(1+r)^s} + a_t\right)$$

- Agents save out of current assets for tomorrow

$$\frac{a_{t+1}}{1+r} = \left(1 - \frac{r}{1+r}\right) a_t + G(y)$$

- Thus, under certainty equivalence: $a_{t+1} = a_t + H(y)$.
- Here agents are prudent, face liquidity constraints, but almost act as if they are certainty equivalence consumers. Why?
  1. With $\sigma = 1$ agents are prudent, but not all that much
  2. Unconditional standard deviation of individual income is roughly 0.2, at the lower end of the estimates used by Aiyagari.
  3. Negative income shocks infrequent, not very persistent.

Endogenous wealth distribution

- Income distribution is input into the model
- Does realistic income process lead to realistic wealth distributions?
- No: it fails to generate the high concentration of wealth at the upper end of the distribution:

	1%	5%	10%	20%	30%	Gini
Data	30	51	64	79	88	0.79
Model	3	11	19	35	46	0.25

- Solutions.

A simpler approach

- Recent proposal by Boppart, Krusell, and Mitman (2018) of exploiting "MIT" shocks.
- It builds on the idea of combining projection and perturbation: Reiter (2008).
- Variation, as well, of the standard transitional dynamics algorithm we already saw for Aiyagari models.
- One can use the path for prices generated by a RA model as an initial guess.
- No theoretical convergence properties, but it works quite well in practice.

Algorithm

1. Assume a path for the exogenous shock.
2. Compute the steady state equilibrium using the techniques we learned from Aiyagari's models, including v^{SS}. This will be the initial and final position of the economy.
3. Chose a time T at which we assume the economy is back into the stationary equilibrium after the exogenous shock.
4. Guess a path of prices for $t = 0, ..., T$ (or, equivalently, of aggregate variables such as capital-labor ratio).
5. Solve the value function (and policy functions) backwards from $t = T - 1, ..., 1$ setting $v^T = v^{SS}$.
6. Compute the associated distributions and market clearing.
7. Update prices (or aggregate variables) until convergence.

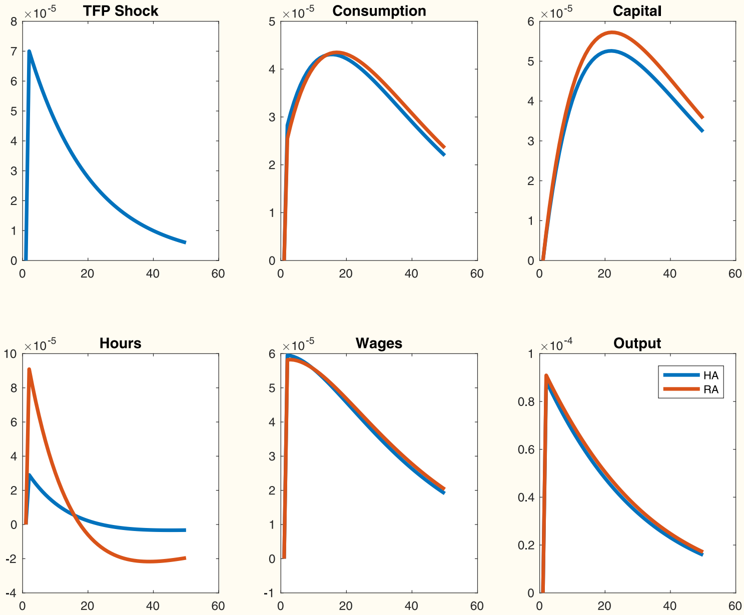


Fig. 2. Impulse response to neutral technology shock for the HA and RA economies.

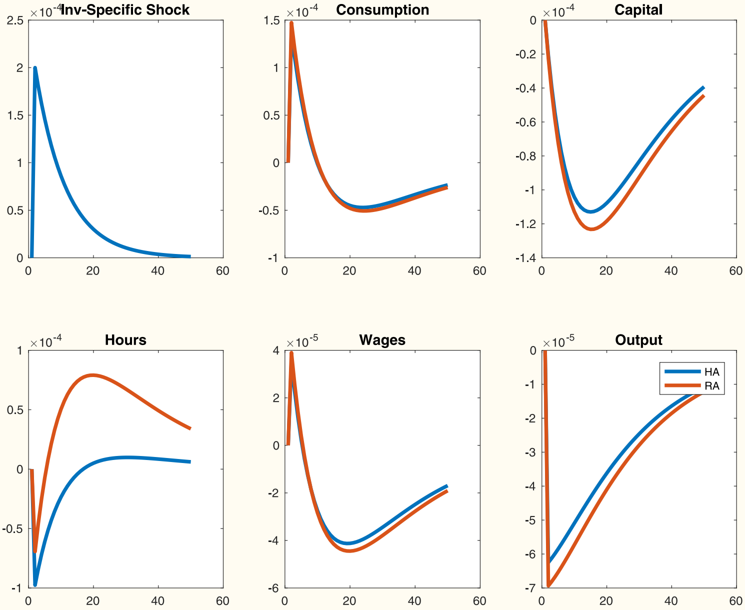


Fig. 3. Impulse response to investment-specific technology shock for the HA and RA economies.

New developments

1. Xpa algorithm: Algan, Allais, Den Haan, and Rendahl (2010).
2. Perturbation: Roca and Preston (2007).
3. Function-Valued States: Childers (2016).
4. Low-dimensional smooth approximation of cross-sectional distributions: Winberry (2018).
5. Linearization with the assumption of a cross-sectional distribution with constant copula across household variables: Bayer and Luetticke (2018).
6. Aggregation of households not by capital, but by shock history: aggregation of households not by capital, but by shock history: Grand and Ragot (2019).
7. Continuous time: Achdou, Lasry, Lions, and Moll (2013).