A Model with Explicit Solution

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Motivation

- Brunnermeier and Sannikov, 2012.
- Full equilibrium dynamics of an economy with financial frictions:
  1. Nonlinearity: model will respond very differently to small and large shocks.
  2. Volatility paradox: lower values of exogenous risk may lead to higher levels of endogenous risk.
- Features:
  1. Continuous time.
  2. We have more productive but less patient agents borrowing from less productive but more patient agents. Financial frictions difficult the flow of funds between both groups.

Preferences

- Continuum of infinitely lived, risk-neutral agents:
  1. Experts, $\mathbb{I} = [0, 1]$:
  $$\mathbb{E}_0 \int e^{-\rho t} dc_t$$
  where c_t is cumulative consumption until time t.
  We impose $dc_t \geq 0$.
  2. Households, $\mathbb{J} = [0, 1]$:
  $$\mathbb{E}_0 \int e^{-rt} d\underline{c}_t$$
  where $\underline{c}_t$ is cumulative consumption until time t.
  We do not impose $d\underline{c}_t \geq 0$ (negative consumption can be thought as additional labor effort): hence r is risk-free rate.
- Assumption : $r < \rho$.

Technology I

- Experts with efficiency units of capital k_t produce output:

$$y_t = ak_t$$

- Experts can invest:

$$dk_t = (\phi(\iota_t) - \delta) k_t dt + \sigma k_t dZ_t$$

where

1. ι_t is investment rate per unit of capital.
2. $\phi(\iota_t)$ is an investment technology with adjustment costs ($\phi(0) = 0$, $\phi'(\cdot) = 0$, and $\phi''(\cdot) < 0$).
3. We do not impose $\iota_t > 0$. Concavity of $\phi(\cdot)$ imposes large costs to disinvestment.
4. dZ_t is a Brownian motion.

Technology II

- Households with efficiency units of capital $\underline{k}_t$ produce output:

$$\underline{y}_t = \underline{a}\underline{k}_t$$

where $\underline{a} < a$.

- Households can invest:

$$d\underline{k}_t = \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta}\right)\underline{k}_t dt + \sigma \underline{k}_t dZ_t$$

where $\underline{\delta} > \delta$.

- We take output as numeraire.

Price of Capital

- Capital can be traded at price q_t, which evolves as:

$$dq_t = \mu_t^q q_t dt + \sigma_t^q q_t dZ_t$$

- Boundaries for price of capital:

$$\begin{aligned} \underline{q} &= \max_{\underline{\iota}} \frac{\underline{a} - \underline{\iota}}{r - \phi\left(\underline{\iota}\right) + \underline{\delta}} \\ \overline{q} &= \max_{\underline{\iota}} \frac{a - \iota}{r - \phi\left(\iota\right) + \delta} \end{aligned}$$

Returns I

- Thus, the value of the capital hold by an expert generates:
  1. Capital gains:

$$\frac{d\left(k_t q_t\right)}{k_t q_t} = \left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t$$

  2. Dividend

$$\frac{a - \iota_t}{q_t}$$

  3. Total return:

$$dr_t^k = \frac{a - \iota_t}{q_t} + \left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t$$

- Similarly, return for a household:

$$d\underline{r}_t^k = \frac{\underline{a} - \underline{\iota}_t}{q_t} + \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t$$

Returns II

- Two components of risk on returns:

  1. σdZ_t is exogenous risk caused by stochastic process for capital efficiency.

  2. $\sigma_t^q dZ_t$ is endogenous risk caused by financial frictions. Without them, $\sigma_t^q = 0$ because $q_t = \overline{q}$.

- Even if experts are risk-neutral with respect to consumption, they exhibit risk-averse behavior because the return to investment is time-varying.

- Experts suffer losses when they want to buy more capital: its price is lower.

Budget Constraint

- Because of an agency problem, experts must retain 100 percent of equity and finance the rest of their investment with risk-free debt.

- Experts:

$$\frac{dn_t}{n_t} = x_t dr_t^k + (1 - x_t)\, rdt - \frac{dc_t}{n_t}$$

where $n_t \geq 0$ is net wealth, a fraction $x_t \geq 0$ invested in capital and a fraction $1 - x_t$ in the risk-free asset. In general, $x_t > 1$. The solvency constrain: $n_t \geq 0$.

- Households:

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t d\underline{r}_t^k + (1 - \underline{x}_t)\, rdt - \frac{d\underline{c}_t}{\underline{n}_t}$$

with $\underline{x}_t \geq 0$.

Equilibrium I

Definition

For any initial endowments of capital $\left\{k_0^i, \underline{k}_0^j; i \in \mathbb{I}, j \in \mathbb{J}\right\}$ such that:

$$\int_{\mathbb{I}} k_0^i di + \int_{\mathbb{J}} \underline{k}_0^i dj = K_0$$

an equilibrium is described by a group of stochastic processes on the filtered probability space defined by the Brownian motion $\{Z_t, t \geq 0\}$: the price process for capital $\{q_t\}$, net worths $\left\{n_t^i, \underline{n}_t^j \geq 0\right\}$, capital holdings $\left\{k_t^i, \underline{k}_t^j \geq 0\right\}$, investment decisions $\left\{\iota_t^i, \underline{\iota}_t^j\right\}$, and consumption choices $\left\{dc_t^i \geq 0, d\underline{c}_t^j\right\}$ of individual agents $i \in \mathbb{I}, j \in \mathbb{J}$ such that:

1. Given prices, all experts and households maximize.
2. Initial net worths satisfy $n_0^i = k_0^i q_0$ and $\underline{n}_0^i = \underline{k}_0^i q_0$ for all $i \in \mathbb{I}, j \in \mathbb{J}$.

Equilibrium II

Definition

3. Markets clear:

$$\int_{\mathbb{I}} k_t^i di + \int_{\mathbb{J}} \underline{k}_t^i dj = K_t$$

$$\int_{\mathbb{I}} \left(dc_t^i\right) di + \int_{\mathbb{J}} \left(d\underline{c}_t^i\right) dj = \left(\int_{\mathbb{I}} \left(a - \iota_t^i\right) k_t^i di + \int_{\mathbb{J}} \left(\underline{a} - \underline{\iota}_t^j\right) \underline{k}_t^j dj \right) dt$$

$$dK_t = \left(\int_{\mathbb{I}} \left(\phi\left(\iota_t^i\right) - \delta\right) k_t^i di + \int_{\mathbb{J}} \left(\phi\left(\underline{\iota}_t^i\right) - \underline{\delta}\right) \underline{k}_t^j dj \right) dt + \sigma K_t dZ_t$$

Excess Returns I

- First, to maximize experts return with respect to ι_t, from

$$dr_t^k = \frac{a - \iota_t}{q_t} + \left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t$$

  set:

$$\phi'\left(\iota_t\right) = \frac{1}{q_t} \Rightarrow \iota_t = \iota\left(q_t\right)$$

- Similarly, for a household:

$$\phi'\left(\underline{\iota}_t\right) = \frac{1}{q_t} \Rightarrow \underline{\iota}_t = \iota\left(q_t\right)$$

- Thus:

$$\iota_t = \underline{\iota}_t = \iota\left(q_t\right)$$

Excess Returns II

- Now, define expected excess returns:

$$\frac{\mathbb{E}_t dr_t^k}{dt} - r = \frac{a - \iota(q_t)}{q_t} + \phi(\iota(q_t)) - \delta + \mu_t^q + \sigma\sigma_t^q - r$$

$$\frac{\mathbb{E}_t d\underline{r}_t^k}{dt} - r = \frac{\underline{a} - \iota(q_t)}{q_t} + \phi(\iota(q_t)) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - r$$

Optimal Strategies I

- Consider a feasible strategy for the experts $\{x_t, d\zeta_t\}$ with payoff:

$$\theta_t n_t = \mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} dc_s$$

  where $dc_t = n_t d\zeta_t$

- Then, for a price process for capital $\{q_t\}$ that generates a finite maximal payoff, a feasible strategy is optimal if and only if:

$$\max_{\widehat{x}_t \geq 0, d\widehat{\zeta}_t \geq 0} n_t d\widehat{\zeta}_t + \mathbb{E}_t (\theta_t n_t)$$

$$\text{s.t. } \frac{dn_t}{n_t} = \widehat{x}_t dr_t^k + (1 - \widehat{x}_t) rdt - d\widehat{\zeta}_t$$

$$\mathbb{E}_t e^{-\rho t} \theta_t n_t \rightarrow 0$$

Optimal Strategies II

- Moreover, for

$$\frac{d\theta_t}{\theta_t} = \mu_t^{\theta} dt + \sigma_t^{\theta} dZ_t$$

a feasible strategy is optimal if and only if:

1. $\theta_t \geq 1$ and $d\zeta_t > 0$ only when $\theta_t = 1$.
2. $\mu_t^{\theta} = \rho - r$.
3. Either:

$$x_t > 0 \text{ and } \frac{\mathbb{E}_t dr_t^k}{dt} - r = -\left(\sigma + \sigma_t^q\right)\sigma_t^{\theta} \text{ (desire to leverage)}$$

or

$$x_t = 0 \text{ and } \frac{\mathbb{E}_t dr_t^k}{dt} - r \leq -\left(\sigma + \sigma_t^q\right)\sigma_t^{\theta} \text{ (flight to quality)}$$

4. $\mathbb{E} e^{-\rho t}\theta_t n_t \to 0$.

Optimal Strategies III

- For the households,

$$\frac{\mathbb{E}_t d\underline{r}_t^k}{dt} - r \leq 0$$

with equality if

$$1 - \psi_t = \frac{1}{K_t} \int_{\mathbb{J}} \underline{k}_t^i dj > 0$$

- It can be verified that, in equilibrium,

$$\psi_t q_t K_t > N_t = \int_{\mathbb{I}} n_t^i di$$

Wealth Distribution I

- Aggregate wealth:

$$N_t = \int_{\mathbb{I}} n_t^i di$$

$$q_t K_t - N_t = \int_{\mathbb{J}} \underline{n}_t^i dj$$

- Hence, experts's wealth share:

$$\eta_t = \frac{N_t}{q_t K_t} \in [0, 1]$$

Wealth Distribution II

- Law of motion:

$$\begin{aligned}\frac{d\eta_t}{\eta_t} &= \left(\frac{\psi_t - \eta_t}{\eta_t}\right)\left(dr^k - rdt - \left(\sigma + \sigma_t^q\right)^2 dt\right) \\ &+ \left(\frac{a - \iota\left(q_t\right)}{q_t} + \left(1 - \psi_t\right)\left(\underline{\delta} - \delta\right)\right) dt - d\zeta_t\end{aligned}$$

- If $\psi_t > 0 \Rightarrow x_t = 0$ and we have

$$r = \frac{\mathbb{E}_t dr_t^k}{dt} + \left(\sigma + \sigma_t^q\right)\sigma_t^\theta \Rightarrow rdt = \mathbb{E}_t dr_t^k + \left(\sigma + \sigma_t^q\right)\sigma_t^\theta dt$$

- Then:

$$\begin{aligned}dr^k - rdt &= dr^k - \mathbb{E}_t dr_t^k - \left(\sigma + \sigma_t^q\right)\sigma_t^\theta dt \\ &= \left(\sigma + \sigma_t^q\right) dZ_t - \left(\sigma + \sigma_t^q\right)\sigma_t^\theta dt \\ &= \left(\sigma + \sigma_t^q\right)\left(dZ_t - \sigma_t^\theta dt\right)\end{aligned}$$

Wealth Distribution III

- We can substitute

$$
\begin{aligned}
\frac{d\eta_t}{\eta_t} &= \left(\frac{\psi_t - \eta_t}{\eta_t}\right)\left(\sigma + \sigma_t^q\right)\left(dZ_t - \sigma_t^\theta dt - \left(\sigma + \sigma_t^q\right)dt\right) \\
&\quad + \left(\frac{a - \iota\left(q_t\right)}{q_t} + \left(1 - \psi_t\right)\left(\underline{\delta} - \delta\right)\right)dt - d\zeta_t \\
&= \left(\begin{array}{c} -\frac{\psi_t - \eta_t}{\eta_t}\left(\sigma + \sigma_t^q\right)\left(\sigma + \sigma_t^\theta + \sigma_t^q\right) \\ +\frac{a - \iota\left(q_t\right)}{q_t} + \left(1 - \psi_t\right)\left(\underline{\delta} - \delta\right) \end{array}\right)dt \\
&\quad + \frac{\psi_t - \eta_t}{\eta_t}\left(\sigma + \sigma_t^q\right)dZ_t - d\zeta_t
\end{aligned}
$$

Wealth Distribution IV

- By defining

$$\mu_t^n = -\frac{\psi_t - \eta_t}{\eta_t}\left(\sigma + \sigma_t^q\right)\left(\sigma + \sigma_t^q + \sigma_t^\theta\right) + \frac{a - \iota\left(q_t\right)}{q_t} + \left(1 - \psi_t\right)\left(\underline{\delta} - \delta\right)$$

$$\sigma_t^\eta = \frac{\psi_t - \eta_t}{\eta_t}\left(\sigma + \sigma_t^q\right)$$

we get

$$\frac{d\eta_t}{\eta_t} = \mu_t^n dt + \sigma_t^\eta dZ_t - d\zeta_t$$

Markov Equilibria

- Search for functions:

$$
\begin{aligned}
q_t &= q(\eta_t) \\
\theta_t &= \theta(\eta_t) \\
\psi_t &= \psi(\eta_t)
\end{aligned}
$$

- Then, once we know η_t, we can get q_t, θ_t, ψ_t, and from

$$
\frac{d\eta_t}{\eta_t} = \mu_t^n dt + \sigma_t^\eta dZ_t - d\zeta_t
$$

we get $d\eta_t$.

A Proposition I

Let us suppose that we know $\left(\eta, q\left(\eta\right), q'\left(\eta\right), \theta\left(\eta\right), \theta'\left(\eta\right)\right)$.

1. Find $\psi \in \left(\eta, \eta + \frac{q'(\eta)}{q(\eta)}\right)$ such that:

$$\frac{a - \underline{a}}{q\left(\eta\right)} + \underline{\delta} - \delta + \sigma_t^{\theta}\left(\sigma + \sigma_t^{q}\right) = 0$$

where:

$$\begin{aligned}
\sigma_t^{\eta} &= \frac{1}{\eta}\frac{\left(\psi - \eta\right) a}{1 - \left(\psi - \eta\right)\frac{q'(\eta)}{q(\eta)}} \\
\sigma_t^{q} &= \frac{q'\left(\eta\right)}{q\left(\eta\right)}\sigma_t^{\eta}\eta \\
\sigma_t^{\theta} &= \frac{\theta'\left(\eta\right)}{\theta\left(\eta\right)}\sigma_t^{\eta}\eta
\end{aligned}$$

A Proposition II

2. If $\psi > 1$, set $\psi = 1$ and recalculate σ_t^η, σ_t^q, σ_t^θ.
3. Compute:

$$\mu_t^n = -\sigma_t^\eta \left(\sigma + \sigma_t^q + \sigma_t^\theta\right) + \frac{a - \iota\left(q\left(\eta\right)\right)}{q\left(\eta\right)} + \left(1 - \psi\right)\left(\underline{\delta} - \delta\right)$$

$$\mu_t^q = r - \frac{a - \iota\left(q\left(\eta\right)\right)}{q\left(\eta\right)} - \phi\left(q\left(\eta\right)\right) + \delta - \sigma\sigma_t^q - \sigma_t^\theta\left(\sigma + \sigma_t^q\right)$$

$$\mu_t^\theta = \rho - r$$

$$q''\left(\eta\right) = \frac{2\left(\mu_t^q q\left(\eta\right) - q'\left(\eta\right)\mu_t^n \eta\right)}{\left(\sigma_t^\eta\right)^2 \eta^2}$$

$$\theta''\left(\eta\right) = \frac{2\left(\mu_t^\theta \theta\left(\eta\right) - \theta'\left(\eta\right)\mu_t^n \eta\right)}{\left(\sigma_t^\eta\right)^2 \eta^2}$$

A Proposition III

4. Use boundary conditions

$$q'(\eta^*) = 0,\ q(0) = \underline{q}$$
$$\theta(\eta^*) = 1,\ \theta'(\eta^*) = 0$$
$$\lim \theta(\eta) = \infty$$

where η^* is the reflecting boundary when experts consume.

An Algorithm

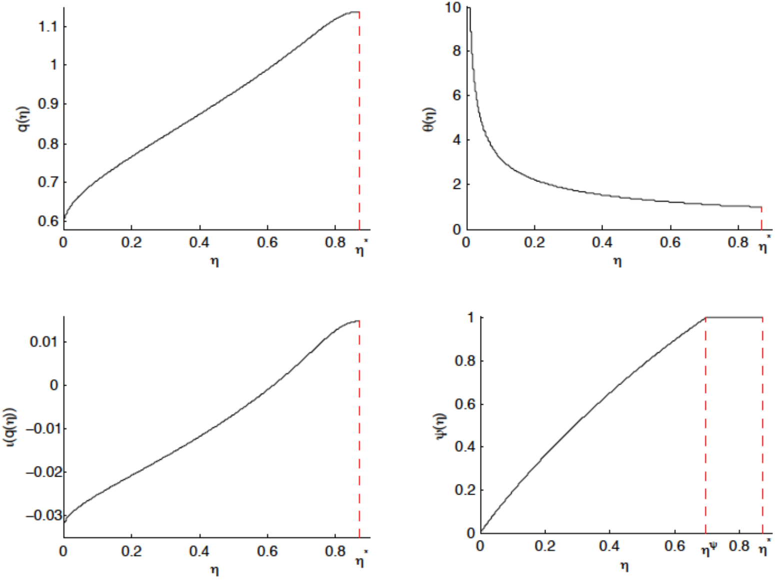
- Set $q(0) = \underline{q}$, $\theta(0) = 1$, and $\theta'(0) =$small number.
- Set $q_L = 0$ and $q_H =$large number.
- Guess $q'(0) = \frac{q_H + q_L}{2}$ and solve for $q(\eta)$ and $\theta(\eta)$ until the first of the three conditions holds:
    1. $q(\eta) = \overline{q}$.
    2. $\theta'(\eta) = 0$.
    3. $q'(\eta) = 0$.
- If $q'(\eta) = 0$, set $q_L = q'(0)$, otherwise $q_H = q'(0)$.
- Iterate until convergence.
- Check that $q'(\eta)$ and $\theta'(\eta)$ reach 0 at the same point η^*.
- Normalize $\theta(\eta) = \frac{\theta(\eta)}{\theta(\eta^*)}$ to match boundary condition.

Calibration

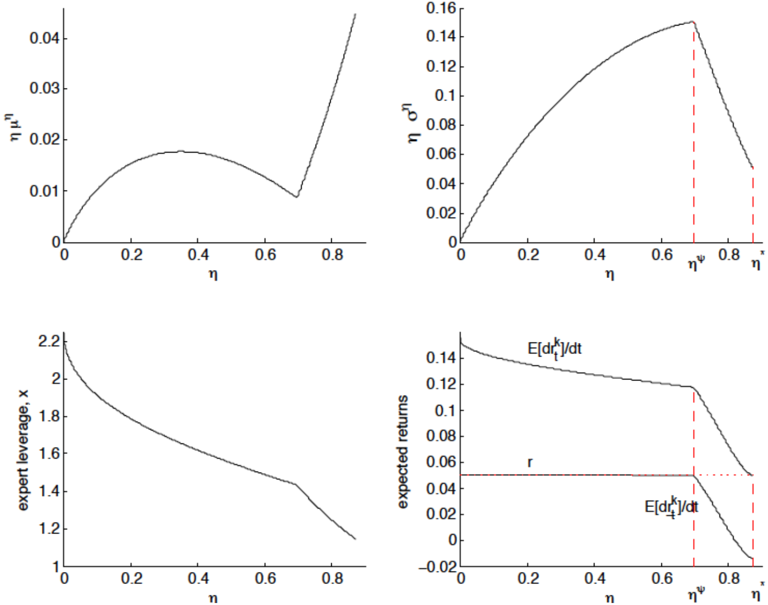
- Parameters

ρ	r	a	$\underline{a}$	δ	$\underline{\delta}$	σ	$\phi(\iota)$
0.06	0.05	0.1	0.05	0.03	0.05	0.4	$0.1\sqrt{1+20\iota}$

- This implies $\underline{q}$= 0.5858 and $\overline{q} = 1.3101$.



q(η)
1.1
1
0.9
0.8
0.7
0.6
0
0.2
0.4
0.6
0.8
η*
η
θ(η)
10
8
6
4
2
0
0
0.2
0.4
0.6
0.8
η*
η
ι(q(η))
0.01
0
−0.01
−0.02
−0.03
0
0.2
0.4
0.6
0.8
η*
η
ψ(η)
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
ηψ
0.8
η*
η



$\eta \mu^\eta$
η
$\eta \sigma^\eta$
η^ψ
η^*
expert leverage, x
expected returns
$E[dr_t^k]/dt$
r
$E[d\tilde{r}_t^k]/dt$

Three Inefficiencies

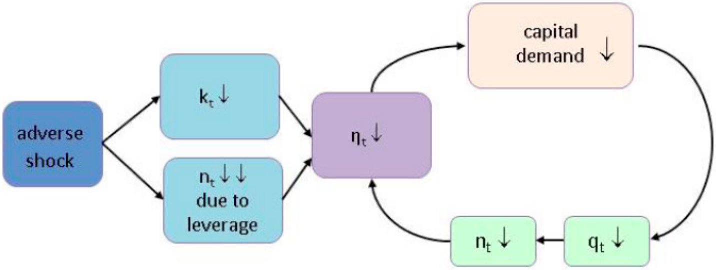
1. Capital misallocation: for low η_t, households manage part of the capital $(\psi < 1)$.

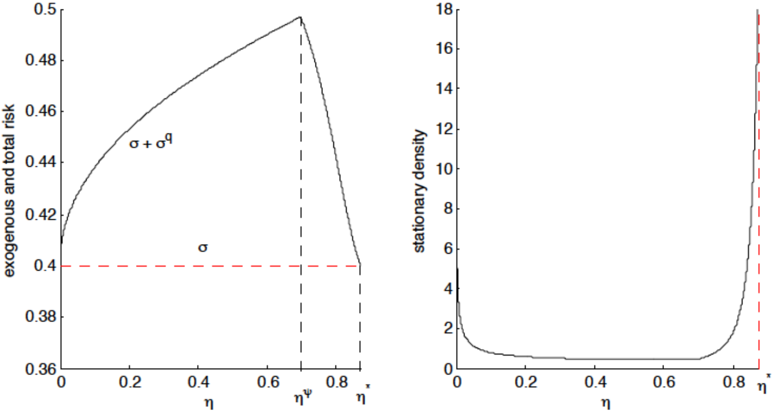
2. Under investment: $\iota(q_t) < \iota(\overline{q})$.

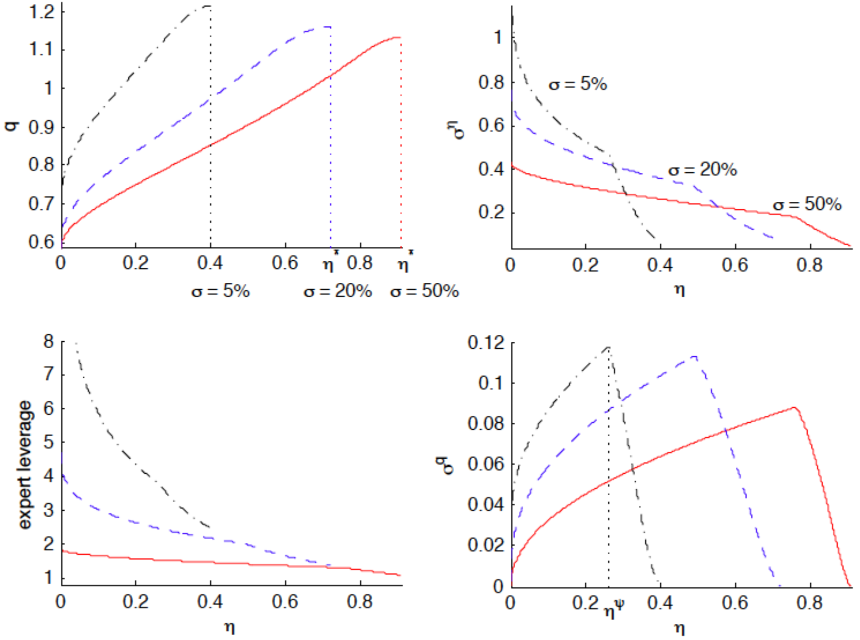
3. Consumption distortions: experts should only consume at time 0.

- Note: inefficiencies get worse for low η_t.

Adverse Feedback Loop




0.5
0.48
0.46
0.44
0.42
0.4
0.38
0.36
exogenous and total risk
σ + σ^q
σ
0
0.2
0.4
0.6
0.8
η
η^ψ
η*
18
16
14
12
10
8
6
4
2
0
stationary density
0
0.2
0.4
0.6
0.8
η*
η



q
1.2
1.1
1
0.9
0.8
0.7
0.6
0
0.2
0.4
0.6
η*
0.8
η*
σ = 5%
σ = 20%
σ = 50%
σ^η
σ = 5%
σ = 20%
σ = 50%
η
expert leverage
σ^q
η^ψ

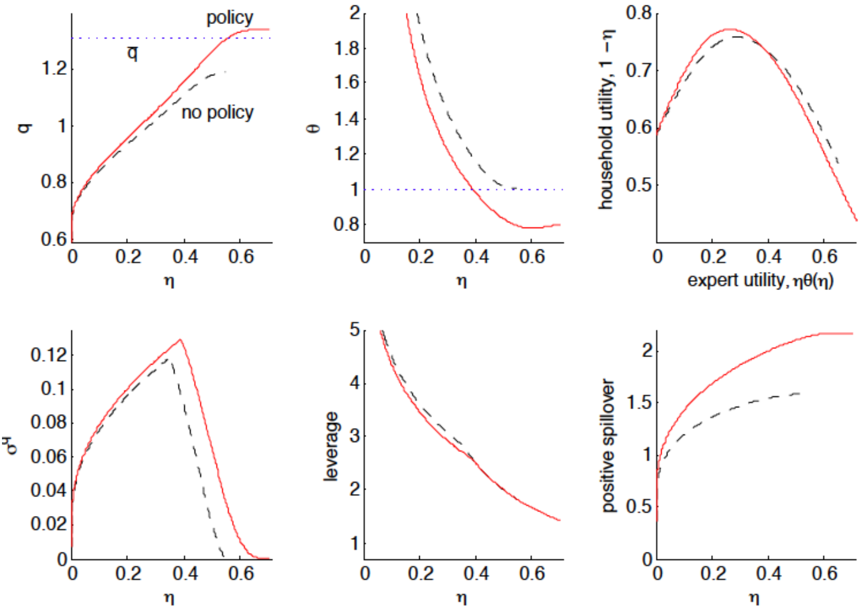

policy
q̄
no policy
q
η
θ
η
household utility, 1 − η
expert utility, ηθ(η)
σ^q
η
leverage
η
positive spillover
η

Figure 9: The peculiar dynamics of VIX: 2004-2012. Source: CBOE