# Optimal policies with heterogeneous agents

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# Prelude: Infinite dimensional control

# Solving infinite dimensional control problems

- The standard theory of optimization in finite dimensional spaces (e.g., $\mathbb{R}^n$) can be extended to function spaces.
- The core was developed by Euler and Lagrange in the 18th century: calculus of variations.
- Put in more rigorous footing in the 20th century.
- Milestones in the history of optimization theory: http://www.mitrikitti.fi/opthist.html.

# Gâteaux differential

- Let $J(g)$ be a functional and let $h$ be arbitrary in $L^2(\Phi)$. If the limit:

$$\delta J(g;h) = \lim_{\alpha \to 0} \frac{J(g+\alpha h) - J(g)}{\alpha}$$

  exists, it is called the *Gâteaux derivative* of $J$ at $g$ in the direction $h$.

- If the limit exists for each $h \in L^2(\Phi)$, the functional $J$ is said to be *Gâteaux differentiable* at $g$.

- If the limit exists, it can be expressed as $\delta J(g;h) = \frac{d}{d\alpha} J(g+\alpha h)\,|_{\alpha=0}$.

## Fréchet differential

- Let $h$ be arbitrary in $L^2(\Phi)$. If for fixed $g \in L^2(\Phi)$ there exists $\delta J(g;h)$ which is linear and continuous with respect to $h$ such that:

$$\lim_{\|h\|_{L^2(\Phi)} \to 0} \frac{|J(g+h) - J(g) - \delta J(g;h)|}{\|h\|_{L^2(\Phi)}} = 0,$$

  then, $J$ is said to be *Fréchet differentiable* at $g$ and $\delta J(g;h)$ is the*Fréchet differential* of $J$ at $g$ with increment $h$.

- If the Fréchet differential of $J$ exists at $g$, then the Gâteaux differential exists at $g$ and they are equal.

- See Luenberger (1969, p. 173).

## Infinite dimensional optimization

- A function $g \in L^2(\Phi)$ is a maximum of $J(g)$ if for all functions $h$, $\|h - g\|_{L^2(\Phi)} < \varepsilon$, then $J(g) \geq J(h)$.

- Fundamental Theorem of Calculus: Let $J$ have a Gâteaux differential, a necessary condition for $J$ to have an maximum at $g$ is that $\delta J(g; h) = 0$ for all $h \in L^2(\Phi)$.

- See Luenberger (1969, p. 173), Gelfand and Fomin (1991, pp. 13-14), or Sagan (1992, p. 34).

# Constrained optimization

- Let $H$ be a mapping from $L^2(\Phi)$ into $\mathbb{R}$.
- If $J$ has a continuous Fréchet differential, a necessary condition for $J$ to have a maximum at $g$ under the constraint $H(g) = 0$ is that there exists a function $\lambda \in L^2(\Phi)$ such that:

$$\mathcal{L}(g) = J(g) + \langle \lambda, H(g) \rangle_{\Phi}$$

is stationary in $g$, i.e., $\delta\mathcal{L}(g; h) = 0$.

- See Luenberger (1969, p. 243).
- Example

# Application 1: Social optima with heterogeneous agents

# Optimal allocations with heterogeneous agents

- Often questions in economics require computing the **optimal allocation** produced by a benevolent social planner:
  - This is relatively straightforward with a representative agent...
  - ...but what about a continuum of **heterogeneous agents**?
- Other problems may also have an infinite-dimensional space state.
  - Examples: Oil extraction with a distribution of reserves, spatial AK models, ...

# How to do it

- Nuño and Moll (2018) analyze **optimal control problems** with a continuum of **heterogeneous agents**.
- Example: constrained-efficient equilibrium in the Aiyagari model with stochastic lifetimes.

## Related literature

- Constrained-efficient problems in discrete-time models with incomplete markets and idiosyncratic risk.
    - Dávila, Hong, Krusell, and Ríos-Rull (2012).
- Optimal control problems in continuous time:
    - Lucas and Moll (2014) or Afonso and Lagos (2015).
- Mean field control:
    - Bensoussan, Frehse, and Yam (2013).

# We illustrate the method with an example: Aiyagari model with finite lifetimes

- Continuous-time Aiyagari economy with stochastic lifetimes *à la* Blanchard-Yaari.
- A benevolent social planner chooses the individual levels of consumption, while respecting all budget constraints
- With infinite lifetimes optimal allocation depends on the calibration (Dávila, Hong, Krusell, and Ríos-Rull, 2012).
  - No ergodic distribution under the original Aiyagari's calibration.
  - No Pareto distribution in the competitive equilibrium.

# Households

- Household's utility:

$$\mathbb{E}_0 \left[ \int_0^\infty e^{-(\rho+\eta)t} \frac{c_t^{1-\chi}}{1-\chi} dt \right]$$

  where $\eta$ is the death arrival (Poisson).

- Asset dynamics (per capita), assuming insurance sector:

$$da_t = (w_t z_t + (r_t + \eta)\, a_t - c_t)\, dt$$

- Borrowing limit:

$$a_t \geq 0$$

- Idiosyncratic labor productivity:

$$dz_t = \theta(\hat{z} - z_t)dt + \sigma dB_t, \;\; z_t \in [\underline{z}, \bar{z}]$$

# The KF equation

$$
\begin{aligned}
\frac{\partial g}{\partial t} = & -\frac{\partial}{\partial a}\left(s\left(a, z, w_t, r_t, c\right) g\right) \\
& -\frac{\partial}{\partial z}\left(\theta(\hat{z}-z) g\right)+\frac{1}{2} \frac{\partial^2}{\partial z^2}\left(\sigma^2 g\right)-\eta g+\eta \delta_{a_0, z_0},
\end{aligned}
$$

where $-\eta g_t(a, z)$ is the outflow of agents due to death and $\eta \delta_{a_0, z_0}=\eta \delta\left(a\right) \delta\left(z-\underline{z}\right)$ is the inflow of newborn agents with zero assets and productivity $\underline{z}$.

# Firms and market clearing

- Competitive firms:

$$
\begin{aligned}
r_t &= \alpha k_t^{\alpha-1} - \delta_K \\
w_t &= (1-\alpha)\, k_t^{\alpha}
\end{aligned}
$$

- Market clearing:

$$k_t = \int \int_{\underline{z}}^{\bar{z}} a g_t(a,z) da dz$$

# Constrained-efficient allocation

- The planner chooses individual consumption $c(\cdot)$ in order to maximize:

$$J(g(0,\cdot)) = \max_{c(\cdot)\in\mathcal{C}(t,a,z)} \int_0^\infty e^{-\rho t} \int \int_{\underline{z}}^{\bar{z}} u(c)\, g_t(a,z)\, dadzdt$$

  subject to the law of motion of the aggregate density, to the factor prices and to the market clearing condition.

- The planner cannot redistribute among agents (she has to respect individual budget constraints).

# The Lagrangian

$$\begin{aligned}\mathcal{L}_{ce}(g,\tau,c,j,\lambda) \quad = \quad & \int_0^\infty e^{-\rho t} \int u\left(c_t\left(a,z\right)\right) g_t\left(a,z\right) dadzdt \\ & + \int_0^\infty e^{-\rho t} \int \int_{\underline{z}}^{\bar{z}} j_t\left(a,z\right)\left[-\frac{\partial g}{\partial t} + \mathcal{A}^* g_t\left(a,z\right) + \eta \delta_{a_0,z_0}\right] dadzdt \\ & + \int_0^\infty e^{-\rho t} \lambda_t \left[-k_t + \int_0^\infty \int_{\underline{z}}^{\bar{z}} a g_t(a,z) dadz\right] dt\end{aligned}$$

# Solution

- The social value function $j_t(a, z)$ solves the planner's HJB equation:

$$
\begin{aligned}
(\rho+\eta) j \quad = \quad & \max_{c \geq 0} \frac{c^{1-\gamma}}{1-\gamma} + \lambda (a - k_t) + (w_t z_t + r_t a - c) \frac{\partial j}{\partial a} \\
& + \theta(\hat{z} - z) \frac{\partial j}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 j}{\partial z^2} + \frac{\partial j}{\partial t},
\end{aligned}
$$

with $a \geq 0$.

- The Lagrange multiplier $\lambda_t$ is:

$$
\lambda_t \quad = \quad \int \int_{\underline{z}}^{\bar{z}} \frac{\partial j}{\partial a} \left( \frac{\partial r}{\partial K} a + \frac{\partial w}{\partial K} z \right) g_t(a, z) dz da
$$

# First-best allocation

- The planner is able to fully redistribute among agents.

- Individual wealth is now given by:

$$da_t = \left(w_t z_t + \left(r_t + \eta\right) a_t - c_t + \tau_t\right) dt,$$

  where $\tau_t$ are transfers across agents.

- The aggregate amount of transfers is zero:

$$\int \int_{\underline{z}}^{\bar{z}} \tau_t(a,z) g_t(a,z) dz da = 0$$

# The Lagrangian: First best

$$
\begin{aligned}
\mathcal{L}_{fb}(g,\tau,c,j,\lambda) \quad = \quad & \int_0^\infty e^{-\rho t}\int u\left(c_t(a,z)\right)g_t(a,z)\,dadzdt \\
& + \int_0^\infty e^{-\rho t}\int\int_{\underline{z}}^{\bar{z}} j_t(a,z)\left[-\frac{\partial g}{\partial t}+\mathcal{A}^* g_t(a,z)+\eta\delta_{a_0,z_0}\right]dadzdt \\
& + \int_0^\infty e^{-\rho t}\lambda_t\left[-k_t+\int_0^\infty\int_{\underline{z}}^{\bar{z}} a g_t(a,z)dadz\right]dt \\
& + \int_0^\infty e^{-\rho t}\varphi_t\left[\int_0^\infty\int_{\underline{z}}^{\bar{z}}\tau_t(a,z)g_t(a,z)dadz\right]dt
\end{aligned}
$$

## Solution: First best

- The social value function $j_t(a,z)$ solves the planner's HJB equation:

$$
\begin{aligned}
(\rho+\eta)j &= \max_{c\geq 0} \frac{c^{1-\gamma}}{1-\gamma} + \lambda(a-k_t) \\
&+\varphi_t \tau_t(a,z) + (w_t z_t + r_t a - c)\frac{\partial j}{\partial a} \\
&+\theta(\hat{z}-z)\frac{\partial j}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j}{\partial z^2} + \frac{\partial j}{\partial t}
\end{aligned}
$$

- The Lagrange multiplier $\lambda_t = \int\int_{\underline{z}}^{\bar{z}} \frac{\partial j}{\partial a}\left(\frac{\partial r}{\partial K}a + \frac{\partial w}{\partial K}z\right) g_t(a,z)dzda$ and the Lagrange multiplier $\varphi_t$ is pinned-down by the market clearing condition $\int\int_{\underline{z}}^{\bar{z}} \tau_t(a,z)g_t(a,z)dzda = 0$.

# Capital under-accumulation in the competitive equilibrium

| | Constrained-efficient | Competitive equilibrium | First-best |
|---|---|---|---|
| Aggregate capital, $K$ | 13.82 | 5.04 | 5.57 |
| Output, $Y$ | 2.57 | 1.79 | 1.86 |
| Capital-output ratio, $K/Y$ | 5.37 | 2.82 | 1.57 |
| Aggregate Consumption, $C$ | 1.45 | 1.39 | 1.41 |
| Wages, $w$ | 1.65 | 1.15 | 1.19 |
| Interest rate (%), $r$ | -1.29 | 4.79 | 4.00 |
| Tail exponent, $\zeta$ | 2.83 | 5.08 | 0.33 |
| Welfare (% cons. c.e.),$\Theta$ | 15.13 | - | 15.41 |

# Results

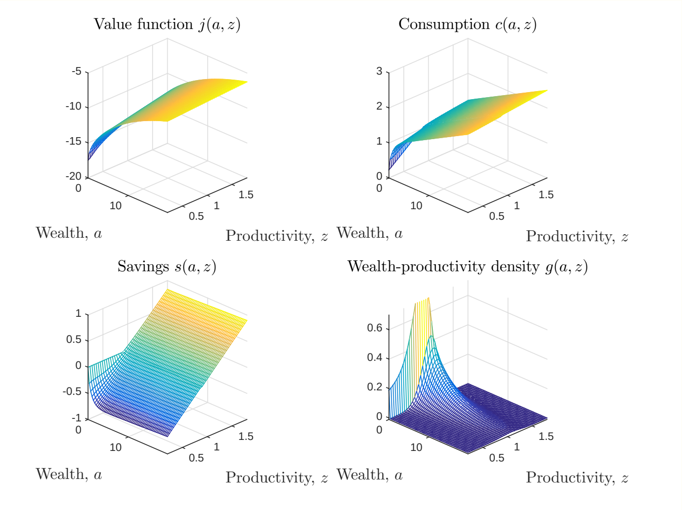



**Figure 1:** Competitive equilibrium

# Results

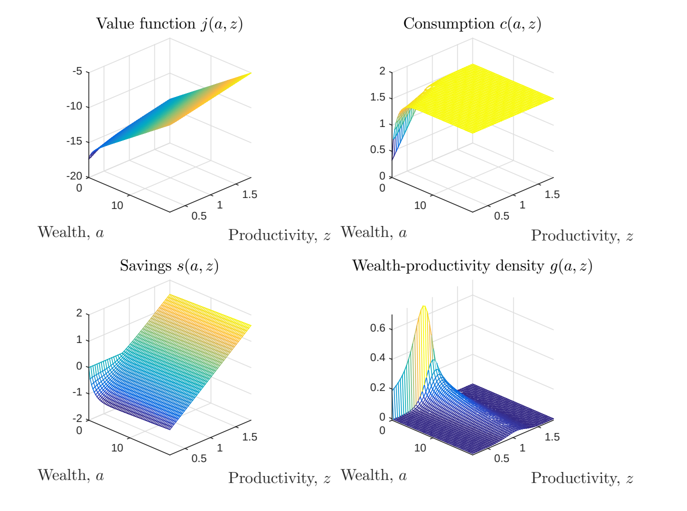



**Figure 2:** Constrained-efficient allocation

# Application 2: Optimal monetary policy with heterogeneous agents

# How does household heterogeneity affect optimal monetary policy?

- Emerging positive literature about the redistributive effects of monetary policy in incomplete-markets models with non-trivial heterogeneity.
- Examples: Auclert (2019), Kaplan, Moll, and Violante (2018),Gornemann, Kuester and Nakajima (2016), McKay, Nakamura and Steinsson (2016), Luetticke (2018), ...
- Less progress on the normative front: the entire wealth distribution is a state in the policy-maker's problem.
- Nuño and Thomas (2020) solve the optimal monetary policy with commitment in a model with non-trivial heterogeneity.

# The economy in a nutshell

- Incomplete markets economy *à la* Huggett (1993).
- Nominal, long-term, non-contingent financial assets.
- Small open-economy with risk-neutral foreign investors.
- Disutility costs of inflation (nominal rigidities).

# Transmission channels of monetary policy

- Redistributive channels:

  1. Fisher: surprise inflation reduces the initial (time-0) r market price of the long-term nominal bond $\rightarrow$ redistributes wealth from creditors to debtors.

  2. Liquidity: expected and unexpected lower inflation raises asset prices $\rightarrow$ relaxing borrowing limit in market-value.

- Costly inflation (due to nominal rigidities).

# Related literature

1. Ramsey policies in incomplete-market models in which the policy-maker does not need to keep track of the wealth distribution or the latter is finite-dimensional:
   - Gottardi, Kajii, and Nakajima (2011), Bilbiie and Ragot (2017), Le Grand and Ragot (2017), Challe (2019), and Acharya et al. (2019).
2. Ex-ante parametric form for the optimal policy + numerical optimization:
   - Dyrda and Pedroni (2018) and Itskhoki and Moll (2019).
3. Finite-dimensional Lagrangian methods:
   - Bhandari et al. (2018) and Açikgöz et al. (2018).
4. Infinite-dimensional calculus in problems with non-degenerate distributions:
   - Dávila et al. (2012), Lucas and Moll (2014), and Nuño and Moll (2018).

# Model: output and prices

- Household $k \in [0,1]$ is endowed with $y_{kt}$ units of output:
  - $y_{kt}$ follows 2-state Poisson process, $y_1 < y_2$, with intensities $\lambda_1$ and $\lambda_2$.
- Domestic price level $P_t$ follows

$$dP_t = \pi_t P_t dt$$

# Assets

- Long-term bond issued at time $t$ pays stream of geometrically-decaying *nominal* coupons $\left\{\delta e^{-\delta(s-t)}\right\}_{s\geq t}$.
- Nominal face value of net wealth follows:

$$dA_{kt} = \left(A_{kt}^{new} - \delta A_{kt}\right) dt$$

- Budget constraint:

$$Q_t A_{kt}^{new} = P_t \left(y_{kt} - c_{kt}\right) + \delta A_{kt}$$

- Define $a_{kt} \equiv A_{kt}/P_t$: *real* face value of net wealth.

# Household problem

- The household solves:

$$v_t(a,y) = \max_{\{c_s\}_{s\in[t,\infty)}} \mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} \left[u(c_{ks}) - x(\pi_s)\right] ds$$

subject to:

$$\dot{a}_{kt} = s_t\left(a_k, y_k\right) = \frac{1}{Q_t}\left[\underbrace{\left(y_{kt} - c_{kt} + \delta a_{kt}\right)}_{URE_t(a,y)} - \left(\delta + \pi_s\right) \underbrace{Q_t a_{kt}}_{NNP_t(a,y)\equiv a_t^m}\right],$$

and the exogenous borrowing limit

$$a_{kt} \geq \phi, \quad \phi \leq 0$$

# International investors (bond pricing)

- Risk-neutral investors can invest elsewhere at riskless real rate $\bar{r}$.

- Unit price of the nominal non-contingent bond:

$$Q(t) = \int_t^{\infty} \delta e^{-(\bar{r}+\delta)(s-t) - \int_t^s \pi_s du} ds$$

# Dynamics of the income-wealth density

- Density $f_{it}(a) \equiv f_t(a, y_i)$ dynamics given by Kolmogorov Forward equation:

$$\frac{\partial f_{it}(a)}{\partial t} = -\frac{\partial}{\partial a}\left[s_{it}\left(a\right) f_{it}(a)\right] - \lambda_i f_{it}(a) + \lambda_j f_{jt}(a),$$

$i, j = 1, 2,\ j \neq i$.

# Central bank

- Central bank can trade a short-term nominal claim with foreign investors.
  - It sets the instantaneous nominal rate $R_t$ of that facility. By no-arbitrage: $R_t = \bar{r} + \pi_t$.
  - Equivalent to assume that the central bank chooses directly the inflation rate $\{\pi_t\}_{t\geq 0}$.
- Central bank's utilitarian welfare criterion:

$$
\begin{aligned}
U_0^{CB} &\equiv \sum_{i=1}^{2} \int_{\phi}^{\infty} v_0(a, y_i)\, f_0(a, y_i)\, da \\
&= \mathbb{E}_{f_0(a,y)}\left[v_0(a, y)\right] \\
&= \int_0^{\infty} e^{-\rho t} \mathbb{E}_{f_t(a,y)}\left[u\left(c_t(a, y), \pi_t\right)\right] dt
\end{aligned}
$$

# Central bank problem

- The Ramsey problem is:

$$J^R\left[f_0\left(\cdot\right)\right] = \max_{\{\pi_t\}_{t\geq 0}} U_0^{CB}$$

subject to:

  - the law of motion of the distribution.
  - the bond pricing equation.
  - the individual HJB equation.
  - the first-order condition of households.

- $J^R$ and $\pi$ are not ordinary functions, but functionals as they map a distribution $f_t\left(\cdot\right)$ into $\mathbb{R}$.
- The problem is time-inconsistent.

## How can we solve it?

- We construct a functional Lagragian $\mathcal{L}_0[f, \pi, Q, v, c]$.
- This is a problem of constrained optimization in an infinite-dimensional Hilbert space $\rightarrow$ Gâteaux derivative.
- Example, the Gâteaux derivative with respect to density $f$ is:

$$\lim_{\alpha \to 0} \frac{\mathcal{L}_0[f + \alpha h, \pi, Q, v, c] - \mathcal{L}_0[f, \pi, Q, v, c]}{\alpha}$$

where $h$ is an arbitrary function in the same function space as $f$.

# Optimal inflation

$$x'(\pi_t) = \overbrace{\text{cov}_{f_t(a,y)}\left[-NNP_t(a,y), MUC_t(a,y)\right]}^{\text{Domestic net nominal position motive}}$$
$$+\overbrace{\mathbb{E}_{f_t(a,y)}\left[-NNP_t(a,y)\right]\mathbb{E}_{f_t(a,y)}\left[MUC_t(a,y)\right]}^{\text{Cross-border net nominal position motive}} + \mu_t Q_t,$$

and:

$$\mu_t = \int_0^t e^{-\int_s^t(\bar{r}+\pi_z+\delta-\rho)dz}\frac{1}{Q_s}\{\overbrace{\text{cov}_{f_s(a,y)}\left[URE_s(a,y), MUC_s(a,y)\right]}^{\text{Domestic interest rate exposure motive}}$$
$$+\overbrace{\mathbb{E}_{f_s(a,y)}\left[URE_s(a,y)\right]\mathbb{E}_{f_s(a,y)}\left[MUC_s(a,y)\right]}^{\text{Cross-border interest rate exposure motive}}\}ds,$$

where $MUC_t(a,y) \equiv u'(c_t(a,y))$ denotes the marginal utility of consumption.

# Redistributive inflationary bias at time 0

- Provided the aggregate NNP is non-positive, $\mathbb{E}_{f_0(a,y)}\left[-NNP_0(a,y)\right]/Q_0 = -\bar{a}_0 \geq 0$, optimal inflation at time-0 is strictly positive, $\pi_0 > 0$.
  - Even if economy's aggregate NNP is zero, as long as $u''(c) < 0$ (concave preferences) and there is net wealth dispersion, the central bank has a reason to inflate.
  - Different from classical "inflationary bias" in NK models.

$$\overbrace{\psi\pi_0}^{\text{disutility of inflation}} = -Q_0 \sum_{i=1}^{2} \int_{\phi Q_0}^{\infty} \overbrace{v_{i0}\left(\frac{a^m}{Q_0}\right)}^{\text{time-0 value function}} \frac{d}{dQ_0} \left[\overbrace{\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)}^{\text{initial wealth dist. in market value}}\right] da^m$$

$$= \mathbb{E}_{f_0^m(a^m,y)}\left[-NNP_0(a^m)\, MUC_0(a^m,y)\right]$$

# Optimal long-run inflation

- In the limit as $\rho \to \bar{r}$, the optimal steady-state inflation rate under commitment tends to zero: $\lim_{\rho \to \bar{r}} \pi_{\infty} = 0$.

- Reminiscent of same result in NK models, but very different reason (two counteracting redistributive motives).

# Numerical solution and calibration

- Given an initial distribution, the model is solved using finite difference methods as in Achdou et al. (2017).
- Calibrate to a prototypical European small open economy, time unit $= 1$ year.
- $u(c) = \log(c)$, $x(\pi) = \frac{\psi}{2}\pi^2$ (Rotemberg pricing).
- $f_0(\cdot) = f_{ss}(\cdot)$ under $\pi = 0$.

# Calibration

| | Value | Description | Source/Target |
|---|---|---|---|
| $\bar{r}$ | 0.03 | world real interest rate | standard |
| $\psi$ | 5.5 | scale inflation disutility | slope NKPC in Calvo model |
| $\delta$ | 0.19 | bond amortization rate | Macaulay duration $=$ 4.5 yrs |
| $\lambda_1$ | 0.72 | transition rate U to E | monthly job finding rate 0.1% |
| $\lambda_2$ | 0.08 | transition rate E to U | unemployment rate 10% |
| $y_1$ | 0.73 | income in U state | Hall & Milgrom (2008) |
| $y_2$ | 1.03 | income in E state | $E(y) = 1$ |
| $\rho$ | 0.0302 | subjective discount rate | { NIIP/GDP (-25%) |
| $\phi$ | -3.6 | borrowing limit | HH debt/GDP (90%) |

# Time-0 optimal policy

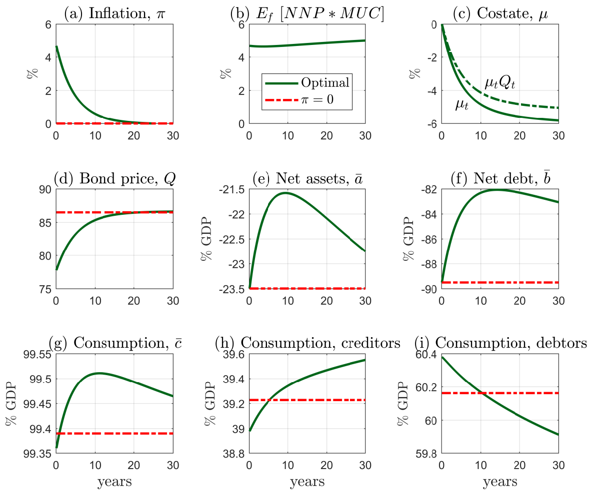
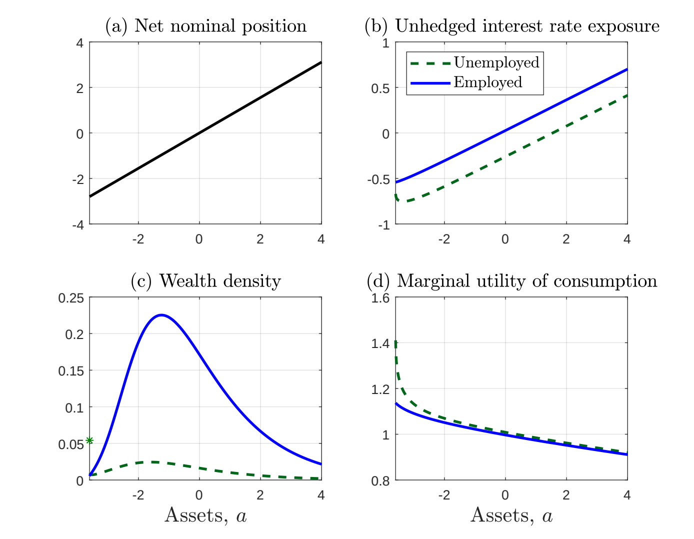



**Figure 3:** Transitional dynamics.

# Understanding the redistributive motives



# Redistributive effects of optimal inflation

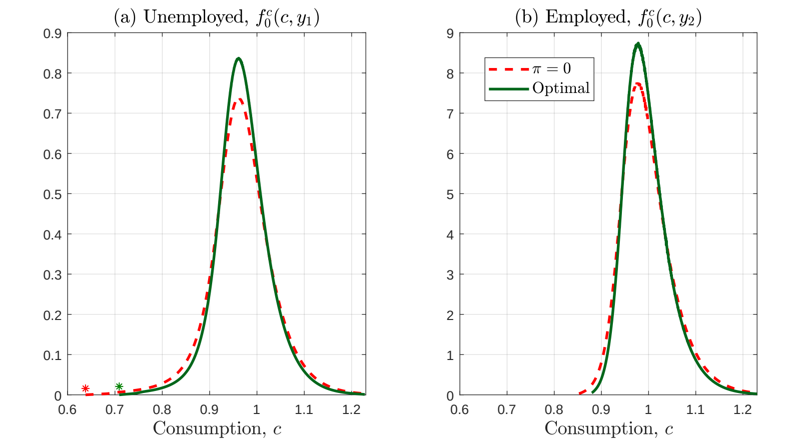



**Figure 5:** Consumption density at time 0.

# Welfare analysis

Aggregate welfare is defined as:

$$\mathbb{E}_{f_0(a,y)}\left[v_0(a,y)\right] = \int_0^\infty e^{-\rho t}\mathbb{E}_{f_t(a,y)}\left[u\left(c_t(a,y)\right) - x\left(\pi_t\right)\right] dt \equiv W[c]$$

Welfare losses of a zero-inflation policy relative to the optimal commitment

| Economy-wide | Lending HHs | Indebted HHs |
|---|---|---|
| 0.05 | -0.17 | 0.22 |

Note: welfare losses are expressed as a % of permanent consumption

# The importance of debt duration

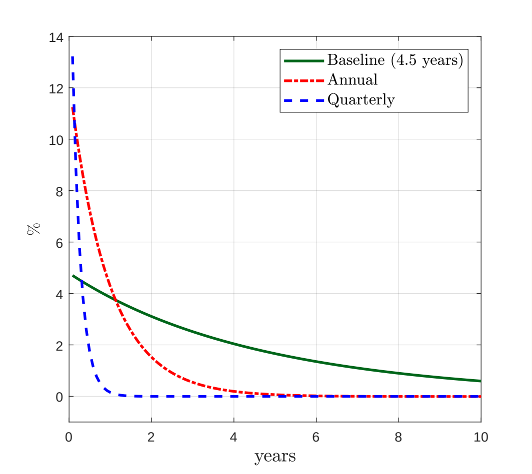



**Figure 6:** Optimal inflation under different debt durations.

# Optimal response to shocks from a timeless perspective

- Let individual income now be given by $\{y_1 Y_t, y_2 Y_t\}$, with

$$dY_t = \eta_Y (1 - Y_t)\, dt + \sigma dZ_t,$$

where $Z_t$ a Brownian motion $(\eta_Y = 0.5, \sigma = 0.01)$.

- We apply the results in Boppart, Krusell and Mittman (2018) to show how this is equivalent to analyze an MIT shock with amplitude $\sigma$.

- We consider policy 'from a *timeless perspective*,' in the sense of Woodford (2003):

  1. the initial wealth distribution is the stationary distribution implied by the optimal commitment $f_0(\cdot) = f_\infty(\cdot)$.

  2. the initial condition $\mu_0 = 0$ is replaced by $\mu_0 = \mu_\infty$.

# Optimal response to a negative TFP shock

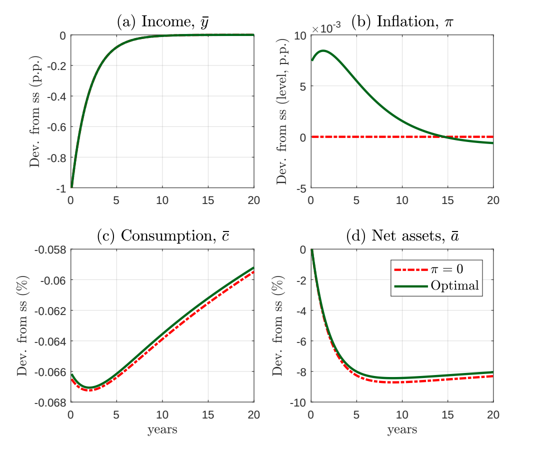



**Figure 7:** Generalized impulse response function of an aggregate income shock.

# Summing up

- Lots of cool new applications.
- Venture into your own research!

# Example: an alternative derivation of the HJB

- An agent maximizes:

$$V_0(x) = \max_{\{\alpha_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(\alpha_t, X_t)\, dt,$$

subject to:

$$dX_t = \mu_t(X_t, \alpha_t)\, dt + \sigma_t(X_t, \alpha_t)\, dW_t, \qquad X_0 = x_0$$

- $\sigma(\cdot) : \mathbb{X} \times \mathbb{A} \to \mathbb{R}^N$. We consider feedback control laws $\alpha_t = \alpha_t(X_t)$.

# If we express it in terms of probabilities

$$\max_{\{\alpha_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} \int u(\alpha_t, x)\, g_t(x)\, dx dt,$$

subject to:

$$\frac{\partial g_t}{\partial t} = \mathcal{A}^* g$$

$$g_0(x) = \delta(x - x_0)$$

# The Lagrangian

$$\mathcal{L}(g,\alpha) = \int_0^\infty e^{-\rho t} \int u(\alpha_t, x)\, g_t(x)\, dxdt + \int_0^\infty e^{-\rho t} \int v_t(x) \left(-\frac{\partial g_t}{\partial t} + \mathcal{A}^* g\right) dxdt,$$

where $v_t(x)$ is the Lagrange multiplier associated to the KF equation.

# Integrating by parts

$$\begin{aligned}
\int_0^\infty e^{-\rho t} \int v_t(x) \left(-\frac{\partial g_t}{\partial t}\right) dx dt &= \int_0^\infty e^{-\rho t} \int g_t(x) \frac{\partial \left(v_t e^{-\rho t}\right)}{\partial t} dx dt \\
&+ \int g_0(x) v_0(x) dx \\
&- \lim_{T \to \infty} \int g_T(x) e^{-\rho T} v_T(x) dx
\end{aligned}$$

# Applying the definition of adjoint operator

$$\int_0^\infty e^{-\rho t} \int v_t(x) \mathcal{A}^* g dx dt = \int_0^\infty e^{-\rho t} \int \mathcal{A} v g_t(x) \, dx dt$$

# Wrapping up

$$
\begin{aligned}
\mathcal{L}(g,\alpha) &= \int_0^\infty e^{-\rho t} \int u\left(\alpha_t, x\right) g_t\left(x\right) dxdt \\
&+ \int_0^\infty e^{-\rho t} \int \left(\mathcal{A}v + \frac{\partial v}{\partial t} - \rho v\right) g_t\left(x\right) dxdt \\
&+ \int \delta\left(x - x_0\right) v_0\left(x\right) dx \\
&- \lim_{T\to\infty} \int g_T\left(x\right) e^{-\rho T} v_T\left(x\right) dx
\end{aligned}
$$

## Gâteaux derivative with respect to the density

$$
\begin{aligned}
\frac{d}{d\epsilon}\mathcal{L}(g+\epsilon h,\alpha)\Big|_{\epsilon=0} \quad = \quad & \frac{d}{d\epsilon}\int_0^{\infty} e^{-\rho t}\int u\left(\alpha_t,x\right)\left[g_t\left(x\right)+\epsilon h_t\left(x\right)\right]dxdt\Big|_{\epsilon=0} \\
+ \quad & \frac{d}{d\epsilon}\int_0^{\infty} e^{-\rho t}\int \left(\mathcal{A}v+\frac{\partial v}{\partial t}-\rho v\right)\left[g_t\left(x\right)+\epsilon h_t\left(x\right)\right]dxdt\Big|_{\epsilon=0} \\
- \quad & \frac{d}{d\epsilon}\lim_{T\to\infty}\int \left[g_T\left(x\right)+\epsilon h_T\left(x\right)\right]e^{-\rho T}v_T\left(x\right)dx\Big|_{\epsilon=0}
\end{aligned}
$$

# Taking derivatives

$$
\begin{aligned}
\frac{d}{d\epsilon}\mathcal{L}(g+\epsilon h,\alpha)\Big|_{\epsilon=0} &= \int_0^\infty e^{-\rho t}\int u\left(\alpha_t, x\right) h_t\left(x\right) dxdt \\
&+ \int_0^\infty e^{-\rho t}\int \left(\mathcal{A}v + \frac{\partial v}{\partial t} - \rho v\right) h_t\left(x\right) dxdt \\
&- \lim_{T\to\infty}\int h_T\left(x\right) e^{-\rho T} v_T\left(x\right) dx
\end{aligned}
$$

# Applying the Fundamental Theorem of Calculus

- $\frac{d}{d\epsilon}\mathcal{L}(g+\epsilon h,\alpha)|_{\epsilon=0}=0$ for any function $h$ belonging to Sobolev space $H^2(\Phi)$.
- Then:

$$-\rho V+\frac{\partial v}{\partial t}+u(\alpha,x)+\mathcal{A}v=0$$
$$\lim_{T\to\infty} e^{-\rho T}v_T(x)=0$$

# Proceeding the same for the optimal control

- We recover the HJB equation:

$$\rho V = \frac{\partial V}{\partial t} + \max_{\alpha} \{u(\alpha, x) + \mathcal{A}v\},$$

where the value function is the Lagrange multiplier associated to the KF equation. Back