Heterogeneous agent models

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Prelude: the Kolmogorov Forward equation

The Kolmogorov Forward equation: overview

- Given a stochastic process X_t with an associated infinitinesimal generator $\mathcal{A}$, its probability density function $f_t\left(x\right)$ defined as:

$$\mathbb{P}_{t_0}\left[X_t \in \Omega\right] = \int_{\Omega} f_t\left(x\right) dx,$$

for any $\Omega \in X$ follows the dynamics:

$$\frac{\partial f}{\partial t} = \mathcal{A}^* f_t,$$

where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$.

Proof Operator

Example 1: A diffusion

- Let X_t be a stochastic process given by the SDE:

$$dX_t = \mu_t(X_t)\,dt + \sigma_t(X_t)\,dW_t, \qquad X_0 = x_0.$$

- The evolution of the associated density is given by:

$$\frac{\partial f}{\partial t} \quad = \quad \mathcal{A}^* f = -\frac{\partial}{\partial x}\left[\mu_t(x)\, f_t(x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma_t^2(x)\, f_t(x)\right],$$

with initial value $f_0(x) = \delta(x - x_0)$.

Example 2: A Markov chain

- Let X_t be a stochastic process given by:

$$dX_t = \mu(X_t, Z_t)\,dt, \qquad X_0 = x, \quad Z_0 = z_1.$$

where Z_t is a two-state continuous-time Markov Chain $Z_t \in \{z_1, z_2\}$ with intensities λ_1 and λ_2.

- The evolution of the density is given by:

$$\frac{\partial f_{it}}{\partial t} \quad = \quad \mathcal{A}^* f = -\frac{\partial}{\partial x}\left[\mu(x, z_i)\, f_{it}(x)\right] - \lambda_i f_{it}(x) + \lambda_j f_{jt}(x),$$

$i, j = 1, 2, j \neq i$, with initial value $f_{10}(x) = \delta(x - x_0)\,,\ f_{20}(x) = 0$.

The Aiyagari-Bewley-Huggett model

The workhorse model of heterogeneity

- Extension of the neoclassical model with a representative agent/complete markets to heterogeneous agents/incomplete markets.
- Basic framework to analyze questions related to income and wealth distributions (inequality, transmission of monetary and fiscal policies, etc.).
- Methodology can be easily extended to heterogeneous firms, heterogeneous countries, heterogeneous banks, ...
- Perfect foresight first (aggregate shocks later).

Hugget model with Poisson shocks: Households

- There is a continuum of mass unity of agents that are heterogeneous in their wealth a and endowment z.

- Households solve:

$$\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} u(c_t) dt \right],$$

  subject to

$$da_t = (z_t + r_t a_t - c_t)\, dt, \qquad a_0 = \bar{a}.$$

  where idiosyncratic endowment $z_t \in \{z_1, z_2\}$ follows a Markov chain with intensities $z_1 \to z_2 : \lambda_1$ and $z_2 \to z_1 : \lambda_2$.

- Exogenous borrowing limit:

$$a_t \geq -\phi < 0.$$

Market clearing

- Aggregate income normalize to one: $\mathbb{E}[z_t] = 1$.
- Total assets in zero net supply:

$$\sum_{i=1}^{2} \int a f_t(a, z_i) da = 0,$$

where $f_t(a, z_i)$ is the income-wealth density (infinite dimensional object).

Competitive equilibrium

A competitive equilibrium is composed by an interest rate r_t, a value function $v_t(a,z)$, a consumption policy $c_t(a,z)$, and a density $f_t(a,z)$ such that:

1. Given r, v is the solution of the household's HJB equation and the optimal control is c:

$$\rho v_{it}(a) = \frac{\partial v}{\partial t} + \max_c \left\{ u(c) + (z_i + r_t a - c) \frac{\partial v}{\partial a} \right\} + \lambda_i \left(v_{jt}(a) - v_{it}(a) \right).$$

2. Given r and c, f is the solution of the KF equation , $i, j = 1, 2, j \neq i$,

$$\frac{\partial f_{it}}{\partial t} = -\frac{\partial}{\partial a} \left[(z_i + r_t a - c_{it}(a)) f_{it}(x) \right] - \lambda_i f_{it}(a) + \lambda_j f_{jt}(a).$$

3. Given f, the capital market clears $\sum_{i=1}^{2} \int a f_{it}(a) da = 0$.

Stationary equilibrium (deterministic steady state)

- A stationary equilibrium of this model is a time-invariant competitive equilibrium.

- Characterized by three equations (stationary HJB + KF +market clearing):

$$\rho v_i(a) = \max_c \left\{ u(c) + (z_i + ra - c) \frac{\partial v}{\partial a} \right\} + \lambda_i (v_j(a) - v_i(a))$$

$$0 = -\frac{\partial}{\partial a} \left[(z_i + ra - c_i(a)) f_i(x) \right] - \lambda_i f_i(a) + \lambda_j f_j(a)$$

$$\sum_{i=1}^{2} \int a f_i(a) da = 0$$

How can we solve it?

- The stationary equilibrium does not yield to analytical solutions.

- We need to employ a (simple) algorithm. We begin with a guess for the interest rate $r^0 \in \mathbb{R}$. We set $n := 0$.

  1. HJB. Given r^n we solve households' HJB equation and obtain c^n.

  2. KF. Given r^n and c^n we obtain f^n.

  3. Market clearing. We compute the excess demand $\mathcal{D}(r^n) = \sum_{i=1}^{2} \int a f_i^n(a) da$. If $\mathcal{D}(r^n) = 0$, stop, otherwise update r^{n+1} and go back to 1.

Solving the KF equation using the finite difference method, I

- We have to solve the ODE $0 = -\frac{d}{dx}\left[\left(z_i + ra - c_i\left(a\right)\right) f_i\left(x\right)\right] - \lambda_i f_i\left(a\right) + \lambda_j f_j\left(a\right)$, using finite differences. We use the notation $f_{i,j} \equiv f_i(a_j)$.

- The ODE is approximated by:

$$
\begin{aligned}
0 \quad = \quad & -\frac{f_{i,j} s_{i,j,F} \mathbf{1}_{s_{i,j,F}>0} - f_{i,j-1} s_{i,j-1,F} \mathbf{1}_{s_{i,j-1,F}>0}}{\Delta a} \\
& -\frac{f_{i,j+1} s_{i,j+1,B} \mathbf{1}_{s_{i,j+1,B}<0} - f_{i,j} s_{i,j,B} \mathbf{1}_{s_{i,j,B}<0}}{\Delta a} \\
& -\lambda_i f_{i,j} + \lambda_{-i} f_{-i,j}
\end{aligned}
$$

Solving the KF equation using the finite difference method, II

- Collecting terms, we obtain:

$$f_{i,j-1}z_{i,j} + f_{i,j+1}x_{i,j} + f_{i,j}y_{i,j} + \lambda_{-i}f_{-i,j} = 0.$$

where:

$$\begin{aligned}
x_{i,j} &\equiv -\frac{s_{i,j,B}\mathbf{1}_{s_{i,j,B}<0}}{\Delta a}, \\
y_{i,j} &\equiv -\frac{s_{i,j,F}\mathbf{1}_{s_{i,j,F}>0}}{\Delta a} + \frac{s_{i,j,B}\mathbf{1}_{s_{i,j,B}<0}}{\Delta a} - \lambda_i, \\
z_{i,j} &\equiv \frac{s_{i,j,F}\mathbf{1}_{s_{i,j,F}>0}}{\Delta a}.
\end{aligned}$$

Matrix formulation

- This is also a system of $2J$ linear equations:

$$\boldsymbol{A}^T \boldsymbol{f} = \mathbf{0},$$

where $\boldsymbol{A}^T$ is the transpose of $\boldsymbol{A} = \lim_{n\to\infty} \boldsymbol{A}^n$.

- In order to impose the normalization constraint we fix one value of the $\mathbf{0}$ vector equal to 0.1. Alternatively compute the eigenvectors of $\boldsymbol{A}^T$.

- We then solve the system and obtain a solution $\hat{f}$. Finally, we renormalize as:

$$f_{i,j} = \frac{\hat{f}_{i,j}}{\sum_{j=1}^{J} \hat{f}_{1,j} \Delta a + \sum_{j=1}^{J} \hat{f}_{2,j} \Delta a}$$

Adjoint operator = Transpose matrix

- Advantage of finite difference. Notice how the system

$$\rho v = u(c) + \mathcal{A} v.$$

$$\mathcal{A}^* f = 0,$$

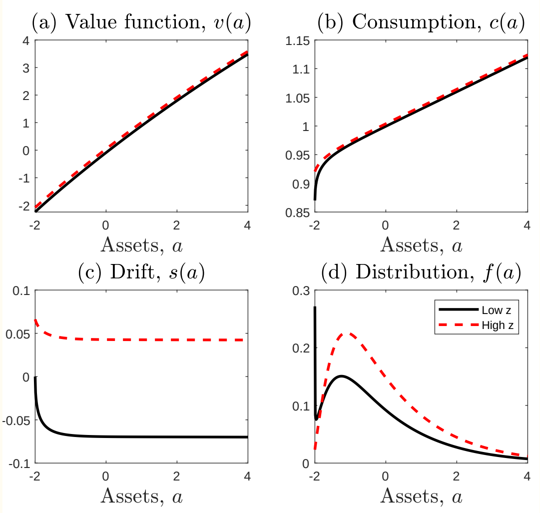
is approximated by:

$$\rho \boldsymbol{v} = \boldsymbol{u} + \boldsymbol{A} \boldsymbol{v}.$$

$$\boldsymbol{A}^T \boldsymbol{f} = \mathbf{0}.$$

Results

A condition for equilibrium is $r < \rho$.



Aiyagari model with diffusion: firms

- Introduce firms in the economy: a representative firm with production function $Y = F(K, L) = AK^{\alpha} L^{1-\alpha}$. Capital depreciates at rate δ_K.

- Competitive factor markets:

$$
\begin{aligned}
r_t &= \frac{\partial F(K_t, 1)}{\partial K} - \delta_K = \alpha \frac{Y_t}{K_t} - \delta_K, \\
w_t &= \frac{\partial F(K_t, 1)}{\partial L} = (1-\alpha) \frac{Y_t}{L_t}.
\end{aligned}
$$

Households

- Assume now that labor productivity evolves according to a Ornstein–Uhlenbeck process:

$$dz_t = \theta(\hat{z} - z_t)dt + \sigma dB_t,$$

on a bounded interval $[\underline{z}, \bar{z}]$ with $\underline{z} \geq 0$, where B_t is a Brownian motion.

- The HJB is now:

$$\begin{aligned} \rho V_t(a, z) &= \frac{\partial V}{\partial t} + \max_{c \geq 0} u(c) + [w_t z + r_t a - c] \frac{\partial V}{\partial a} \\ &+ \theta(\hat{z} - z)\frac{\partial V}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2}. \end{aligned}$$

Market clearing and the KF equation

- Aggregate productivity normalize to one: $\mathbb{E}[z_t] = 1$.
- Total assets equal aggregate capital:

$$\int a f_t(a,z) da dz = K_t,$$

where $f_t(a,z)$ is the wealth-productivity density.

- The KF equation:

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{\partial}{\partial a}\left(\left[w_t z + r_t a - c\left(a,z\right)\right] f_t\left(a,z\right)\right) \\ &\quad -\frac{\partial}{\partial z}\left(\theta(\hat{z}-z) f_t\left(a,z\right)\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2 f_t\left(a,z\right)\right). \end{aligned}$$

The stationary equilibrium: Algorithm

- We begin with a guess for the aggregate capital $K^0 \in \mathbb{R}$. We set $n := 0$.

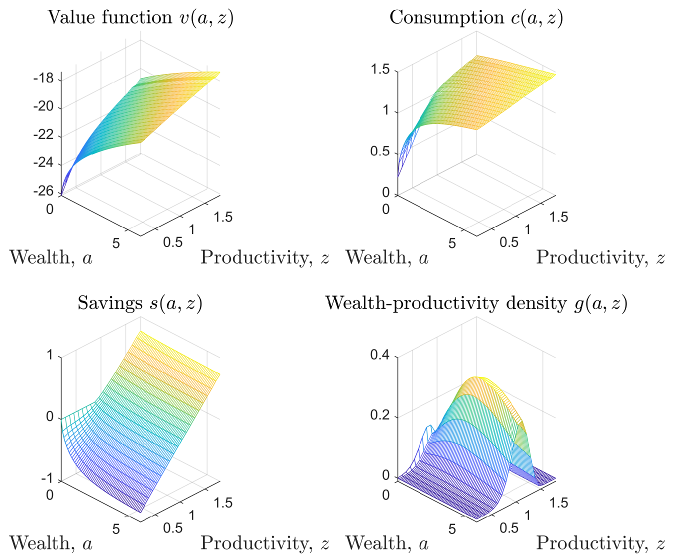
1. HJB. Given K^n we solve households' HJB equation and obtain c^n.
2. KF. Given K^n and c^n we obtain f^n.
3. Market clearing. We update aggregate capital using a relaxation algorithm $K^{n+1} = (1-\chi)\int_0^\infty \int_{\underline{z}}^{\bar{z}} a f_i^n(a) da dz + \chi K^n$. If $K^{n+1} = K^n$, stop, otherwise go back to 1.

The KF with diffusions: Finite difference methods

- Proceeding as above, we obtain

$$\boldsymbol{A}^T \boldsymbol{f} = \mathbf{0}.$$

Results


Value function $v(a, z)$
-18
-20
-22
-24
-26
0
5
Wealth, a
0.5
1
1.5
Productivity, z
Consumption $c(a, z)$
1.5
1
0.5
0
0
5
Wealth, a
0.5
1
1.5
Productivity, z
Savings $s(a, z)$
1
0
-1
0
5
Wealth, a
0.5
1
1.5
Productivity, z
Wealth-productivity density $g(a, z)$
0.4
0.2
0
0
5
Wealth, a
0.5
1
1.5
Productivity, z

Transitional dynamics

- For the transitional dynamics we need an algorithm that iterates backward-forward:

    1. Backwards: Given the steady state value function update backwards using the HJB to obtain the policies
    2. Forward: Given the initial distribution, update forward the KF to propagate the distribution.

Algorithm with finite differences

- We begin with a guess for the aggregate capital path $\boldsymbol{K}^0 = \left\{K_n^0\right\}_{n=1}^N \in \mathbb{R}^N$. We set $s := 0$ and define a time step Δt.

  1. HJB. Given $\boldsymbol{K}^s$ and $\boldsymbol{v}^N$ we solve households' HJB equation backwards

  $$\rho \boldsymbol{v}^n = \boldsymbol{u}^{n+1} + \boldsymbol{A}^{n+1}\boldsymbol{v}^n + \frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^n}{\Delta t},$$

  and obtain $\{\boldsymbol{A}^n\}_{n=1}^N$.

  2. KF. Given $\boldsymbol{f}^0$, and $\{\boldsymbol{A}^n\}_{n=1}^N$ we compute the KF forward

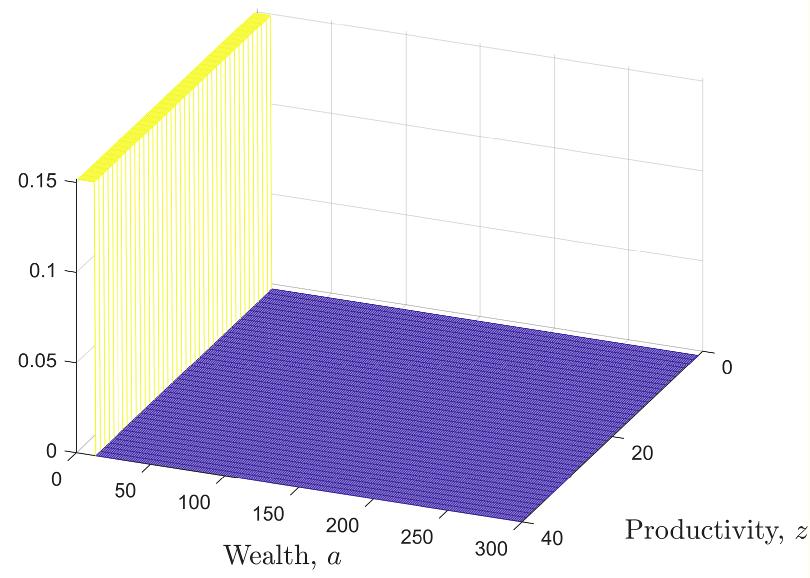
  $$\frac{\boldsymbol{f}^{n+1} - \boldsymbol{f}^n}{\Delta t} = \left(\boldsymbol{A}^{\mathrm{n}}\right)^T \boldsymbol{f}^{n+1}.$$

  3. Market clearing. We update aggregate capital $\boldsymbol{K}^{s+1}$. If $\boldsymbol{K}^{s+1} = \boldsymbol{K}^s$, stop, otherwise go back to 1.

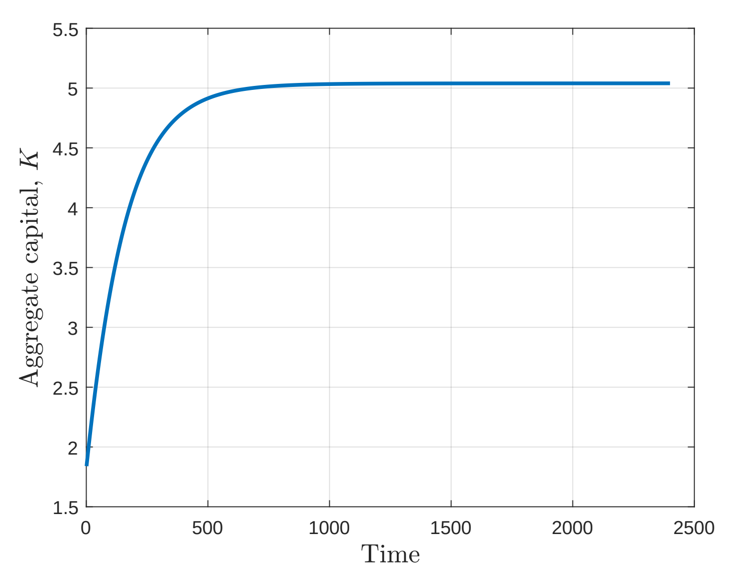
How to update capital

- The simplest strategy is to use a relaxation algorithm, giving the same weight to any time point.
- It is often faster (and sometimes the only option) to use some nonlinear equation solver (Newton's method or similar).

Example: initial distribution



Transitional dynamics



Precautionary savings

- Savings in the HA model with incomplete markets $>$ RA model with complete markets:
  - Steady state interest rate $r < \rho$, whereas in a RA $r = \rho$.
- This is due to precautionary savings:
  - Households save to get some insurance against the possibility of hitting the borrowing limit.

The problem of introducing aggregate shocks

Aiyagari model with aggregate shocks

- Consider the Aiyagari model that we explained above and assume that the production function is now $Y = F(Z, K, L) = ZK^{\alpha}L^{1-\alpha}$ with aggregate TFP Z following a diffusion:

$$dZ_t = \mu_z(Z_t)\,dt + \sigma_z(Z_t)\,dW_t$$

where W_t is a Brownian motion.

The key difference with the case without aggregate shocks

- Without aggregate shocks, the aggregate state of the economy $f_t(\cdot)$ is absorbed into time t:
  - The aggregate impact on individual households is captured by $\frac{\partial v}{\partial t}$.
- This is no longer the case with aggregate shocks. The aggregate state is now $(Z_t, f_t(\cdot))$.

Household HJB with aggregate shocks

- The HJB results in:

$$\rho V_t(a,z,Z,f) = \max_{c\geq 0} u(c) + [wz + ra - c]\frac{\partial V}{\partial a} + \theta(\hat{z} - z)\frac{\partial V}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2}$$
$$+ \mu_z(Z)\frac{\partial V}{\partial Z} + \frac{\sigma_z^2(Z)}{2}\frac{\partial^2 V}{\partial Z^2} + \frac{\delta V}{\delta f}\frac{\partial f}{\partial t}.$$

- The term $\frac{\delta V}{\delta Z}$ is a functional derivative (more on this later) and cannot be treated as a standard derivative.

- The numerical techniques explained so far are not suitable for this case.

A solution: perturbation

A solution is to work with the linearized system

- We can compute the linear dynamics around the deterministic steady state.
- This allows us to obtain a solution, albeit one only valid when aggregate dynamics are approximately linear
- Ahn, Kaplan, Moll, Winberry, and Wolf (2017) in continuous time. Original discrete time method by Reiter (2009).

The algorithm

1. Compute the deterministic steady state.

2. Compute the first-order Taylor expansion around steady state.

3. Solve linear stochastic differential equation (SDE).

The linear SDE

- We have:

$$
\begin{aligned}
d\boldsymbol{v}_t &= \left[-\boldsymbol{u}\left(\boldsymbol{v}_{\mathbf{t}}\right) - \boldsymbol{A}\left(\boldsymbol{v}_t; \boldsymbol{p}_t\right)\boldsymbol{v}_t + \rho\boldsymbol{v}_t\right] dt \\
d\boldsymbol{g}_t &= \left[\boldsymbol{A}^{\boldsymbol{T}}\left(\boldsymbol{v}_t; \boldsymbol{p}_t\right)\boldsymbol{g}_{\mathbf{t}}\right] dt \\
\boldsymbol{p}_t &= \boldsymbol{F}\left(\boldsymbol{g}_t; Z_t\right) \\
dZ_t &= \boldsymbol{\mu}_z\left(Z_t\right) dt + \boldsymbol{\sigma}_{\boldsymbol{z}}\left(Z_t\right) dW_t
\end{aligned}
$$

A simpler alternative

- The solution to this problem may require some dimensionality reduction techniques to solve the resulting high-dimensional SDE.

- A simpler alternative was proposed by Boppart, Krusell, and Mitman (2018): employ "MIT shocks."

- This alternative obtains the first-order perturbation solution just by computing transitional dynamics in an economy without aggregate uncertainty.

MIT shocks

- Consider the model without aggregate shocks. The initial state is the deterministic steady state $f_{ss}(\cdot)$.
- The parameter that substitutes the aggregate variables (TFP) evolves with time according to:

$$\begin{aligned} \Delta Z_0 &= \mu_z(Z_0)\,\Delta t + \sigma_z(Z_0)\sqrt{\Delta t} \\ \Delta Z_t &= \mu_z(Z_t)\,\Delta t,\ t > 0, \end{aligned}$$

  where $Z_0 = Z_{ss}$.

- Compute the transitional dynamics of this system.

The response to a MIT shock is the impulse response function of the model

- If the model is approximately linear, the response to a MIT shock is the impulse response function of the model.
- These methodology is easily scalable to problems with N shocks, just compute N impulse responses.
- But what can we do when the model is strongly nonlinear?

A solution for nonlinear models: bounded rationality

Prelude: The original Krusell-Smith methodology

- Krusell and Smith (1998) proposed an alternative solution concept: bounded rationality.
- Households in the model approximate the distribution by a number of its moments., e.g., the mean $\int_0^\infty \int_{\underline{z}}^{\bar{z}} a f_t(a,z) da dz = K_t$.
- The HJB simplifies to:

$$\begin{aligned} \rho V_t(a,z,Z,K) &= \max_{c \geq 0} u(c) + [w_t z + r_t a - c] \frac{\partial V}{\partial a} + \theta(\hat{z} - z) \frac{\partial V}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial z^2} \\ &+ \mu_z(Z) \frac{\partial V}{\partial Z} + \frac{\sigma_z^2(Z)}{2} \frac{\partial^2 V}{\partial Z^2} + K \mu_K(K,Z) \frac{\partial V}{\partial K} \end{aligned}$$

How can we compute the PLM of capital $\mu_K(K, Z)$?

Propose a parametric form of the perceived law of motion (PLM) $\mu_K(K, Z; \boldsymbol{\theta}) = \theta_0 + \theta_1 K + \theta_2 KZ + \theta_3 Z$.

Begin with an initial guess of $\boldsymbol{\theta}^0 = \left(\theta_0^0, \theta_1^0, \theta_2^0, \theta_3^0\right)$.

Set $n := 0$.

1. Given $\mu_K\left(K, Z; \boldsymbol{\theta}^0\right)$ solve the HJB equation and obtain matrix $\boldsymbol{A}$.

2. Simulate using Monte Carlo $\{Z_s\}_{s=0}^{S}$: $\Delta Z_s = \mu_z(Z_{s-1})\Delta t + \sigma_z(Z_{s-1})\sqrt{\Delta t}\varepsilon_s$, where $\varepsilon_s \sim \mathcal{N}(0, 1)$.

3. Compute the dynamics of the distribution using the KF equation and use it to obtain aggregate capital: $\int_0^\infty \int_{\underline{z}}^{\bar{z}} a f_t(a, z) da dz = K_t$.

4. Run an OLS $\frac{\Delta K_s}{K_s \Delta t} = \mu_K(K_s, Z_s; \boldsymbol{\theta})$ over the simulated sample $\{Z_s, K_s\}_{s=0}^{S}$ to update coefficients $\boldsymbol{\theta}^{n+1}$. If $\boldsymbol{\theta}^{n+1} = \boldsymbol{\theta}^n$ stop, otherwise go back to step 1.

How does it perform?

- In the Aiyagari model with aggregate shocks the Krusell-Smith methodology performs quite well. This is due to two different forces:

    1. Approximate aggregation. Aggregate capital provides a good approximation to the distribution because the individual consumption policy rules are approximately linear (except for households very close to the borrowing limit),

    2. Linear aggregate dynamics. The (log)linear law of motion provides a good approximation of the aggregate dynamics of capital because the model is quite linear.

- But this latter kind of problems can be solved more efficiently (the number of states does not grow with the number of shocks) using the Boppart, Krusell, and Mitman (2018) methodology already described.

What if the aggregate dynamics are strongly nonlinear?

- The Krusell and Smith (1998) methodology can be extended to analyze models with aggregate nonlinear dynamics: Fernández-Villaverde, Hurtado, and Nuño (2020).
- The key difference is to have a non-parametric perceived law of motion, updated using machine learning.
- It allows us to analyze the interactions between precautionary savings and endogenous aggregate risk.

Example: Financial frictions and the wealth distribution

Motivation

- Recently, many papers have documented the nonlinear relations between financial variables and aggregate fluctuations.

- For example, Jordà et al. (2016) have gathered data from 17 advanced economies over 150 years to show how output growth, volatility, skewness, and tail events all seem to depend on the levels of leverage in an economy.

- Similarly, Adrian et al. (2019a) have found how, in the U.S., sharply negative output growth follows worsening financial conditions associated with leverage.

- Can a fully nonlinear DSGE model account for these observations?

- To answer this question, we postulate, compute, and estimate a continuous-time DSGE model with a financial sector, modeled as a representative financial expert, and households, subject to uninsurable idiosyncratic labor productivity shocks.

The main takeaway

- The interaction between the supply of bonds by the financial sector and the precautionary demand for bonds by households produces significant *endogenous aggregate risk*.

- This risk induces an endogenous regime-switching process for output, the risk-free rate, excess returns, debt, and leverage.

- Mechanism: *endogenous aggregate risk* begets multiple stochastic steady states or SSS(s), each with its own stable basin of attraction.

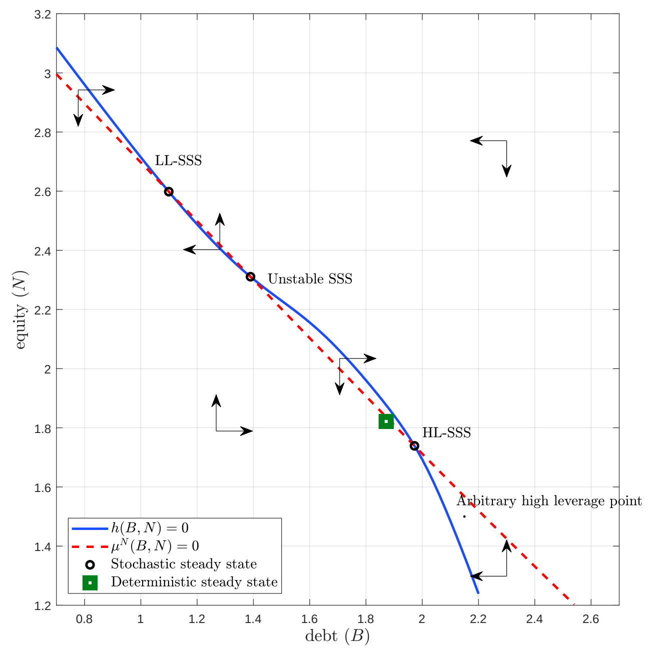
- Intuition: different persistence of wages and risk-free rates in each basin.

- The regime-switching generates:

    1. Multimodal distributions of aggregate variables (Adrian et al., 2019b).

    2. Time-varying levels of volatility and skewness for aggregate variables (Fernández-Villaverde and Guerrón, 2020).

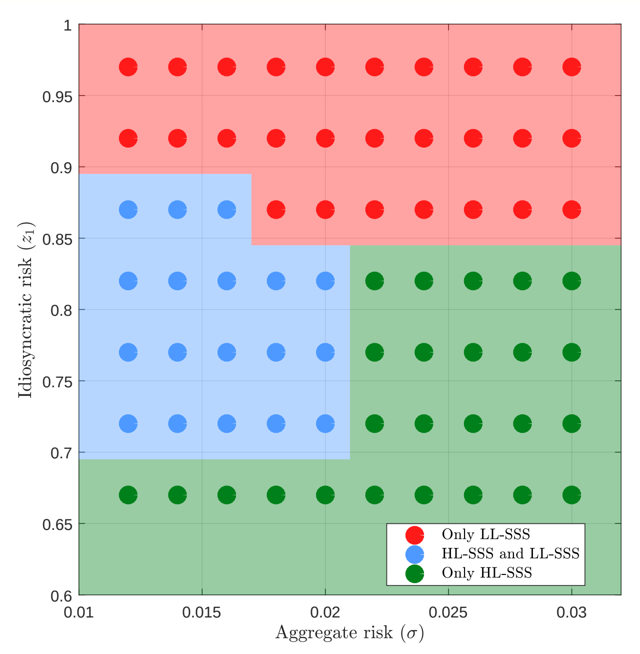
    3. Supercycles of borrowing and deleveraging (Reinhart and Rogoff, 2009).



equity (N)
debt (B)
LL-SSS
Unstable SSS
HL-SSS
Arbitrary high leverage point
$h(B, N) = 0$
$\mu^N(B, N) = 0$
Stochastic steady state
Deterministic steady state
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6

HA vs. RA

- Our findings are in contrast with the properties of the representative household version of the model.
- While the consumption decision rule of the households is close to linear with respect to the household state variables, it is sharply nonlinear with respect to the aggregate state variables.
- This point is more general: agent heterogeneity might matter even if the decision rules of the agents are linear with respect to individual state variables.
- Thus, changes in the forces behind precautionary savings affect aggregate variables, and we can offer a novel and simultaneous account of:
    1. The recent heightened fragility of the advanced economies to adverse shocks.
    2. The rise in wealth inequality witnessed before the 2007-2008 financial crisis.
    3. The increase in debt and leverage experienced during the same period.
    4. The low risk-free interest rates of the last two decades.



1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
Idiosyncratic risk (z_1)
0.01
0.015
0.02
0.025
0.03
Aggregate risk (σ)
Only LL-SSS
HL-SSS and LL-SSS
Only HL-SSS

Methodological contribution

- New approach to (globally) compute and estimate with the likelihood approach HA models:

    1. Computation: we use tools from machine learning.
    2. Estimation: we use tools from inference with diffusions.

- Strong theoretical foundations and many practical advantages.

    1. Deal with a large class of arbitrary operators efficiently.
    2. Algorithm that is i) easy to code, ii) stable, iii) scalable, and iv) massively parallel.
    3. Examples and code at `https://github.com/jesusfv/financial-frictions`

The firm

- Representative firm with technology:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

- Competitive input markets:

$$w_t = (1-\alpha) K_t^{\alpha} L_t^{-\alpha}$$
$$rc_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

- Aggregate capital evolves:

$$\frac{dK_t}{K_t} = (\iota_t - \delta)\, dt + \sigma dZ_t$$

- Instantaneous return rate on capital dr_t^k:

$$dr_t^k = (rc_t - \delta)\, dt + \sigma dZ_t$$

The expert I

- Representative expert holds capital $\widehat{K}_t$ and issues risk-free debt $\widehat{B}_t$ at rate r_t to households.
- Expert can be interpreted as a financial intermediary.
- Financial friction: expert cannot issue state-contingent claims (i.e., outside equity) and must absorb all risk from capital.
- Expert's net wealth (i.e., inside equity): $\widehat{N}_t = \widehat{K}_t - \widehat{B}_t$.
- Together with market clearing, our assumptions imply that economy has a risky asset in positive net supply and a risk-free asset in zero net supply.

The expert II

- The law of motion for expert's net wealth $\widehat{N}_t$:

$$\begin{aligned} d\widehat{N}_t &= \widehat{K}_t dr_t^k - r_t \widehat{B}_t dt - \widehat{C}_t dt \\ &= \left[\left(r_t + \widehat{\omega}_t \left(rc_t - \delta - r_t \right) \right) \widehat{N}_t - \widehat{C}_t \right] dt + \sigma \widehat{\omega}_t \widehat{N}_t dZ_t \end{aligned}$$

where $\widehat{\omega}_t \equiv \frac{\widehat{K}_t}{\widehat{N}_t}$ is the leverage ratio.

- The law of motion for expert's capital $\widehat{K}_t$:

$$d\widehat{K}_t = d\widehat{N}_t + d\widehat{B}_t$$

- The expert decides her consumption levels and capital holdings to solve:

$$\max_{\left\{ \widehat{C}_t, \widehat{\omega}_t \right\}_{t \geq 0}} \mathbb{E}_0 \left[\int_0^\infty e^{-\widehat{\rho} t} \log(\widehat{C}_t) dt \right]$$

given initial conditions and a NPG condition.

Households I

- Continuum of infinitely-lived households with unit mass.
- Heterogeneous in wealth a_m and labor supply z_m for $m \in [0,1]$.
- $G_t(a,z)$: distribution of households conditional on realization of aggregate variables.
- Preferences:

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma} - 1}{1-\gamma} dt \right]$$

- We could have more general Duffie and Epstein (1992) recursive preferences.
- $\rho > \widehat{\rho}$. Intuition from Aiyagari (1994) (and different from BGG class of models!).

Households II

- z_t units of labor valued at wage w_t.
- Labor productivity evolves stochastically following a Markov chain:
  1. $z_t \in \{z_1, z_2\}$, with $z_1 < z_2$.
  2. Ergodic mean of z_t is 1.
  3. Jump intensity from state 1 to state 2: λ_1 (reverse intensity is λ_2).
- Households save $a_t \geq 0$ in the riskless debt issued by experts with an interest rate r_t. Thus, their wealth follows:

$$da_t = (w_t z_t + r_t a_t - c_t)\, dt = s\left(a_t, z_t, K_t, G_t\right) dt$$

- Optimal choice: $c_t = c\left(a_t, z_t, K_t, G_t\right)$.
- Total consumption by households:

$$C_t \equiv \int c\left(a_t, z_t, K_t, G_t\right) dG_t\left(a, z\right)$$

Market clearing

1. Total amount of labor rented by the firm is equal to labor supplied:

$$L_t = \int z dG_t = 1$$

   Then, total payments to labor are given by w_t.

2. Total amount of debt of the expert equals the total households' savings:

$$B_t \equiv \int a dG_t (da, dz) = \widehat{B}_t$$

   with law of motion $d\widehat{B}_t = dB_t = (w_t + r_t B_t - C_t) \, dt$.

3. The total amount of capital in this economy is owned by the expert:

$$K_t = \widehat{K}_t$$

   Thus, $d\widehat{K}_t = dK_t = \left(Y_t - \delta K_t - C_t - \widehat{C}_t \right) dt + \sigma K_t dZ_t$ and $\widehat{\omega}_t = \frac{K_t}{N_t}$, where $\widehat{N}_t = N_t = K_t - B_t$.

4. Also:

$$\iota_t = \frac{Y_t - C_t - \widehat{C}_t}{K_t}$$

Density

- The households distribution $G_t\left(a, z\right)$ has density (i.e., the Radon-Nikodym derivative) $g_t(a, z)$.
- The dynamics of this density conditional on the realization of aggregate variables are given by the Kolmogorov forward (KF) equation:

$$\frac{\partial g_{it}}{\partial t} = -\frac{\partial}{\partial a}\left(s\left(a_t, z_t, K_t, G_t\right) g_{it}(a)\right) - \lambda_i g_{it}(a) + \lambda_j g_{jt}(a), \;\; i \neq j = 1, 2$$

where $g_{it}(a) \equiv g_t(a, z_i)$, $i = 1, 2$.

- The density satisfies the normalization:

$$\sum_{i=1}^{2} \int_0^{\infty} g_{it}(a) da = 1$$

Equilibrium

An equilibrium in this economy is composed by a set of prices $\left\{w_t, rc_t, r_t, r_t^k\right\}_{t\geq 0}$, quantities $\left\{K_t, N_t, B_t, \widehat{C}_t, c_{mt}\right\}_{t\geq 0}$, and a density $\{g_t(\cdot)\}_{t\geq 0}$ such that:

1. Given w_t, r_t, and g_t, the solution of the household m's problem is $c_t = c\left(a_t, z_t, K_t, G_t\right)$.
2. Given r_t^k, r_t, and N_t, the solution of the expert's problem is $\widehat{C}_t$, K_t, and B_t.
3. Given K_t, firms maximize their profits and input prices are given by w_t and rc_t.
4. Given w_t, r_t, and c_t, g_t is the solution of the KF equation.
5. Given g_t and B_t, the debt market clears.

Characterizing the equilibrium I

- First, we proceed with the expert's problem. Because of log-utility:

$$\widehat{C}_t = \widehat{\rho} N_t$$

$$\omega_t = \widehat{\omega}_t = \frac{rc_t - \delta - r_t}{\sigma^2}$$

- We can use the equilibrium values of rc_t, L_t, and ω_t to get the wage:

$$w_t = (1-\alpha) K_t^{\alpha}$$

the rental rate of capital:

$$rc_t = \alpha K_t^{\alpha-1}$$

and the risk-free interest rate:

$$r_t = \alpha K_t^{\alpha-1} - \delta - \sigma^2 \frac{K_t}{N_t}$$

Characterizing the equilibrium II

- Expert's net wealth evolves as:

$$dN_t = \underbrace{\left(\alpha K_t^{\alpha-1} - \delta - \widehat{\rho} - \sigma^2 \left(1 - \frac{K_t}{N_t}\right) \frac{K_t}{N_t}\right)}_{\mu_t^N(B_t, N_t)} N_t dt + \underbrace{\sigma K_t}_{\sigma_t^N(B_t, N_t)} dZ_t$$

- And debt as:

$$dB_t = \left((1-\alpha) K_t^{\alpha} + \left(\alpha K_t^{\alpha-1} - \delta - \sigma^2 \frac{K_t}{N_t}\right) B_t - C_t\right) dt$$

- Nonlinear structure of law of motion for dN_t and dB_t.
- We need to find:

$$C_t \equiv \int c\left(a_t, z_t, K_t, G_t\right) g_t\left(a, z\right) dadz$$

$$\frac{\partial g_{it}}{\partial t} = -\frac{\partial}{\partial a}\left(s\left(a_t, z_t, K_t, G_t\right) g_{it}(a)\right) - \lambda_i g_{it}(a) + \lambda_j g_{jt}(a), \;\; i \neq j = 1, 2$$

The DSS

- No aggregate shocks ($\sigma = 0$), but we still have idiosyncratic household shocks.
- Then:

$$r = r_t^k = rc_t - \delta = \alpha K_t^{\alpha-1} - \delta$$

and

$$dN_t = \left(\alpha K_t^{\alpha-1} - \delta - \widehat{\rho}\right) N_t dt$$

- Since in a steady state the drift of expert's wealth must be zero, we get:

$$K = \left(\frac{\widehat{\rho} + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

and:

$$r = \widehat{\rho} < \rho$$

- The value of N is given by the dispersion of the idiosyncratic shocks (no analytic expression).

How do we find aggregate consumption?

- As in Krusell and Smith (1998), households only track a finite set of n moments of $g_t(a, z)$ to form their expectations.
- No exogenous state variable (shocks to capital encoded in K). Instead, two endogenous states.
- For ease of exposition, we set $n = 1$. The solution can be trivially extended to the case with $n > 1$.
- More concretely, households consider a *perceived law of motion* (PLM) of aggregate debt:

$$dB_t = h\left(B_t, N_t\right) dt$$

where

$$h\left(B_t, N_t\right) = \frac{\mathbb{E}\left[dB_t | B_t, N_t\right]}{dt}$$

A new HJB equation

- Given the PLM, the household's Hamilton-Jacobi-Bellman (HJB) equation becomes:

$$\begin{aligned}\rho V_i\left(a, B, N\right) &= \max_c \frac{c^{1-\gamma}-1}{1-\gamma} + s\frac{\partial V_i}{\partial a} + \lambda_i\left[V_j(a, B, N) - V_i(a, B, N)\right] \\ &\quad + h\left(B, N\right)\frac{\partial V_i}{\partial B} + \mu^N\left(B, N\right)\frac{\partial V_i}{\partial N} + \frac{\left[\sigma^N\left(B, N\right)\right]^2}{2}\frac{\partial^2 V_i}{\partial N^2}\end{aligned}$$

  $i \neq j = 1, 2$, and where

$$s = s\left(a, z, N + B, G\right)$$

- We solve the HJB with a first-order, implicit upwind scheme in a finite difference stencil.
- Sparse system. Why?
- Alternatives for solving the HJB? Meshfree, FEM, deep learning, ...

An algorithm to find the PLM

1) Start with h_0, an initial guess for h.

2) Using current guess h_n, solve for the household consumption, c_m, in the HJB equation.

3) Construct a time series for B_t by simulating by J periods the cross-sectional distribution of households with a constant time step Δt (starting at DSS and with a burn-in).

4) Given B_t, find N_t, K_t, and:

$$\widehat{\mathbf{h}} = \left\{ \widehat{h}_1, \widehat{h}_2 ..., \widehat{h}_j \equiv \frac{B_{t_j+\Delta t} - B_{t_j}}{\Delta t}, ..., \widehat{h}_J \right\}$$

5) Define $\mathbf{S} = \{\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_J\}$, where $\mathbf{s}_j = \left\{s_j^1, s_j^2\right\} = \left\{B_{t_j}, N_{t_j}\right\}$.

6) Use $\left(\widehat{\mathbf{h}}, \mathbf{S}\right)$ and a universal nonlinear approximator to obtain h_{n+1}, a new guess for h.

7) Iterate steps 2)-6) until h_{n+1} is sufficiently close to h_n.

A universal nonlinear approximator

- We approximate the PLM with a neural network (NN):

$$h(\mathbf{s};\theta) = \theta_0^1 + \sum_{q=1}^{Q} \theta_q^1 \phi\left(\theta_{0,q}^2 + \sum_{i=1}^{D} \theta_{i,q}^2 s^i\right)$$

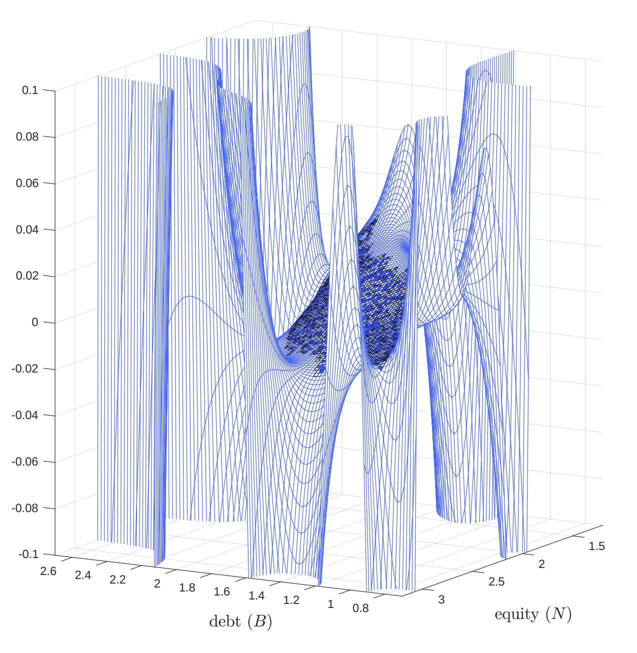
  where $Q = 16$, $D = 2$, and $\phi(x) = \log(1 + e^x)$.

- θ is selected as:

$$\theta^* = \arg\min_{\theta} \frac{1}{2} \sum_{j=1}^{J} \left\| h(\mathbf{s}_j;\theta) - \widehat{h}_j \right\|^2$$

- Easy to code, stable, and good extrapolation properties.

- You can flush the algorithm to a GPU, a TPU, a FPGA, or a AI accelerator instead of a standard CPU.


0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
debt (B)
1.5
2
2.5
3
equity (N)

A universal nonlinear approximator

- We approximate the PLM with a neural network (NN):

$$h(\mathbf{s};\theta) = \theta_0^1 + \sum_{q=1}^{Q} \theta_q^1 \phi\left(\theta_{0,q}^2 + \sum_{i=1}^{D} \theta_{i,q}^2 s^i\right)$$

  where $D = 2$ and $\phi(\cdot)$ is an activation function.

- We choose the *softplus* function: $\phi(x) = \log(1 + e^x)$. Robustness to *ReLUs*.

- $Q = 16$ is set by regularization.

- Note difference with a projection or a series approximation:

$$h(\mathbf{s};\theta) = \theta_0 + \sum_{q=1}^{Q} \theta_q \psi_q(\mathbf{s})$$

- When we have many hidden layers, the network is called *deep*.

- When do we want to have deep networks? Poggio et al. (2017).

Determining coefficients

- θ is selected to minimize the quadratic error function $\mathcal{E}\left(\theta; \mathbf{S}, \widehat{\mathbf{h}}\right)$:

$$
\begin{aligned}
\theta^* &= \arg\min_\theta \mathcal{E}\left(\theta; \mathbf{S}, \widehat{\mathbf{h}}\right) \\
&= \arg\min_\theta \sum_{j=1}^{J} \mathcal{E}\left(\theta; \mathbf{s_j}, \widehat{h}_j\right) \\
&= \arg\min_\theta \frac{1}{2} \sum_{j=1}^{J} \left\| h\left(\mathbf{s}_j; \theta\right) - \widehat{h}_j \right\|^2
\end{aligned}
$$

- We use steepest descent with line search (we tried stochastic and mini-batch gradient descent as well).
- In practice, we do not need a global min ($\neq$ likelihood).
- You can flush the algorithm to a graphics processing unit (GPU) or a tensor processing unit (TPU) instead of a standard CPU.

Estimation with aggregate variables I

- $D+1$ observations of Y_t at fixed time intervals $[0, \Delta, 2\Delta, .., D\Delta]$:

$$Y_0^D = \{Y_0, Y_\Delta, Y_{2\Delta}, ..., Y_D\}.$$

- More general case: sequential Monte Carlo approximation to the Kushner-Stratonovich equation (Fernández-Villaverde and Rubio Ramírez, 2007).
- We are interested in estimating a vector of structural parameters Ψ.
- Likelihood:

$$\mathcal{L}_D\left(Y_0^D|\Psi\right) = \prod_{d=1}^{D} p_Y\left(Y_{d\Delta}|Y_{(d-1)\Delta};\Psi\right),$$

where

$$p_Y\left(Y_{d\Delta}|Y_{(d-1)\Delta};\Psi\right) = \int f_{d\Delta}(Y_{d\Delta}, B)dB.$$

given a density, $f_{d\Delta}(Y_{d\Delta}, B)$, implied by the solution of the model.

Estimation with aggregate variables II

- After finding the diffusion for Y_t, $f_t^d(Y, B)$ follows the Kolmogorov forward (KF) equation in the interval $[(d-1)\Delta, d\Delta]$:

$$\begin{aligned}\frac{\partial f_t}{\partial t} &= -\frac{\partial}{\partial Y}\left[\mu^Y(Y,B)f_t(Y,B)\right] - \frac{\partial}{\partial B}\left[h(B, Y^{\frac{1}{\alpha}} - B)f_t^d(Y,B)\right] \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial Y^2}\left[\left(\sigma^Y(Y)\right)^2 f_t(Y,B)\right]\end{aligned}$$

- The operator in the KF equation is the adjoint of the infinitesimal generator of the HJB.

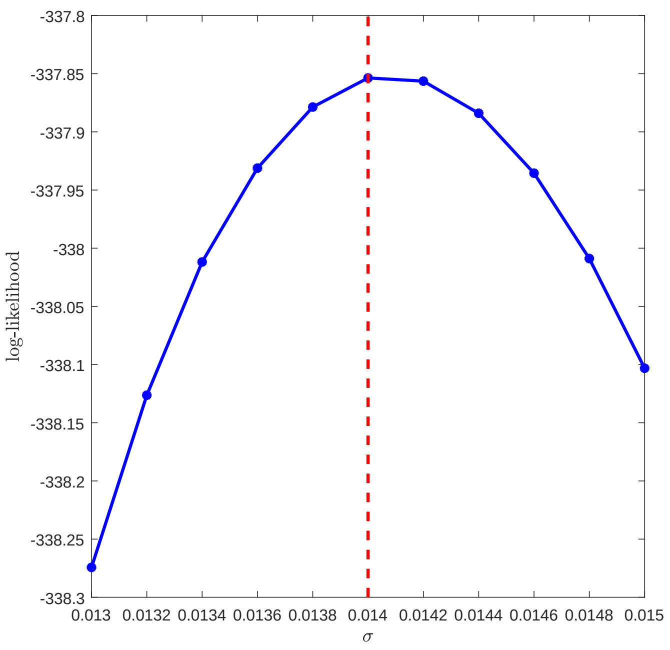
- Thus, the solution of the KF equation amounts to transposing and inverting a sparse matrix that has already been computed.

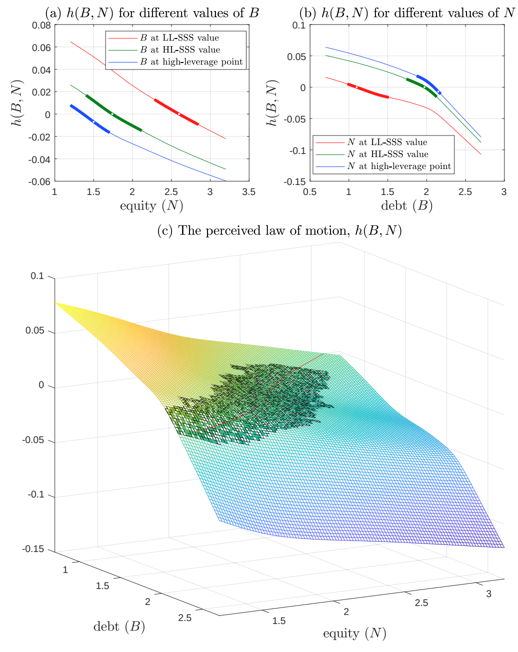
- Our approach provides a highly efficient way of evaluating the likelihood once the model is solved.

- Conveniently, retraining of the neural network is easy for new parameter values.

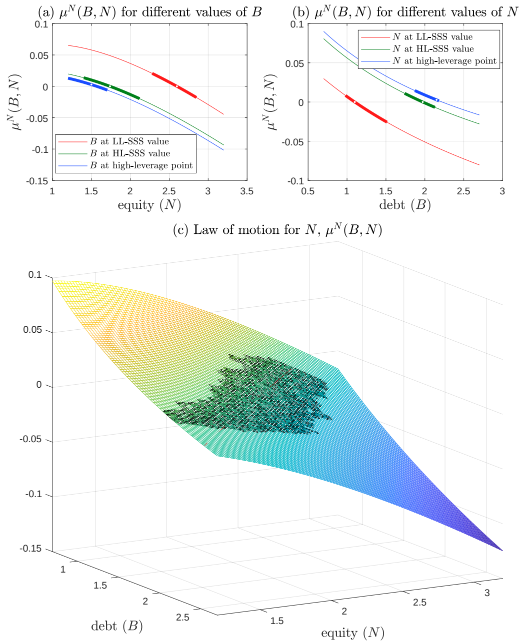
Parametrization

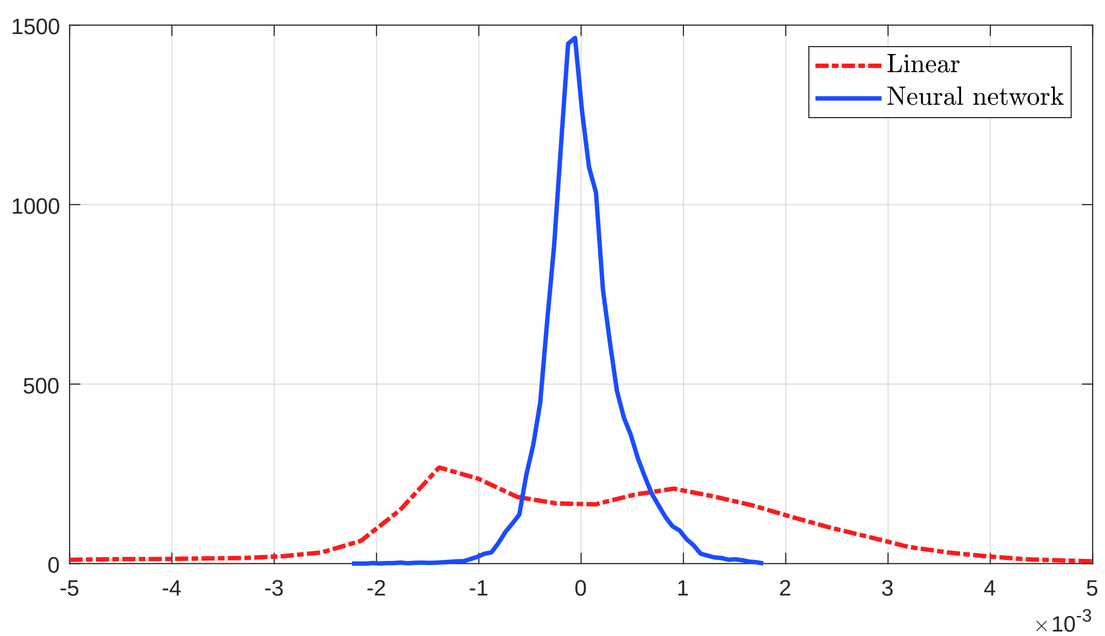
Parameter	Value	Description	Source/Target
α	0.35	capital share	standard
δ	0.1	yearly capital depreciation	standard
γ	2	risk aversion	standard
ρ	0.05	households' discount rate	standard
λ_1	0.986	transition rate u.-to-e.	monthly job finding rate of 0.3
λ_2	0.052	transition rate e.-to-u.	unemployment rate 5 percent
y_1	0.72	income in unemployment state	Hall and Milgrom (2008)
y_2	1.015	income in employment state	$\mathbb{E}(y) = 1$
$\widehat{\rho}$	0.0497	experts' discount rate	$K/N = 2$


-337.8
-337.85
-337.9
-337.95
-338
-338.05
-338.1
-338.15
-338.2
-338.25
-338.3
log-likelihood
0.013
0.0132
0.0134
0.0136
0.0138
0.014
0.0142
0.0144
0.0146
0.0148
0.015
σ

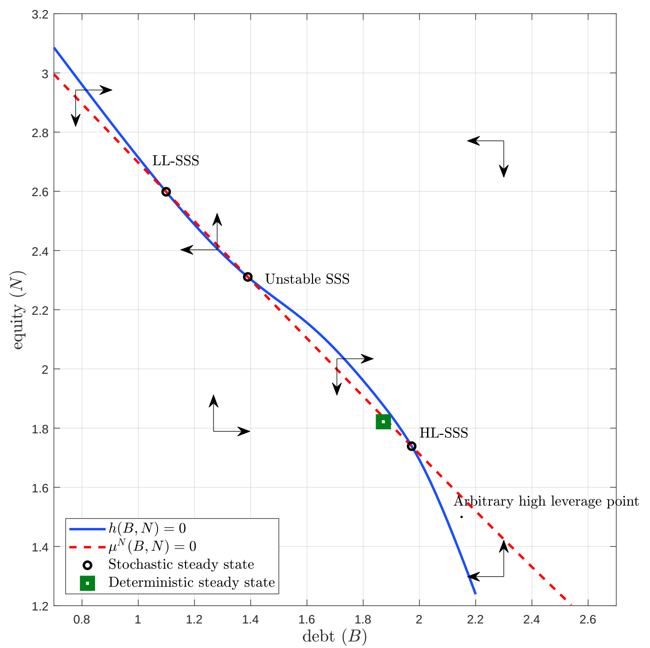


(a) $h(B,N)$ for different values of B
B at LL-SSS value
B at HL-SSS value
B at high-leverage point
$h(B,N)$
equity (N)
(b) $h(B,N)$ for different values of N
N at LL-SSS value
N at HL-SSS value
N at high-leverage point
debt (B)
(c) The perceived law of motion, $h(B,N)$
debt (B)
equity (N)

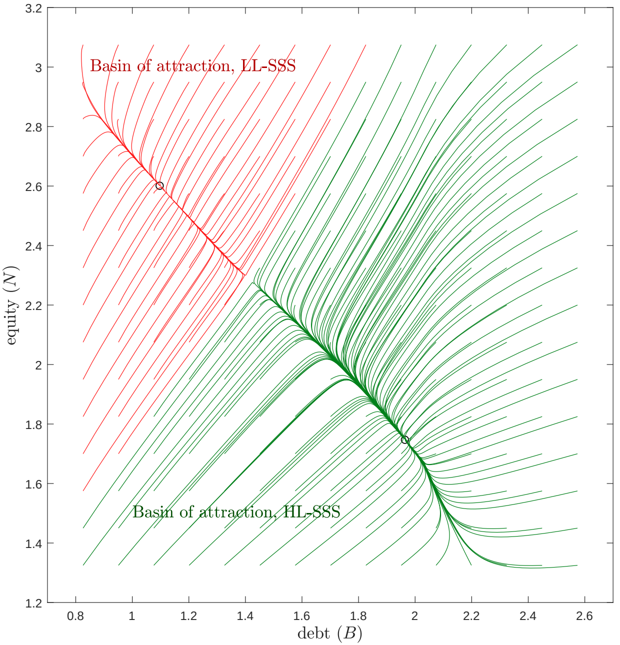

(a) $\mu^N(B,N)$ for different values of B
B at LL-SSS value
B at HL-SSS value
B at high-leverage point
$\mu^N(B,N)$
equity (N)
(b) $\mu^N(B,N)$ for different values of N
N at LL-SSS value
N at HL-SSS value
N at high-leverage point
$\mu^N(B,N)$
debt (B)
(c) Law of motion for N, $\mu^N(B,N)$
debt (B)
equity (N)

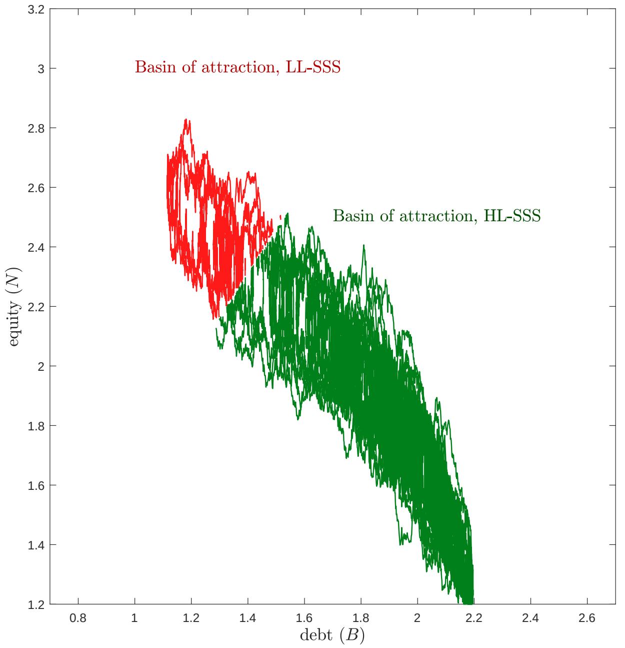


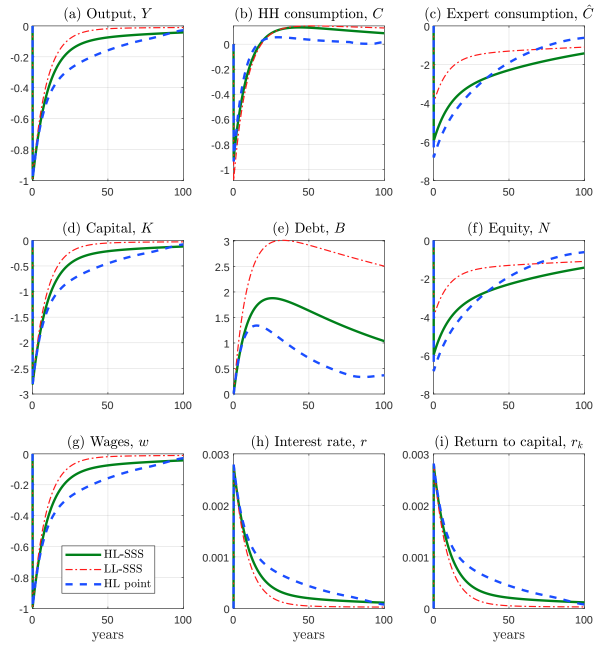
1500
1000
500
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
$\times 10^{-3}$
Linear
Neural network



LL-SSS
Unstable SSS
HL-SSS
Arbitrary high leverage point
$h(B, N) = 0$
$\mu^N(B, N) = 0$
Stochastic steady state
Deterministic steady state
equity (N)
debt (B)
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6

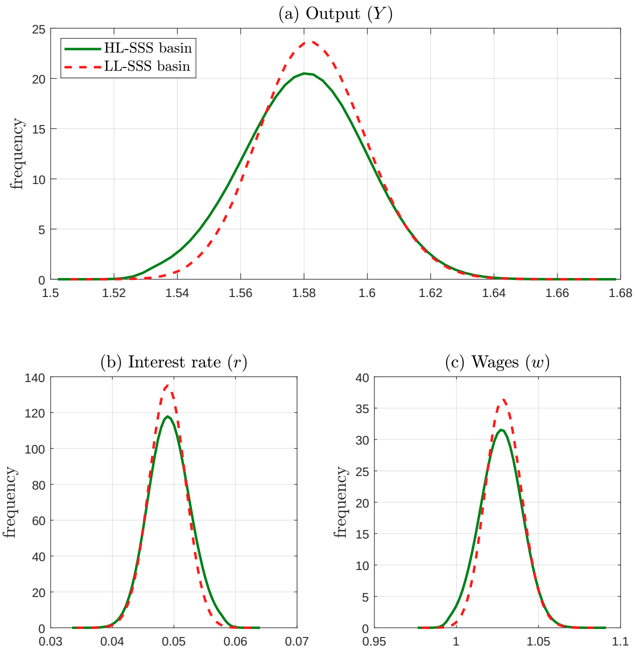

Basin of attraction, LL-SSS
Basin of attraction, HL-SSS
equity (N)
debt (B)
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6


Basin of attraction, LL-SSS
Basin of attraction, HL-SSS
equity (N)
debt (B)
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6

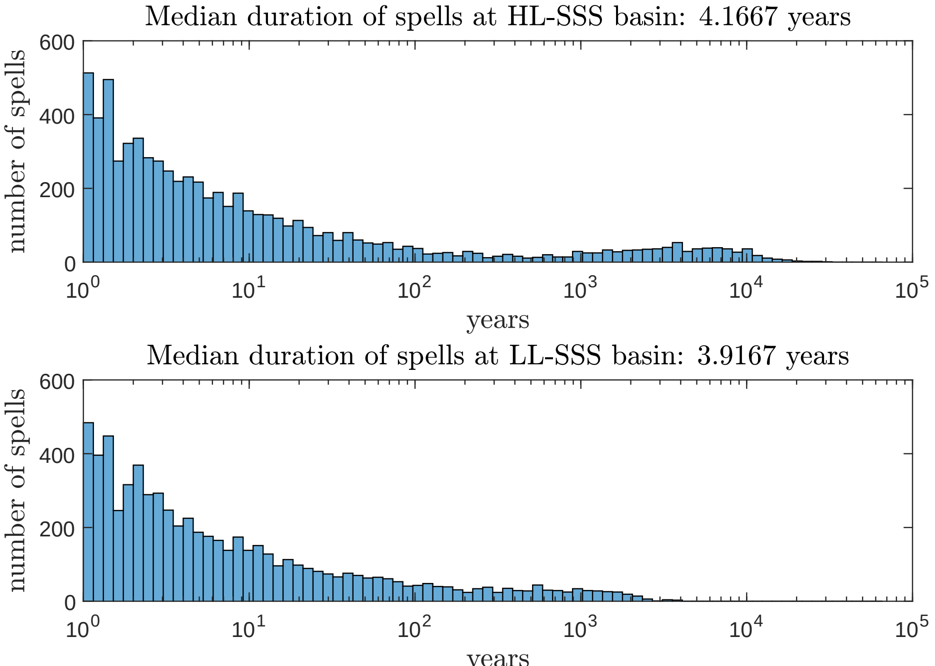

(a) Output, Y
(b) HH consumption, C
(c) Expert consumption, $\hat{C}$
(d) Capital, K
(e) Debt, B
(f) Equity, N
(g) Wages, w
(h) Interest rate, r
(i) Return to capital, r_k
HL-SSS
LL-SSS
HL point
years
years
years

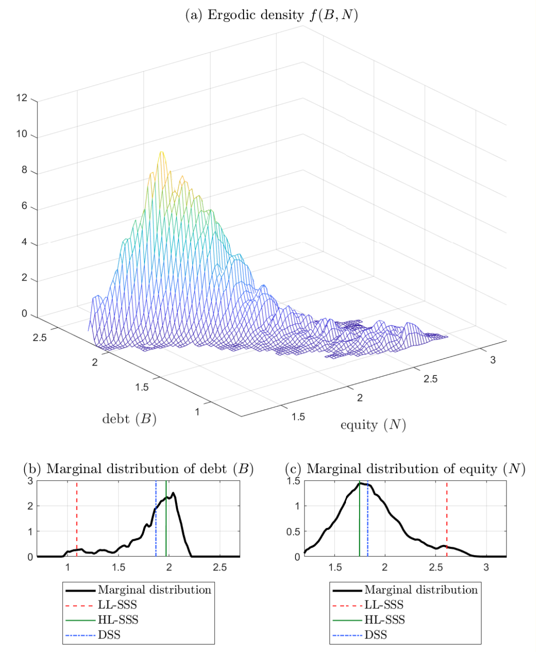
	Mean	Standard deviation	Skewness	Kurtosis
$Y^{\text{basin } HL}$	1.5802	0.0193	0.0014	2.869
$Y^{\text{basin } LL}$	1.5829	0.0169	0.1186	3.0302
$r^{\text{basin } HL}$	4.92	0.3364	0.0890	2.866
$r^{\text{basin } LL}$	4.89	0.2947	-0.0282	3.0056
$w^{\text{basin } HL}$	1.0271	0.0125	0.0014	2.8691
$w^{\text{basin } LL}$	1.0289	0.0111	0.1186	3.0302

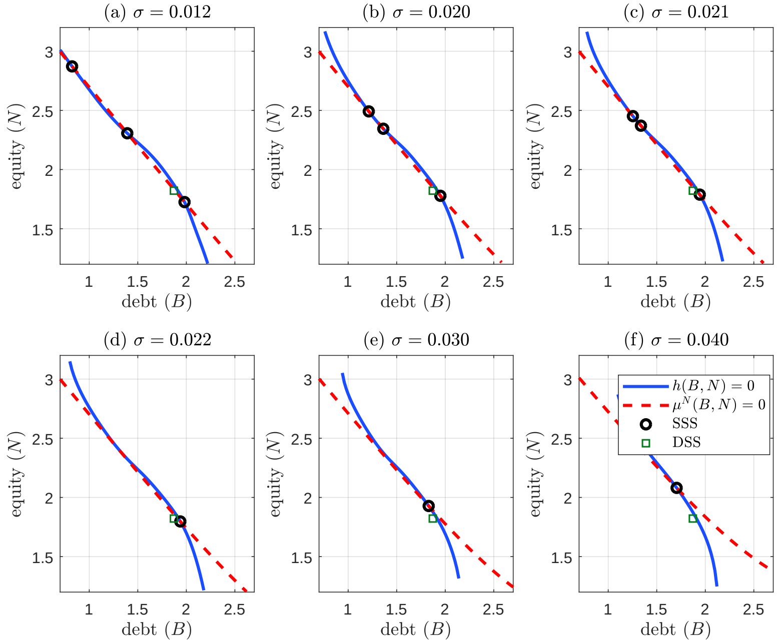
Table 1: Moments conditional on basin of attraction.



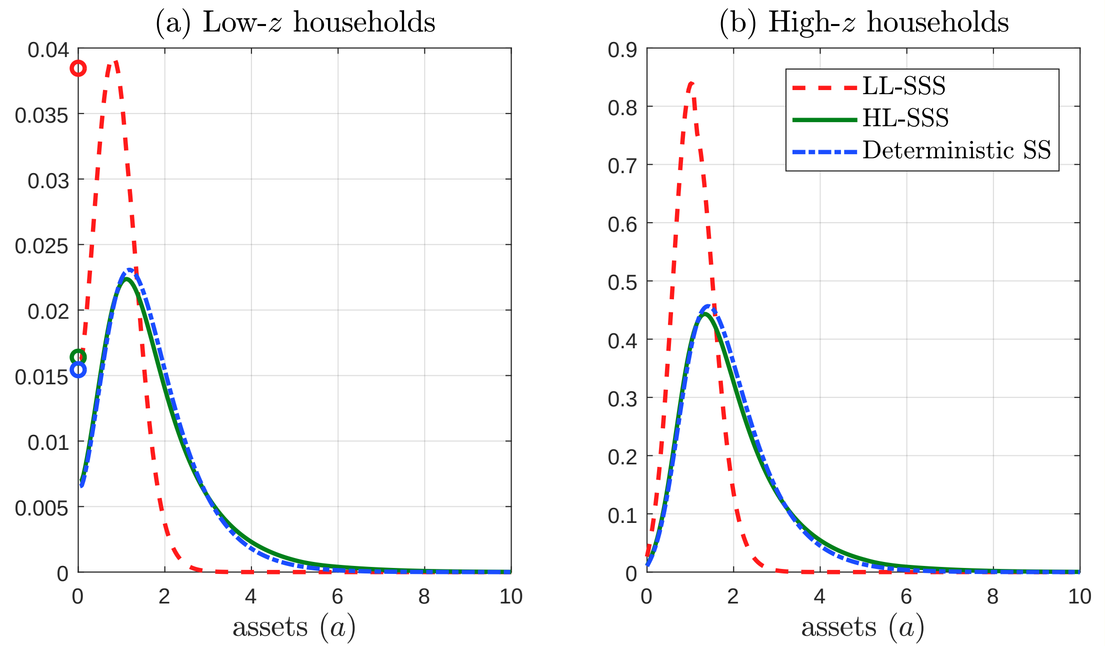
(a) Output (Y)
HL-SSS basin
LL-SSS basin
frequency
25
20
15
10
5
0
1.5
1.52
1.54
1.56
1.58
1.6
1.62
1.64
1.66
1.68
(b) Interest rate (r)
frequency
140
120
100
80
60
40
20
0
0.03
0.04
0.05
0.06
0.07
(c) Wages (w)
frequency
40
35
30
25
20
15
10
5
0
0.95
1
1.05
1.1

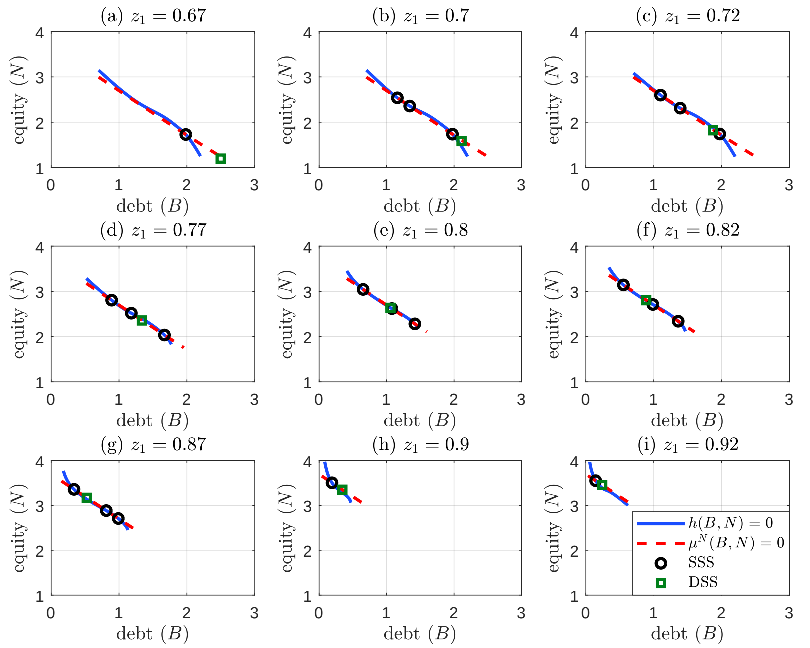

Median duration of spells at HL-SSS basin: 4.1667 years
number of spells
600
400
200
0
10^0
10^1
10^2
10^3
10^4
10^5
years
Median duration of spells at LL-SSS basin: 3.9167 years
number of spells
600
400
200
0
10^0
10^1
10^2
10^3
10^4
10^5
years


(a) Ergodic density $f(B, N)$
12
10
8
6
4
2
0
2.5
2
1.5
1
debt (B)
1.5
2
2.5
3
equity (N)
(b) Marginal distribution of debt (B)
3
2
1
0
1
1.5
2
2.5
Marginal distribution
LL-SSS
HL-SSS
DSS
(c) Marginal distribution of equity (N)
1.5
1
0.5
0
1.5
2
2.5
3
Marginal distribution
LL-SSS
HL-SSS
DSS

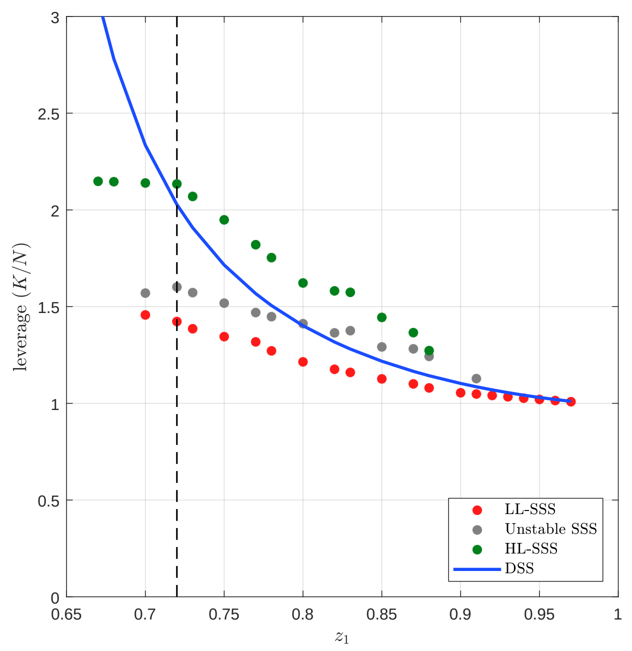


(a) $\sigma = 0.012$
(b) $\sigma = 0.020$
(c) $\sigma = 0.021$
(d) $\sigma = 0.022$
(e) $\sigma = 0.030$
(f) $\sigma = 0.040$
equity (N)
debt (B)
$h(B, N) = 0$
$\mu^N(B, N) = 0$
SSS
DSS

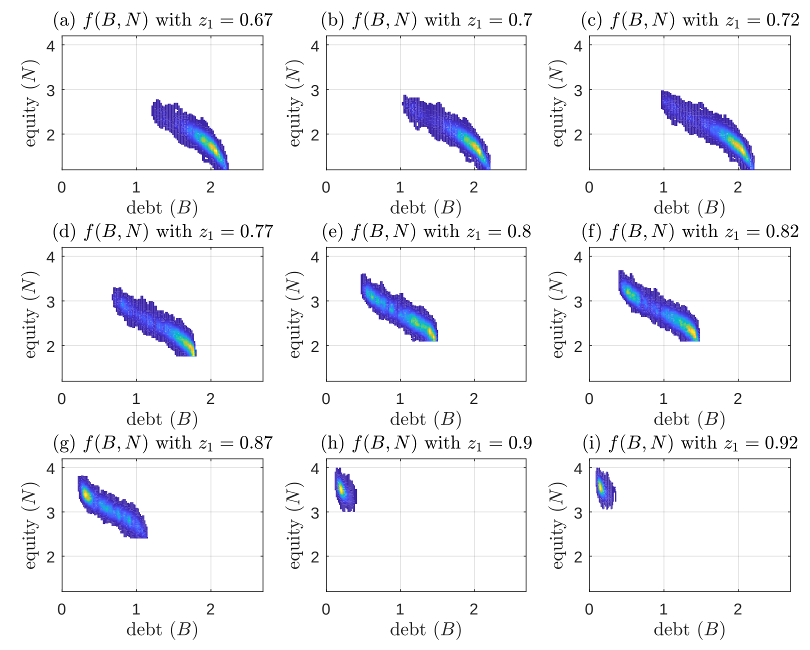

(a) Low-z households
(b) High-z households
LL-SSS
HL-SSS
Deterministic SS
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0 2 4 6 8 10
assets (a)
assets (a)



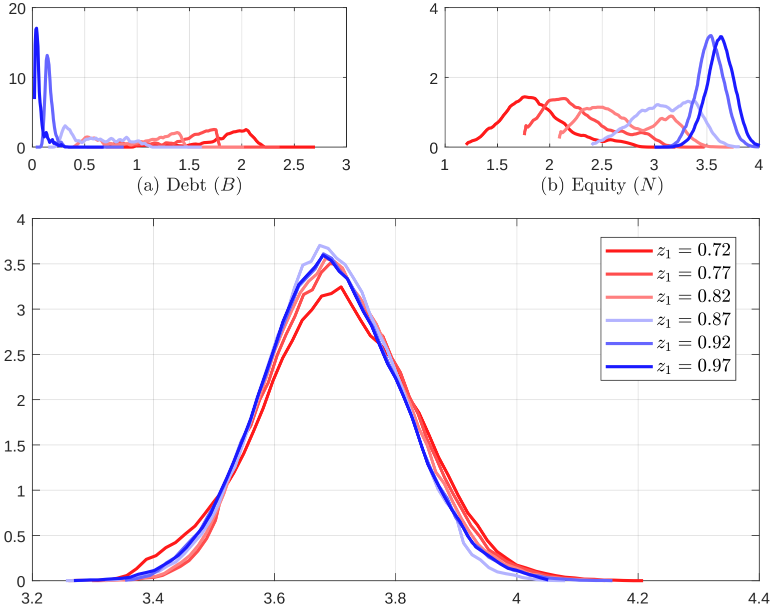
(a) $z_1 = 0.67$
(b) $z_1 = 0.7$
(c) $z_1 = 0.72$
(d) $z_1 = 0.77$
(e) $z_1 = 0.8$
(f) $z_1 = 0.82$
(g) $z_1 = 0.87$
(h) $z_1 = 0.9$
(i) $z_1 = 0.92$
equity (N)
debt (B)
$h(B, N) = 0$
$\mu^N(B, N) = 0$
SSS
DSS



3
2.5
2
1.5
1
0.5
0
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
leverage (K/N)
z_1
LL-SSS
Unstable SSS
HL-SSS
DSS



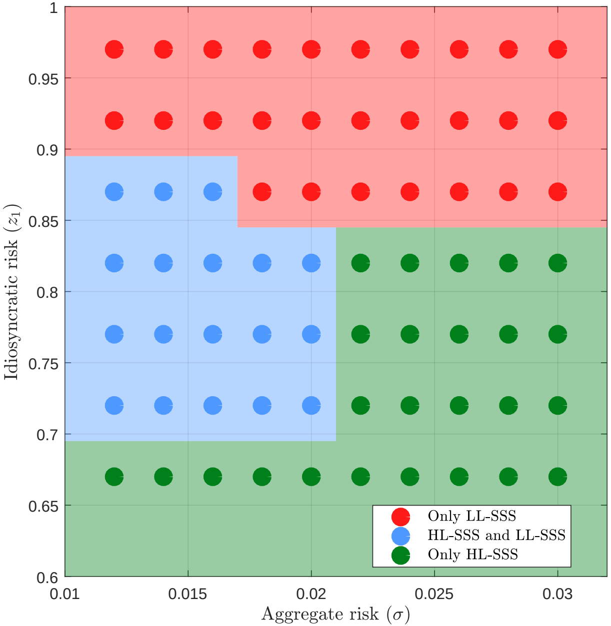
(a) $f(B,N)$ with $z_1 = 0.67$
(b) $f(B,N)$ with $z_1 = 0.7$
(c) $f(B,N)$ with $z_1 = 0.72$
(d) $f(B,N)$ with $z_1 = 0.77$
(e) $f(B,N)$ with $z_1 = 0.8$
(f) $f(B,N)$ with $z_1 = 0.82$
(g) $f(B,N)$ with $z_1 = 0.87$
(h) $f(B,N)$ with $z_1 = 0.9$
(i) $f(B,N)$ with $z_1 = 0.92$
equity (N)
debt (B)



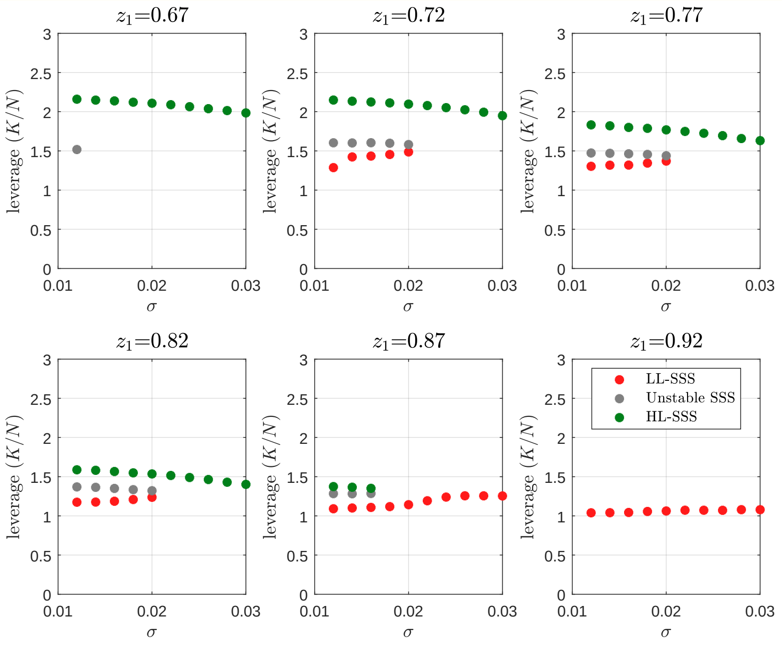
(a) Debt (B)

(b) Equity (N)

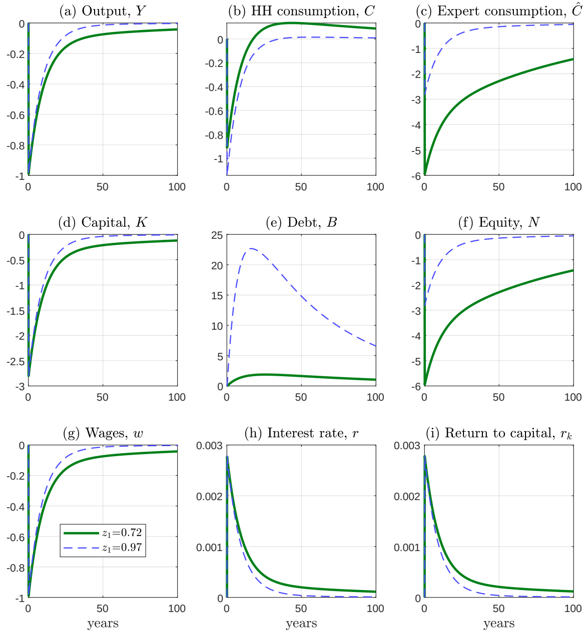
(c) Capital (K)



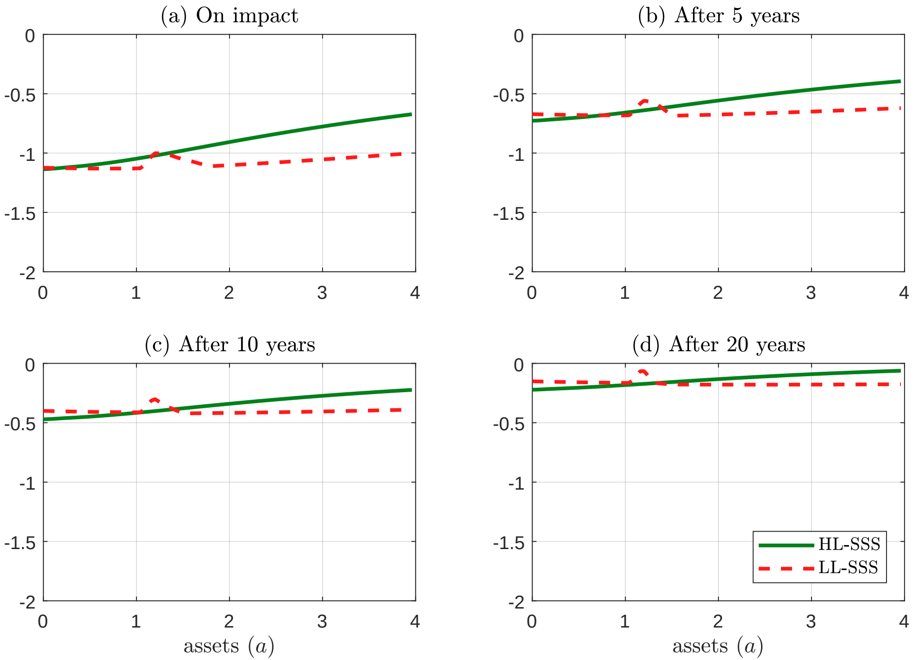
Idiosyncratic risk (z_1)
Aggregate risk (σ)
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.01
0.015
0.02
0.025
0.03
Only LL-SSS
HL-SSS and LL-SSS
Only HL-SSS

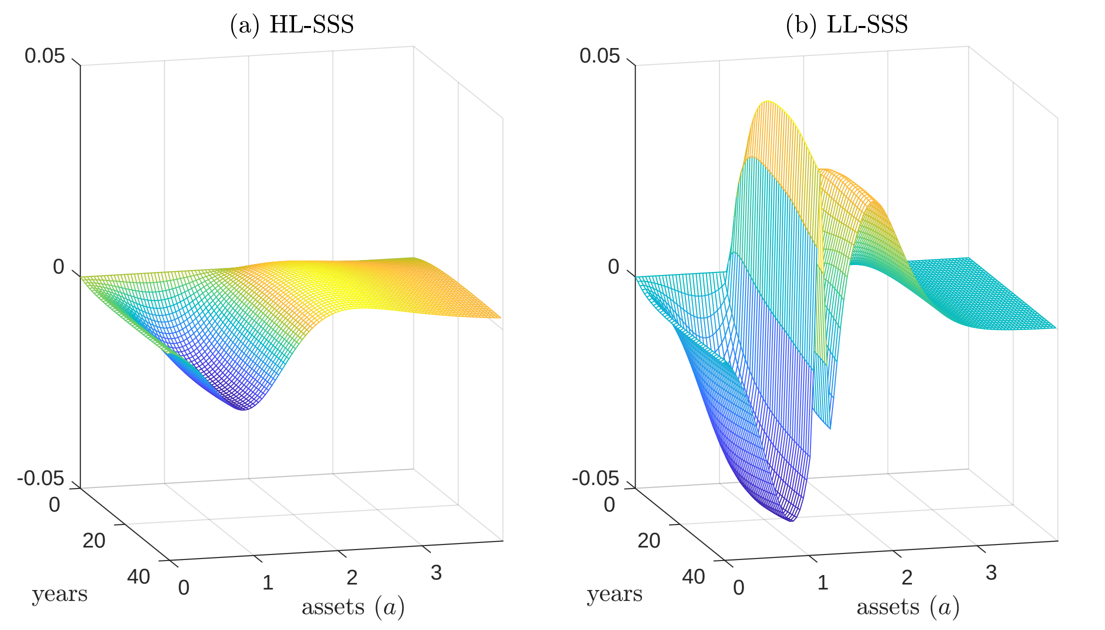


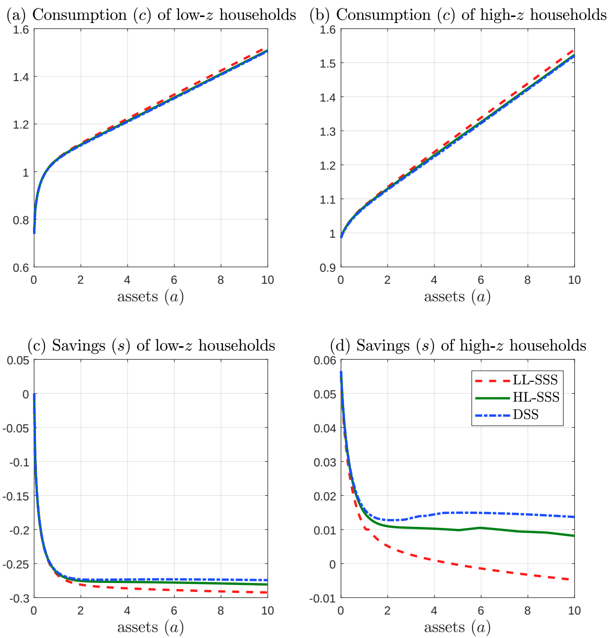
z_1=0.67
z_1=0.72
z_1=0.77
z_1=0.82
z_1=0.87
z_1=0.92
leverage (K/N)
σ
0
0.5
1
1.5
2
2.5
3
0.01
0.02
0.03
LL-SSS
Unstable SSS
HL-SSS



(a) Output, Y
(b) HH consumption, C
(c) Expert consumption, $\hat{C}$
(d) Capital, K
(e) Debt, B
(f) Equity, N
(g) Wages, w
(h) Interest rate, r
(i) Return to capital, r_k
z_1=0.72
z_1=0.97
years


(a) On impact
(b) After 5 years
(c) After 10 years
(d) After 20 years
0
-0.5
-1
-1.5
-2
0 1 2 3 4
assets (a)
HL-SSS
LL-SSS


(a) HL-SSS
0.05
0
-0.05
0
20
40
years
0
1
2
3
assets (a)
(b) LL-SSS
0.05
0
-0.05
0
20
40
years
0
1
2
3
assets (a)


(a) Consumption (c) of low-z households
(b) Consumption (c) of high-z households
(c) Savings (s) of low-z households
(d) Savings (s) of high-z households
assets (a)
LL-SSS
HL-SSS
DSS

Appendix

A quick tour on Hilbert spaces and operators

- Let $L^2(\Phi)$ be the space of functions with a square that is Lebesgue-integrable over $\Phi \subset \mathbb{R}$.

- $L^2(\Phi)$ is a Banach space (a complete normed vector space) with the norm $\|g\|_{L^2(\Phi)} = \sqrt{\int_\Phi |g(x)|^2 \, dx}$.

- The space $L^2(\Phi)$ with the inner product $\langle u, g \rangle_\Phi = \int_\Phi u(x) g(x) dx, \quad \forall u, g \in L^2(\Phi)$, is a Hilbert space.

- An operator T is a mapping from one vector space to another.

- The adjoint operator T^* of a linear operator T in a Hilbert space is defined by the equation:

$$\langle u, Tg \rangle_\Phi = \langle T^* u, g \rangle_\Phi \, .$$

Adjoint operator

- We may verify that $\mathcal{A}$ and $\mathcal{A}^*$, defined as:

$$\mathcal{A}V = \sum_{n=1}^{N}\mu_n\frac{\partial V}{\partial x_n} + \frac{1}{2}\sum_{n_1,n_2=1}^{N}\left(\sigma^2\right)_{n_1,n_2}\frac{\partial^2 V}{\partial x_{n_1}\partial x_{n_2}}.$$

$$\mathcal{A}^* f \equiv -\sum_{n=1}^{N}\frac{\partial}{\partial x_n}\left[\mu_n(x) f(x)\right] + \frac{1}{2}\sum_{n_1,n_2=1}^{N}\frac{\partial^2}{\partial x_{n_1}\partial x_{n_2}}\left[\left(\sigma^2\right)_{n_1,n_2} f_t(x)\right],$$

are adjoint operators in $L^2(\mathbb{R})$:

$$\langle f, \mathcal{A}V\rangle = \int f(x)\left(\sum_{n=1}^{N}\mu_n\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{n_1,n_2=1}^{N}\left(\sigma^2\right)_{n_1,n_2}\frac{\partial^2 V}{\partial x_{n_1}\partial x_{n_2}}\right)dx$$

$$= \int V\left(-\sum_{n=1}^{N}\frac{\partial}{\partial x_n}(f\mu_n)\,dx + \sum_{n_1,n_2=1}^{N}\frac{\partial^2}{\partial x_{n_1}\partial x_{n_2}}\left(f\frac{\left(\sigma\sigma^\top\right)_{i,k}}{2}\right)dx\right)dx$$

$$= \int V\mathcal{A}^* f dx = \langle \mathcal{A}^* f, V\rangle$$

Back

The Kolmogorov Forward equation: proof, I

- We define the density $f_t(x)$ of a stochastic process as:

$$\mathbb{E}_{t_0}[h(X_t)] = \int h(x) f_t(x)\, dx,$$

  or any suitable function $h(x)$.

- Then:

$$\begin{aligned} \int h(x) f_t(x)\, dx \quad = \mathbb{E}_{t_0}[h(X_t)] = \quad & h(x_0) + \mathbb{E}_{t_0}\left[\int_{t_0}^{t} \mathcal{A} h ds\right] \\ & \underbrace{+\mathbb{E}_{t_0}\left[\int_{t_0}^{t} \sigma \frac{\partial h}{\partial x} dW_s\right]}_{0}. \end{aligned}$$

The Kolmogorov Forward equation: proof, II

- Taking derivatives with respect to t:

$$\int h(x)\frac{\partial f}{\partial t}dx = \mathbb{E}_{t_0}\left[\mathcal{A}h\right] = \int \mathcal{A}h\left(x\right) f_t\left(x\right) dx,$$

  or equivalently, $\left\langle \frac{\partial f}{\partial t}, h\right\rangle = \langle \mathcal{A}h, f_t\rangle$.

- Applying the definition of adjoint operator $\left\langle \frac{\partial f}{\partial t}, h\right\rangle = \langle \mathcal{A}^* f_t, h\rangle$ for any function h.

- Hence:

$$\frac{\partial f}{\partial t} = \mathcal{A}^* f_t$$

Back