# Dynamic programming in continuous time

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# Basic ideas

# Dynamic optimization

- Many (most?) macroeconomic models of interest require the solution of dynamic optimization problems, both in deterministic and stochastic environments.
- Two time frameworks:
  1. Discrete time.
  2. Continuous time.
- Three approaches:
  1. Calculus of Variations and Lagrangian multipliers on Banach spaces.
  2. Hamiltonians.
  3. Dynamic Programming.
- We will study dynamic programming in continuous time.

# Why dynamic programming in continuous time?

- Continuous time methods transform optimal control problems into partial differential equations (PDEs):

  1. The Hamilton-Jacobi-Bellman equation, the Kolmogorov Forward equation, the Black-Scholes equation,... they are all PDEs.
  2. Solving these PDEs turns out to be much simpler than solving the Bellman or the Chapman-Kolmogorov equations in discrete time. Also, much knowledge of PDEs in natural sciences and applied math.
  3. Key role of typical sets in the "curse of dimensionality."

- Dynamic programming is a convenient framework:

  1. It can do everything economists could get from calculus of variations.
  2. It is better than Hamiltonians for the stochastic case.

## The development of "continuous-time methods"

- Differential calculus introduced in the 17th century by Isaac Newton and Gottfried Wilhelm Leibniz.
- In the late 19th century and early 20th century, it was extended to accommodate stochastic processes ("stochastic calculus").
  - Thorvald N. Thiele (1880): Introduces the idea of Brownian motion.
  - Louis Bachelier (1900): Formalizes the Brownian motion and applies to the stock market.
  - Albert Einstein (1905): A model of the motion of small particles suspended in a liquid.
  - Norbert Wiener (1923): Uses the ideas of measure theory to construct a measure on the path space of continuous functions.
  - Andrey Kolmogorov (1931): Diffusions depend on drift and volatility, Kolmogorov equations.
  - Wolfgang Döblin (1938-1940): Modern treatment of diffusions with a change of time.
  - Kiyosi Itô (1944): Itô's Lemma.
  - Paul Malliavin (1978): Malliavin calculus.

# The development of "dynamic programming"

- Calculus of variations: Issac Newton (1687), Johann Bernoulli (1696), Leonhard Euler (1733), Joseph-Louis Lagrange (1755).
- 1930s and 1940s: many problems in aerospace engineering are hard to tackle with calculus of variations. Example: minimum time interception problems for fighter aircraft.
- Closely related to the Cold War.
- Lev S. Pontryagin, Vladimir G. Boltyanskii, and Revaz V. Gamkrelidze (1956): Maximum principle.
- Magnus R. Hestenes, Rufus P. Isaacs, and Richard E. Bellman at RAND (1950s):
  1. Distinction between controls and states.
  2. Principle of optimality.
  3. Dynamic programming.

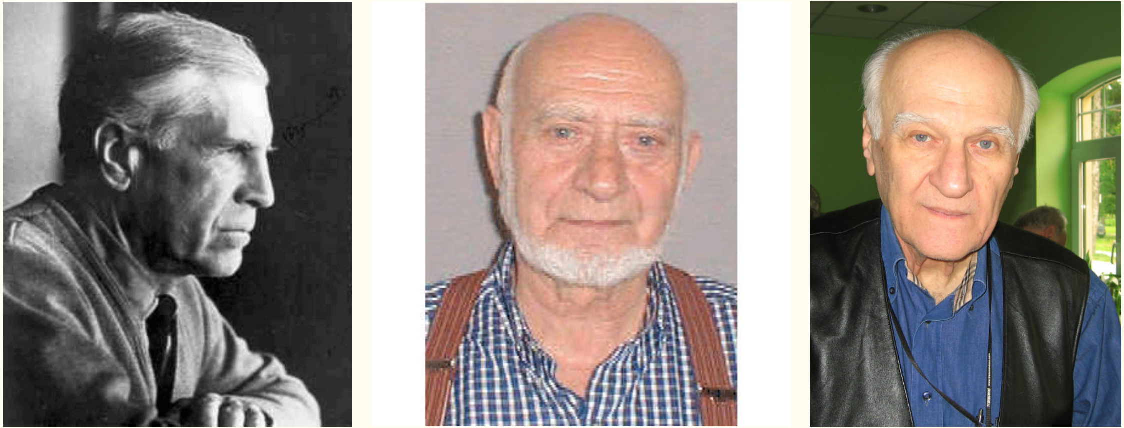


**Figure 1:** Lev S. Pontryagin, Vladimir G. Boltyanskii, and Revaz V. Gamkrelidze

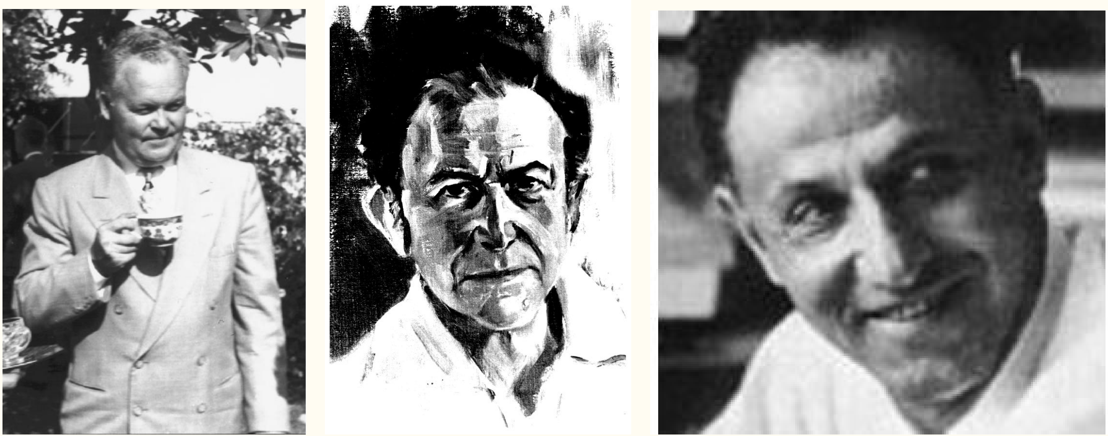
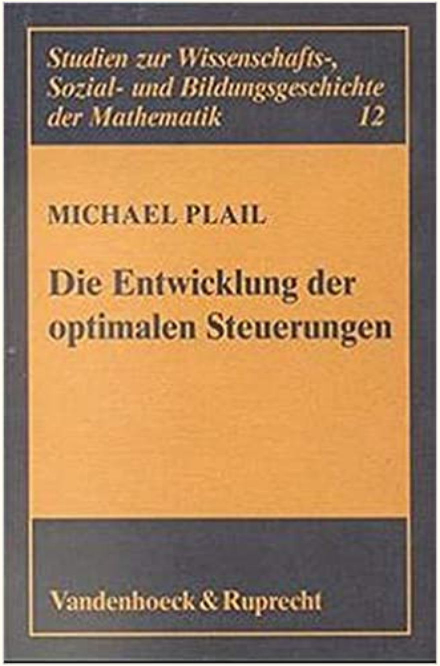
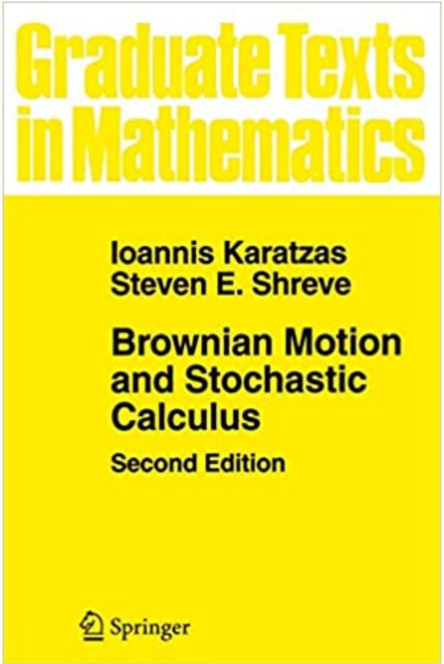
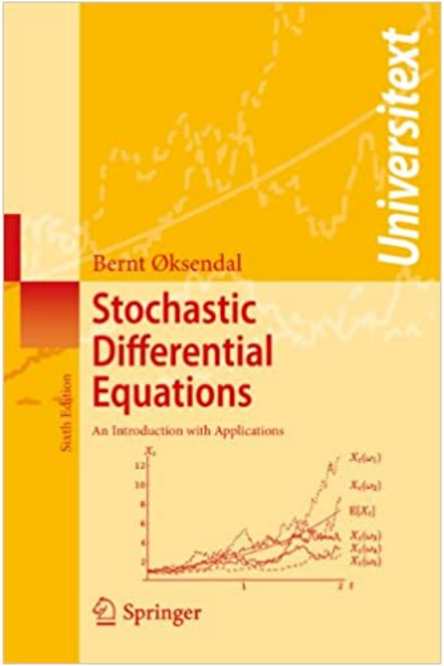
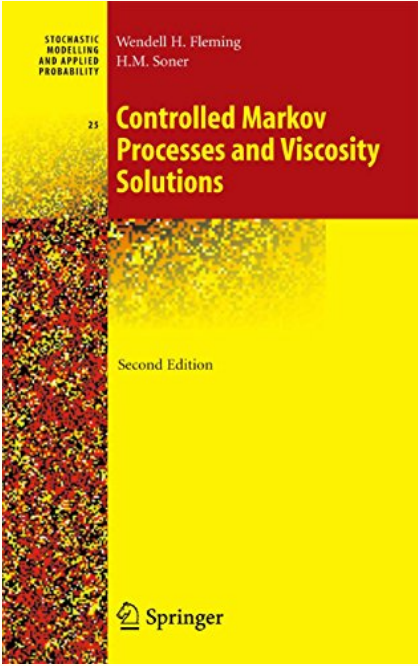


**Figure 2:** Magnus R. Hestenes, Rufus P. Isaacs, and Richard E. Bellman


Studien zur Wissenschafts-, Sozial- und Bildungsgeschichte der Mathematik 12
MICHAEL PLAIL
Die Entwicklung der optimalen Steuerungen
Vandenhoeck & Ruprecht


Graduate Texts in Mathematics
Ioannis Karatzas
Steven E. Shreve
Brownian Motion and Stochastic Calculus
Second Edition
Springer


Universitext
Bernt Øksendal
Stochastic Differential Equations
An Introduction with Applications
Sixth Edition
Springer


STOCHASTIC MODELLING AND APPLIED PROBABILITY
25
Wendell H. Fleming
H.M. Soner
Controlled Markov Processes and Viscosity Solutions
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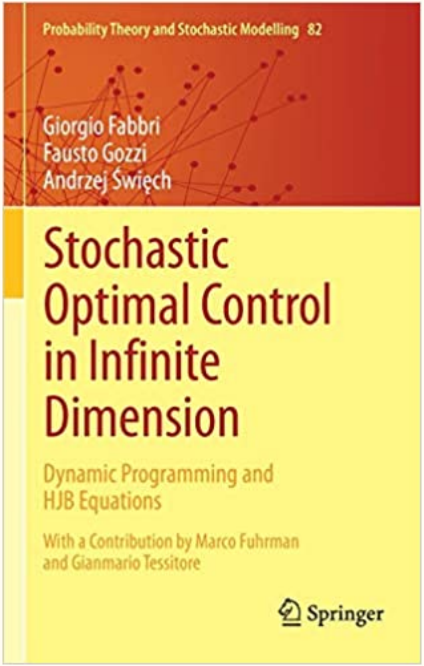
Springer


Probability Theory and Stochastic Modelling 82
Giorgio Fabbri
Fausto Gozzi
Andrzej Święch
Stochastic Optimal Control in Infinite Dimension
Dynamic Programming and HJB Equations
With a Contribution by Marco Fuhrman and Gianmario Tessitore
Springer

# Optimal control

- An agent maximizes:

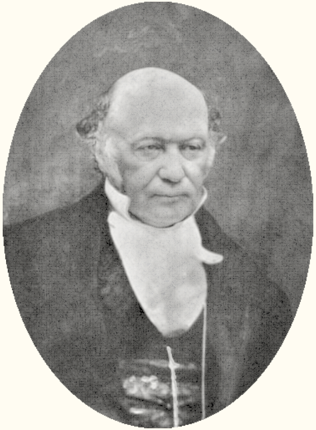
$$\max_{\{\alpha_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(\alpha_t, x_t)\, dt,$$

subject to:

$$\frac{dx_t}{dt} = \mu_t(\alpha_t, x_t), \qquad x_0 = x.$$

- Here, $x_t \in \mathbb{X} \subset \mathbb{R}^N$ is the state, $\alpha_t \in \mathbb{A} \subset \mathbb{R}^M$ is the control, $\rho > 0$ is the discount factor, $\mu(\cdot) : \mathbb{A} \times \mathbb{X} \to \mathbb{R}^N$ the drift, and $u(\cdot) : \mathbb{A} \times \mathbb{X} \to \mathbb{R}$ the instantaneous reward (utility).

# Hamilton-Jacobi-Bellman



William Hamilton (1805-1865)

Carl Jacobi (1804-1851)

Richard Bellman (1920-1984)

# The Hamilton-Jacobi-Bellman equation

- If we define the value function:

$$V(t,x) = \max_{\{\alpha_s\}_{s\geq t}} \int_t^{\infty} e^{-\rho(s-t)} u(\alpha_s, x_s)\, ds,$$

then, under technical conditions, it satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho V_t(x) = \frac{\partial V}{\partial t} + \max_{\alpha} \left\{ u(\alpha, x) + \sum_{n=1}^{N} \mu_{t,n}(x,\alpha) \frac{\partial V}{\partial x_n} \right\},$$

with a transversality condition $\lim_{T\to\infty} e^{-\rho T} V_T(x) = 0$.

# HJB: proof

1. Apply the Bellman optimality principle (and $\lim_{T\to\infty} e^{-\rho T} V_T(x) = 0$):

$$V(t_0, x) \equiv \max_{\{\alpha_s\}_{t_0 \le s \le t}} \left[\int_{t_0}^{t} e^{-\rho(s-t_0)} u(\alpha_s, x_s)\, ds\right] + \left[e^{-\rho(t-t_0)} V(t, x_t)\right]$$

2. Take the derivative with respect to $t$ with the Leibniz integral rule and $\lim_{t\to t_0}$:

$$0 = \lim_{t\to t_0} \left[\max_{\alpha_t} e^{-\rho(t-t_0)} u(\alpha_t, x) + \frac{\left[d\left(e^{-\rho(t-t_0)} V(t, x_t)\right)\right]}{dt}\right]$$

# Example: consumption-savings problem

- A household solves:

$$\max_{\{c_t\}_{t\geq 0}} \int_0^{\infty} e^{-\rho t} \log\left(c_t\right) dt,$$

subject to:

$$\frac{da_t}{dt} = ra_t + y - c_t, \qquad a_0 = \bar{a},$$

where $r$ and $y$ are constants.

- The HJB is:

$$\rho V\left(a\right) = \max_{c} \left\{ \log\left(c\right) + \left(ra + y - c\right) \frac{dV}{da} \right\}$$

- Intuitive interpretation.

## Example: solution

- We guess $V(a) = \frac{1}{\rho}\log\rho + \frac{1}{\rho}\left(\frac{r}{\rho} - 1\right) + \frac{1}{\rho}\log\left(a + \frac{y}{r}\right)$.
- The first-order condition is:

$$\frac{1}{c} = \frac{dV}{da} = \frac{1}{\rho\left(a + \frac{y}{r}\right)},$$

and hence:

$$c = \rho\left(\overbrace{a}^{\text{Financial wealth}} + \overbrace{\frac{y}{r}}^{\text{Human wealth}}\right)$$

- Then, we verify the HJB:

$$\rho V(a) = \log\left(\rho\left(a + \frac{y}{r}\right)\right) + \left(ra + y - \rho\left(a + \frac{y}{r}\right)\right)\frac{dV}{da}$$

# The Hamiltonian

- Assume $\mu_n(x,\alpha) = \mu_{t,n}(x_n,\alpha)$ (to simplify matters).
- Define the costates $\lambda_{nt} \equiv \frac{\partial V}{\partial x_n}(x_t)$ in the HJB.
- Then, the optimal policies are those that maximize the Hamiltonian $\mathcal{H}(\alpha, x, \lambda)$:

$$\max_{\alpha} \left\{ \overbrace{u(\alpha, x) + \sum_{n=1}^{N} \mu_n(x,\alpha)\lambda_{nt}}^{\mathcal{H}(\alpha,x,\lambda)} \right\}$$

- Notice: $\frac{d\lambda_{nt}}{dt} = \frac{\partial^2 V}{\partial t \partial x_n} + \frac{\partial^2 V}{\partial x_n^2}\frac{dx_t}{dt}$.

# Pontryagin maximum principle

- Recall the Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho V_t(x) = \frac{\partial V}{\partial t} + \max_{\alpha} \left\{ u(\alpha, x) + \sum_{n=1}^{N} \mu_n(x, \alpha) \frac{\partial V}{\partial x_n} \right\}$$

- If we take derivatives with respect to $x_n$ in the HJB, we obtain:

$$\rho \frac{\partial V}{\partial x_n} = \frac{\partial^2 V}{\partial t \partial x_n} + \frac{\partial u}{\partial x_n} + \frac{\partial \mu_n}{\partial x_n} \frac{\partial V}{\partial x_n} + \mu_n(x, \alpha) \frac{\partial^2 V}{\partial x_n^2},$$

  which combined with $\frac{d\lambda_{nt}}{dt} = \frac{\partial^2 V}{\partial t \partial x_n} + \frac{\partial^2 V}{\partial x_n^2} \frac{dx_t}{dt}$ yields:

$$\frac{d\lambda_{nt}}{dt} = \rho \lambda_{nt} - \frac{\partial \mathcal{H}}{\partial x_n}$$

- Plus the transversality conditions, $\lim_{T \to \infty} e^{-\rho T} \lambda_{nT} = 0$.

# Example: now with the maximum principle

- The Hamiltonian $\mathcal{H}(c, a, \lambda) = \log(c) + \lambda(ra + y - c)$.
- The first order condition $\frac{\partial \mathcal{H}}{\partial c} = 0$:

$$\frac{1}{c} = \lambda$$

- The dynamics of the costate $\frac{d\lambda_t}{dt} = \rho\lambda_t - \frac{\partial \mathcal{H}}{\partial a} = (\rho - r)\lambda_t$.
- Then, by basic ODE theory:

$$\lambda_t = \lambda_0 e^{(\rho - r)t},$$

  and $c_t = c_0 e^{-(\rho - r)t}$.
- You need to determine the initial value $c_0 = \rho\left(a_0 + \frac{y}{r}\right)$ using the budget constraint.
- But how do you take care of the filtration in the stochastic case?

# Stochastic calculus

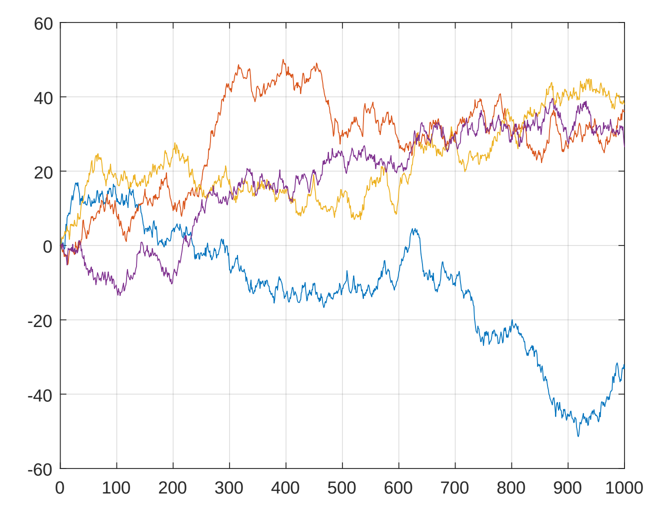
# Brownian motion

- Large class of stochastic processes.
- But stochastic calculus starts with the Brownian motion.
- A stochastic process $W$ is a Brownian motion (a.k.a. Wiener process) if:

  1. $W(0) = 0$.
  2. If $r < s < t < u : W(u) - W(t)$ and $W(s) - W(r)$ are independent random variables.
  3. For $s < t : W(t) - W(s) \sim \mathcal{N}(0, t - s)$.
  4. $W$ has continuous trajectories.

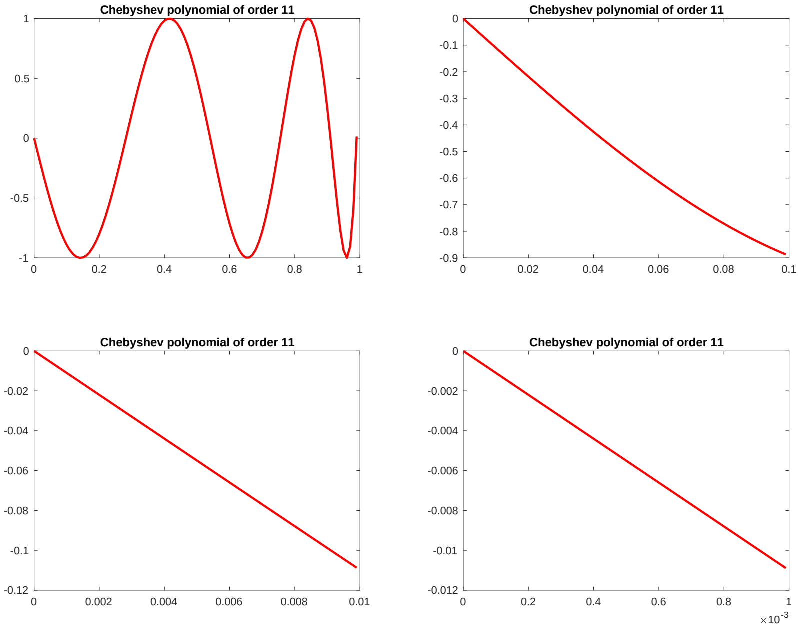
# Simulated paths

- Notice how $\mathbb{E}[W(t)] = 0$ and $Var[W(t)] = t$.



# Why do we need a new concept of integral?

- We will deal with objects such as the expected value function.
- But the value function is now a stochastic function because it depends on stochastic processes.
- How do we think about that expectation?
- More importantly, we need to deal with diffusions, which will include an integral.
- We cannot apply standard rules of calculus: Almost surely, a Brownian motion is nowhere differentiable (even though it is everywhere continuous!).
- Brownian motion exhibits self-similarity (if you know what this means, the Hurst parameter of a Brownian motion is $H = \frac{1}{2} > 0$).
- We need an appropriate concept of integral: Itô stochastic integral.


Chebyshev polynomial of order 11
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
Chebyshev polynomial of order 11
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
0
0.02
0.04
0.06
0.08
0.1
Chebyshev polynomial of order 11
0
-0.02
-0.04
-0.06
-0.08
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-0.12
0
0.002
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Chebyshev polynomial of order 11
0
-0.002
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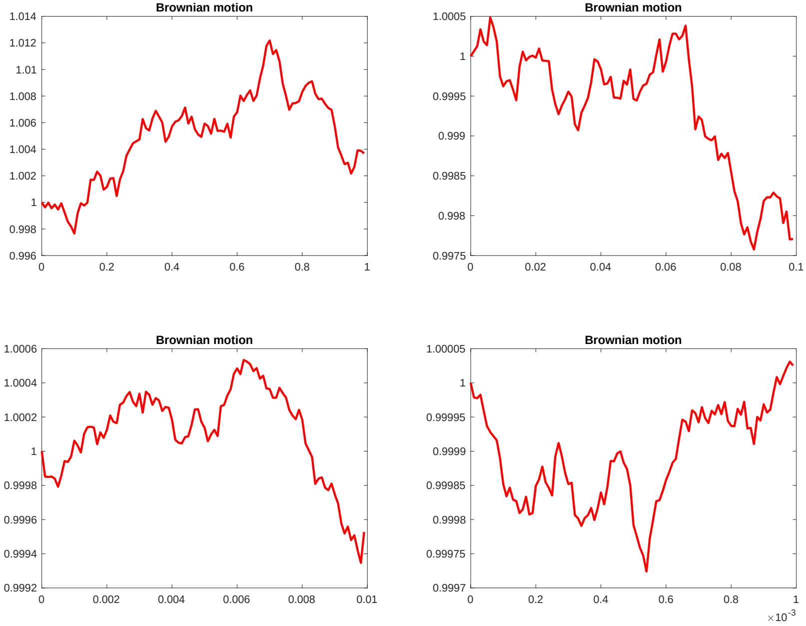
Brownian motion
1.014
1.012
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Brownian motion
1.0005
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0.9995
0.999
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0.02
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Brownian motion
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0.002
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Brownian motion
1.00005
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0.99995
0.9999
0.99985
0.9998
0.99975
0.9997
0
0.2
0.4
0.6
0.8
1
×10^-3

# The stochastic integral

- Recall that the Riemann-Stieltjes integral of a (deterministic) function $g(t)$ with respect to the (deterministic) function $w(t)$ is:

$$\int_0^t g(s)dw(s) = \lim_{n\to\infty} \sum_{k=0}^{n-1} g(t_k)\left[w\left(t_{k+1}\right) - w\left(t_k\right)\right],$$

  where $t_0 = 0$ and $t_n = t$.

- We want to generalize the Riemann-Stieltjes integral to an stochastic environment.

- Given a stochastic process $g(t)$, the stochastic integral with respect to the Brownian motion $W(t)$ is:

$$\int_0^t g(s)dW(s) = \lim_{n\to\infty} \sum_{k=0}^{n-1} g(t_k)\left[W\left(t_{k+1}\right) - W\left(t_k\right)\right],$$

  where $t_0 = 0$ and $t_n = t$ and the limit converges in probability.

- Notice: both the integrand and the integrator are stochastic processes and that the integral is a random variable.

# Mean of the stochastic integral

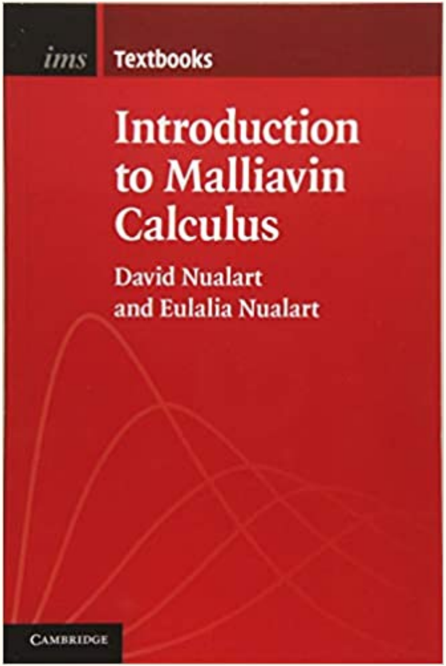
$$
\begin{aligned}
\mathbb{E}\left[\int_0^t g(s)dW(s)\right] &= \mathbb{E}\left[\lim_{n\to\infty}\sum_{k=0}^{n-1} g(t_k)\left[W\left(t_{k+1}\right) - W\left(t_k\right)\right]\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1} g(t_k)\mathbb{E}\left[W\left(t_{k+1}\right) - W\left(t_k\right)\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1} g(t_k)\cdot 0 = 0
\end{aligned}
$$

# Variance of the stochastic integral

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^t g(s)dW(s)\right)^2\right] &= Var\left[\lim_{n\to\infty}\sum_{k=0}^{n-1} g(t_k)\left[W\left(t_{k+1}\right) - W\left(t_k\right)\right]\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1} g^2(t_k)\, Var\left[W\left(t_{k+1}\right) - W\left(t_k\right)\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1} g^2(t_k)\left(t_{k+1} - t_k\right) = \int_0^t g^2(s)ds
\end{aligned}
$$

# Differentiation in stochastic calculus

- In an analogous way that we can define a stochastic integral, we can define a new idea derivative with respect to Brownian motion.
- Malliavin derivative.
- Applications in finance.
- However, in this course, we will not need to use it.


ims
Textbooks
Introduction to Malliavin Calculus
David Nualart
and Eulalia Nualart
CAMBRIDGE

# Stochastic differential equations (SDEs)

- We define a stochastic differential equation (diffusion) as

$$dX\left(t\right)=\mu\left(t,X(t)\right)dt+\sigma\left(t,X(t)\right)dW(t),\qquad X(0)=x,$$

  as a short-cut to express:

$$X\left(t\right)=x+\int_{0}^{t}\mu\left(t,X(s)\right)ds+\int_{0}^{t}\sigma\left(s,X(t)\right)dW(s)$$

- $\mu\left(\cdot\right)$ is the drift and $\sigma\left(t,X(t)\right)$ the volatility.
- Any stochastic process (without jumps) can be approximated by a diffusion.

# Example I: Brownian motion with drift

- Simplest example (random walk with drift in discrete time)

$$dX(t) = \mu dt + \sigma dW(t), \quad X(0) = x_0,$$

where:

$$\begin{aligned} X(t) &= x_0 + \int_0^t \mu ds + \int_0^t \sigma dW(s) \\ &= x_0 + \mu t + \sigma W(t) \end{aligned}$$

- Then $X(t) \sim \mathcal{N}\left(x + \mu t, \sigma^2 t\right)$. This is not stationary.

- Equivalent to a random walk with drift in discrete time.

# Example II: Ornstein-Uhlenbeck process

- Continuous-time counterpart of an AR(1) in discrete time:

$$dX(t) = \theta\left(\overline{X} - X(t)\right)dt + \sigma dW(t), \qquad X(0) = x_0$$

- Named after Leonard Ornstein and George Eugene Uhlenbeck, although in economics and finance is a.k.a. the Vašíček model of interest rates (Vašíček, 1977).

- Stationary process with mean reversion:

$$\mathbb{E}\left[X(t)\right] = x_0 e^{-\theta t} + \overline{X}\left(1 - e^{-\theta t}\right)$$

  and

$$Var\left[X(t)\right] = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right)$$

- Take the limits as $t \to \infty$!

# Euler–Maruyama method

- Except in a few cases (such as the ones before), we do not know how to get an analytic solution for a SDE.

- How do we get an approximate numerical solution of a SDE?

- Euler–Maruyama method: Extension of the Euler method for ODEs.

- Given a SDE:

$$dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t), \qquad X(0) = x,$$

  it can be approximated by:

$$X(t + \Delta t) - X(t) = \mu(t, X(t))\,\Delta t + \sigma(t, X(t))\,\Delta W(t),$$

  where $\Delta W(t) \overset{iid}{\sim} \mathcal{N}(0, \Delta t)$.

# Euler–Maruyama method (Proof)

- If we integrate the SDE:

$$
\begin{aligned}
X(t+\Delta t) - X(t) &= \int_t^{t+\Delta t} \mu(t, X(s))\, ds + \int_t^{t+\Delta t} \sigma(s, X(t))\, dW(s) \\
&\approx \mu(t, X(t))\, \Delta t + \sigma(t, X(t))\, (W(t+\Delta t) - W(t))
\end{aligned}
$$

  where $W(t+\Delta t) - W(t) = \Delta W(t) \overset{iid}{\sim} \mathcal{N}(0, \Delta t)$.

- The smaller the $\Delta t$, the better the method will work.

- Let us look at some code.

# Stochastic calculus

- Now, we need to learn how to manipulate SDEs.
- Stochastic calculus = "normal" calculus + simple rules:

$$(dt)^2 = 0, \quad dt \cdot dW = 0, \quad (dW)^2 = dt$$

- The last rule is the key. It comes from:

$$\mathbb{E}\left[W(t)^2\right] = Var\left[W(t)\right] = t,$$

and:

$$Var\left[W(t)^2\right] = \underbrace{\mathbb{E}\left[W(t)^4\right]}_{3t^2} - \mathbb{E}\left[W(t)^2\right]^2 = 2t^2 \ll t$$

# Functions of stochastic processes: Itô's formula

- Chain rule in standard calculus. Given $f(t,x)$ and $x(t)$:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} \Longrightarrow df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx$$

- Chain rule in stochastic calculus (Itô's lemma). Given $f(t,X)$ and:

$$dX(t) = \mu\left(t, X(t)\right)dt + \sigma\left(t, X(t)\right)dW(t),$$

we get:

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\right)dt + \frac{\partial f}{\partial x}\sigma dW$$

# Itô's formula: proof

- Taylor expansion of $f(t, X)$:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2 + \frac{\partial^2 f}{\partial t \partial X} dt \cdot dX$$

- Given the rules:

$$\begin{aligned} dX &= \mu dt + \sigma dW, \\ (dX)^2 &= \mu^2 (dt)^2 + \sigma^2 (dW)^2 + 2\mu\sigma dt dW = \sigma^2 dt, \\ (dt)^2 &= 0, \\ dt \cdot dX &= \mu (dt)^2 + \sigma dt dW = 0. \end{aligned}$$

- Then:

$$df = \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2 \right) dt + \frac{\partial f}{\partial x} \sigma dW$$

# Multidimensional Itô's formula

- Given $f(t, x_1, x_2, ..., x_n)$, and

$$\begin{aligned} dX_i(t) &= \mu_i(t, X_1(t), ..., X_n(t))\, dt + \sigma_i(t, X_1(t), ..., X_n(t))\, dW_i(t), \\ i &= 1, ..., n, \end{aligned}$$

  then

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dX_i dX_j$$

# Application: Geometric Brownian motion (GBM)

- Basic model for asset prices (non-negative):

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \qquad X(0) = x_0,$$

where

$$X(t) = x_0 + \int_0^t \mu X(s)ds + \int_0^t \sigma X(s)dW(s)$$

- How can we solve it?

# GBM solution using Itôs formula

- Define $Z(t) = \ln(X(t))$, the Itô's formula gives us:

$$
\begin{aligned}
dZ(t) &= \left(\frac{\partial \ln(x)}{\partial t} + \frac{\partial \ln(x)}{\partial x}\mu x + \frac{1}{2}\frac{\partial^2 \ln(x)}{\partial x^2}\sigma^2 x^2\right) dt \\
&\quad + \frac{\partial \ln(x)}{\partial x}\sigma x dW \\
&= \left(0 + \frac{1}{x}\mu x - \frac{1}{2}\frac{1}{x^2}\sigma^2 x^2\right) dt + \frac{1}{x}\sigma x dW \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW \\
\implies &\quad Z(t) = \ln(x_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)
\end{aligned}
$$

- Therefore:

$$X(t) = x_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

# Dynamic programming with stochastic processes

# The problem

- An agent maximizes:

$$V_0(x) = \max_{\{\alpha_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(\alpha_t, X_t)\, dt,$$

subject to:

$$dX_t = \mu_t (X_t, \alpha_t)\, dt + \sigma_t (X_t, \alpha_t)\, dW_t, \qquad X_0 = x.$$

- $\sigma(\cdot) : \mathbb{A} \times \mathbb{X} \to \mathbb{R}^N$.
- We consider feedback control laws $\alpha_t = \alpha_t(X_t)$ (no improvement possible if they depend on the filtration $\mathcal{F}_t$).

# HJB equation with SDEs

$$\rho V_t(x) = \frac{\partial V}{\partial t} + \max_{\alpha}\left\{ u(\alpha, x) + \sum_{n=1}^{N} \mu_{t,n}(x, \alpha) \frac{\partial V}{\partial x_n} + \frac{1}{2} \sum_{n_1, n_2 = 1}^{N} \left(\sigma_t^2(x, \alpha)\right)_{n_1, n_2} \frac{\partial^2 V}{\partial x_{n_1} \partial x_{n_2}} \right\},$$

where $\sigma_t^2(x, \alpha) = \sigma_t(x, \alpha)\, \sigma_t^\top(x, \alpha) \in \mathbb{R}^{N \times N}$ is the variance-covariance matrix.

# HJB with SDEs: proof

1. Apply the Bellman optimality principle:

$$V_{t_0}(x) = \max_{\{\alpha_s\}_{t_0 \le s \le t}} \mathbb{E}_{t_0}\left[\int_{t_0}^{t} e^{-\rho(s-t_0)} u(\alpha_s, X_s)\, ds\right] + \mathbb{E}_{t_0}\left[e^{-\rho(t-t_0)} V_t(X_t)\right]$$

2. Take the derivative with respect to $t$, apply Itô's formula and take $\lim_{t\to t_0}$:

$$0 = \lim_{t\to t_0}\left[\max_{\alpha_t} e^{-\rho(t-t_0)} u(\alpha_t, x) + \frac{\mathbb{E}_{t_0}\left[d\left(e^{-\rho(t-t_0)} V(t, X_t)\right)\right]}{dt}\right]$$

Notice:

$$\begin{aligned}\mathbb{E}_{t_0}\left[d\left(e^{-\rho(t-t_0)} V(t, X_t)\right)\right] &= \mathbb{E}_{t_0}\left[e^{-\rho(t-t_0)}\left(-\rho V + \frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}\right) dt\right] \\ &\quad + e^{-\rho(t-t_0)} \underbrace{\mathbb{E}_{t_0}\left[\sigma \frac{\partial V}{\partial x} dW_t\right]}_{0}\end{aligned}$$

# The infinitesimal generator

- The HJB can be compactly written as:

$$\rho V = \frac{\partial V}{\partial t} + \max_{\alpha} \{u(\alpha, x) + \mathcal{A}V\},$$

where $\mathcal{A}$ is the infinitesimal generator of the stochastic process $X_t$, defined as:

$$\begin{aligned} \mathcal{A}f &= \lim_{t \downarrow 0} \frac{\mathbb{E}_0 [f(X_t)] - f(x)}{t} \\ &= \sum_{n=1}^{N} \mu_n \frac{\partial f}{\partial x_{t,n}} + \frac{1}{2} \sum_{n_1, n_2 = 1}^{N} \left(\sigma^2\right)_{n_1, n_2} \frac{\partial^2 f}{\partial x_{n_1} \partial x_{n_2}} \end{aligned}$$

- Intuitively: the infinitesimal generator describes the movement of the process in an infinitesimal time interval.

# Boundary conditions

- The boundary conditions of the HJB equation are not free to be chosen, they are imposed by the dynamics of the state at the boundary $\partial \mathbb{X}$.

- Only three possibilities:

  1. Reflection barrier: The process is reflected at the boundary: $\frac{dV}{dx}\Big|_{\partial \mathbb{X}} = 0$.

  2. Absorbing barrier: The state jumps at a different point $y$ when the barrier is reached: $V(x)\Big|_{\partial \mathbb{X}} = V(y)$.

  3. State constraint: The policy $\alpha_t$ guarantees that the process does not abandon the boundary.

# Example: Merton portfolio model

- An agent maximize its discounted utility:

$$V(x) = \max_{\{c_t, \Delta_t\}_{t \geq 0}} \mathbb{E} \int_0^\infty e^{-\rho t} \log(c_t)\, dt,$$

by investing in $\Delta_t$ shares of a stock (GBM) and saving the rest in a bond with return $r$:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_t &= r B_t dt \end{aligned}$$

- The value of the portfolio evolves according to:

$$\begin{aligned} dX_t &= \Delta_t dS_t + r\left(X_t - \Delta_t S_t\right) dt - c_t dt \\ &= \Delta_t\left(\mu S_t dt + \sigma S_t dW_t\right) + r\left(X_t - \Delta_t S_t\right) dt - c_t dt \\ &= \left[r X_t + \Delta_t S_t (\mu - r)\right] dt + \Delta_t \sigma S_t dW_t - c_t dt \end{aligned}$$

# Merton model: The HJB equation

- We redefine one of the controls:

$$\omega_t = \frac{\Delta_t S_t}{X_t}$$

- The HJB results in:

$$\rho V(x) = \max_{c,\omega} \left\{ \log(c) + [rx + \omega x(\mu - r) - c] \frac{\partial V}{\partial x} + \frac{\sigma^2 \omega^2 x^2}{2} \frac{\partial^2 V}{\partial x^2} \right\}$$

- The FOC are:

$$\frac{1}{c} - \frac{\partial V}{\partial x} = 0,$$

$$x(\mu - r) \frac{\partial V}{\partial x} + \omega \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} = 0$$

# Solution to Merton portfolio model

- Guess and verify:

$$
\begin{aligned}
V(x) &= \frac{1}{\rho}\log(x) + \kappa_2, \\
\frac{\partial V}{\partial x} &= \frac{1}{\rho x}, \\
\frac{\partial^2 V}{\partial x^2} &= -\frac{1}{\rho x^2}
\end{aligned}
$$

  with $\kappa_1$ and $\kappa_2$ constants.

- The FOC are:

$$
\begin{aligned}
\frac{1}{c} - \frac{1}{\rho x} &= 0 \Longrightarrow c = \rho x, \\
x(\mu - r)\frac{\kappa_1}{x} - \omega\sigma^2 x^2 \frac{\kappa_1}{x^2} &= 0 \Longrightarrow \omega = \frac{(\mu - r)}{\sigma^2}
\end{aligned}
$$

# The case with Poisson processes

- The HJB can also be solved for the case of Poisson shocks.
- The state is now:

$$dX_t = \mu(X_t, \alpha_t, Z_t)\, dt, \qquad X_0 = x, \quad Z_0 = z_0.$$

- $Z_t$ is a two-state continuous-time Markov chain $Z_t \in \{z_1, z_2\}$. The process jumps from state 1 to state 2 with intensity $\lambda_1$ and vice-versa with intensity $\lambda_2$.
- The HJB in this case is

$$\rho V_{ti}(x) = \frac{\partial V_i}{\partial t} + \max_{\alpha} \left\{ u(\alpha, x) + \mu(x, \alpha, z_i) \frac{\partial V_i}{\partial x} \right\} + \lambda_i (V_j - V_i),$$

$i, j = 1, 2,\ i \neq j$, where $V_i(x) \equiv V(x, z_i)$.

- We can have jump-diffusion processes (Lévy processes): HJB includes the two terms (volatility and jumps).

# Viscosity solutions

- Relevant notion of "solutions" to HJB introduced by Pierre-Louis Lions and Michael G. Crandall in 1983 in the context of PDEs.
- Classical solution of a PDE (to be defined below) are too restrictive.
- We want a weaker class of solutions than classical solutions.
- More concretely, we want to allow for points of non-differentiability of the value function.
- Similarly, we want to allow for convex kinks in the value function.
- Different classes of "weaker solutions."

## What is a viscosity solution?

- There are different concepts of what a "solution" to a PDE $F\left(x, Dw(x), D^2 w(x)\right) = 0,\ x \in X$ is:

  1. "Classical" (Strong) solutions. There is a smooth function $u \in C^2(X) \cap C(\bar{X})$ such that $F\left(x, Du(x), D^2 u(x)\right) = 0,\ x \in X$.
     - Hard to find for HJBs.
  2. Weak solutions. There is a function $u \in H^1(X)$ (Sobolev space) such that for any function $\phi \in H^1(X)$, then $\int_X F\left(x, Du(x), D^2 u(x)\right) \phi(x) dx = 0,\ x \in X$.
     - Problem with uniqueness in HJBs.
  3. Viscosity solutions. There is a locally bounded $u$ that is both a subsolution and a supersolution of $F\left(x, Dw(x), D^2 w(x)\right) = 0,\ x \in X$.
     - If it exists, it is unique.

# Subsolutions and supersolutions

- An upper semicontinuous function $u$ in $X$ is a "subsolution" if for any point $x_0 \in X$ and any $C^2$ function $\phi \in C^2(X)$ such that $\phi(x_0) = u(x_0)$ and $\phi \geq u$ in a neighborhood of $x_0$, we have:

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0$$

- An upper semicontinuous function $u$ in $X$ is a "supersolution" if for any point $x_0 \in X$ and any $C^2$ function $\phi \in C^2(X)$ such that $\phi(x_0) = u(x_0)$ and $\phi \leq u$ in a neighborhood of $x_0$, we have:

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$$

# More on viscosity solutions

- Viscosity solution is unique.
- A baby example: consider the boundary value problem $F(u') = |u'| - 1 = 0$, on $(-1, 1)$ with boundary conditions $u(-1) = u(1) = 0$. The unique viscosity solution is the function $u(x) = 1 - |x|$.
- Coincides with solution to sequence problem of optimization.
- Numerical methods designed to find viscosity solutions.
- Check, for more background, *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations* by Michael G. Crandall, Hitoshi Ishii, and Pierre-louis Lions.
- Also, *Controlled Markov Processes and Viscosity Solutions* by Wendell H. Fleming and Halil Mete Soner.

# Finite difference method

# Solving dynamic programming problems = solving PDEs

- We want to numerically solve the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}\rho V_{ti}(x) &= \frac{\partial V_i}{\partial t} + \max_{\alpha}\left\{u(\alpha, x) + \sum_{n=1}^{N} \mu_{nt}(x, \alpha, z_i) \frac{\partial V_i}{\partial x_n}\right. \\ &+ \lambda_i (V_j - V_i) \\ &+ \left. \frac{1}{2} \sum_{n_1, n_2 = 1}^{N} \left(\sigma_t^2(x, \alpha)\right)_{n_1, n_2} \frac{\partial^2 V_i}{\partial x_1 \partial x_{n_2}}\right\},\end{aligned}$$

with a transversality condition $\lim_{T\to\infty} e^{-\rho T} V_T(x) = 0$, and some boundary conditions defined by the dynamics of $X_t$.

# Overview of methods to solve PDEs

1. Perturbation: consider a Taylor expansion of order $n$ to solve the PDEs around the deterministic steady state (not covered here, similar to discrete time).

2. Finite difference: approximate derivatives by differences.

3. Projection (Galerkin): project the value function over a subspace of functions (non-linear version covered later in the course).

4. Semi-Lagragian. Transform it into a discrete-time problem (not covered here, well known to economists)

# A (limited) comparison from Parra-Álvarez (2018)

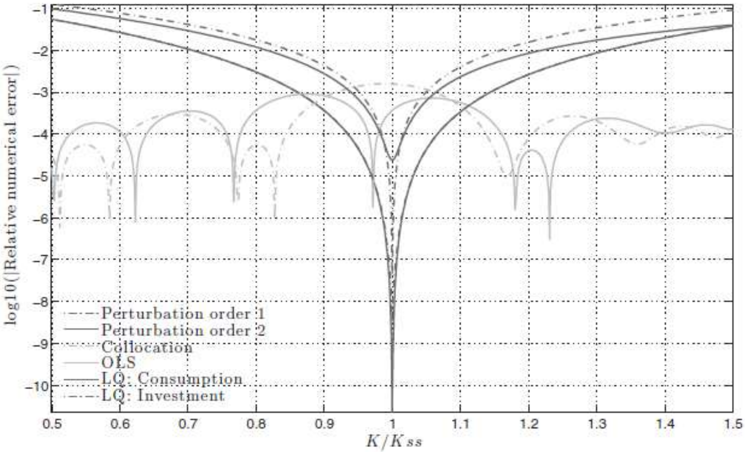



**FIGURE 2.** Numerical error for benchmark model under Proposition 3.1. The graph plots the log10 magnitude of the relative numerical error made by using the approximated value function along the interval $[0.5K^{ss}, 1.5K^{ss}]$. The error is relative to the true value function.

# Numerical advantages of continuous-time methods: Preview

1. “Static” first order conditions. Optimal policies only depend on the current value function:

$$\frac{\partial u}{\partial \alpha} + \sum_{n=1}^{N} \frac{\partial \mu_n}{\partial \alpha} \frac{\partial V}{\partial x_n} + \frac{1}{2} \sum_{n_1, n_2=1}^{N} \frac{\partial}{\partial \alpha} \left( \sigma_t^2 (x, \alpha) \right)_{n_1, n_2} \frac{\partial^2 V}{\partial x_{n_1} \partial x_{n_2}} = 0$$

2. Borrowing constraints only show up in boundary conditions as state constraints.
   - FOCs always hold with equality.
3. No need to compute expectations numerically.
   - Thanks to Itô’s formula.
4. Convenient way to deal with optimal stopping and impulse control problems (more on this later today).
5. Sparsity (with finite differences).

# Our benchmark: consumption-savings with incomplete markets

- An agent maximizes:

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right],$$

subject to:

$$da_t = (z_t + ra_t - c_t)\, dt, \qquad a_0 = \bar{a}$$

where $z_t \in \{z_1, z_2\}$ is a Markov chain with intensities $z_1 \to z_2 : \lambda_1$ and $z_2 \to z_1 : \lambda_2$.

- Exogenous borrowing limit:

$$a_t \geq -\phi$$

# The Hamilton-Jacobi-Bellman equation

- The value function in this problem:

$$v_i(a) = \max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} u(c_t) ds | a_0 = a, z_0 = z_i \right]$$

must satisfy the HJB equation:

$$\rho v_i(a) = \max_c \{u(c) + s_i(a) v_i'(a)\} + \lambda_i (v_j(a) - v_i(a)),$$

where $s_i(a)$ is the drift,

$$s_i(a) = z_i + ra - c(a), \; i = 1, 2$$

- The first-order condition is:

$$u'(c_i(a)) = v_i'(a)$$

# How can we solve it?

- The model proposed above does not yield an analytical solution.
- Therefore we resort to numerical techniques in order to find a solution.
- TIn particular, we employ an upwind finite difference scheme (Achdou et al., 2017).
- This scheme converges to the viscosity solution of the problem.

## Grid

- We approximate the value function $v(a)$ on a finite grid with step $\Delta a : a \in \{a_1, ..., a_J\}$, where

$$a_j = a_{j-1} + \Delta a = a_1 + (j-1)\,\Delta a$$

  for $2 \leq j \leq J$. The bounds are $a_1 = -\phi$ and $a_J = a^*$.

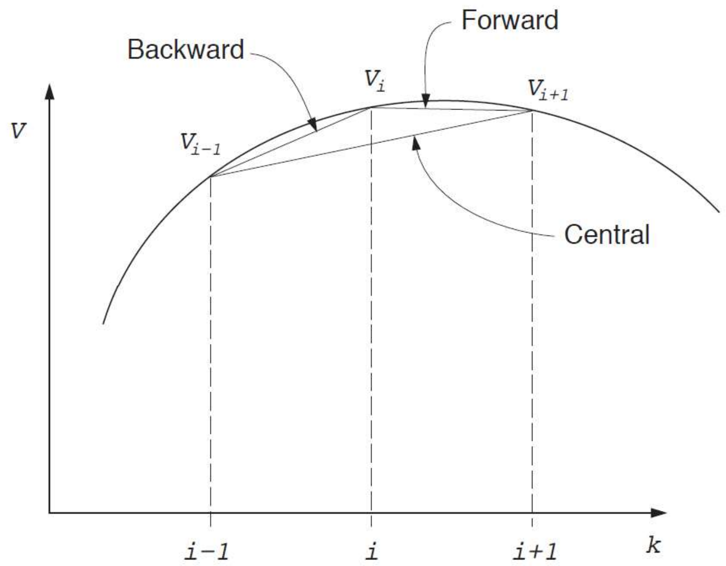
- We use the notation $v_j \equiv v(a_j)$, $j = 1, ..., J$.

# Finite differences

- $v'(a_j)$ can be approximated with a forward $(F)$ or a backward $(B)$ approximation,

$$
\begin{aligned}
v_i'(a_j) &\approx \partial_F v_{i,j} \equiv \frac{v_{i,j+1} - v_{i,j}}{\Delta a} \\
v_i'(a_j) &\approx \partial_B v_{i,j} \equiv \frac{v_{i,j} - v_{i,j-1}}{\Delta a}
\end{aligned}
$$

# Forward and backward approximations



# Upwind scheme

- The choice of $\partial_F v_{i,j}$ or $\partial_B v_{i,j}$ depends on the sign of the drift function $s_i(a) = z_i + ra - (u')^{-1}(v_i'(a))$:
    1. If $s_{iF}(a_j) \equiv z_i + ra_j - (u')^{-1}(\partial_F v_{i,j}) > 0 \longrightarrow c_{i,j} = (u')^{-1}(\partial_F v_{i,j})$.
    2. Else, if $s_{iB}(a_j) \equiv z_i + ra_j - (u')^{-1}(\partial_B v_{i,j}) < 0 \longrightarrow c_{i,j} = (u')^{-1}(\partial_B v_{i,j})$.
    3. Otherwise, $s_i(a) = 0 \longrightarrow c_{i,j} = z_i + ra_j$.
- Why? Key for stability.

# HJB approximation, I

- Let superscript $n$ denote the iteration counter.
- The HJB equation is approximated by:

$$\begin{aligned} \frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} + \rho v_{i,j}^{n+1} &= u(c_{i,j}^{n}) + s_{i,j,F}^{n} \mathbf{1}_{s_{i,j,F}^{n}>0} \partial_F v_{i,j}^{n+1} \\ &\quad + s_{i,j,B}^{n} \mathbf{1}_{s_{i,j,B}^{n}<0} \partial_B v_{i,j}^{n+1} \\ &\quad + \lambda_i \left( v_{-i,j}^{n+1} - v_{i,j}^{n+1} \right), \end{aligned}$$

for $j = 1, ..., J$, where $\mathbf{1}(\cdot)$ is the indicator function and:

$$\begin{aligned} s_{i,j,F}^{n} &= (z_i + r a_j) - (u')^{-1} \left( \partial_F v_{i.j}^{n} \right) \\ s_{i,j,B}^{n} &= (z_i + r a_j) - (u')^{-1} \left( \partial_B v_{i.j}^{n} \right) \end{aligned}$$

# HJB approximation, II

- Collecting terms, we obtain:

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} + \rho v_{i,j}^{n+1} = u(c_{i,j}^{n}) + v_{i,j-1}^{n+1} x_{i,j}^{n} + v_{i,j}^{n+1} y_{i,j}^{n} + v_{i,j+1}^{n+1} z_{i,j}^{n} + v_{-i,j}^{n+1} \lambda_i,$$

where:

$$\begin{aligned}
x_{i,j}^{n} &\equiv -\frac{s_{i,j,B}^{n} \mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a}, \\
y_{i,j}^{n} &\equiv -\frac{s_{i,j,F}^{n} \mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a} + \frac{s_{i,j,B}^{n} \mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a} - \lambda_i, \\
z_{i,j}^{n} &\equiv \frac{s_{i,j,F}^{n} \mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a}
\end{aligned}$$

# Boundary conditions

- State constraint $a \geq 0 \longrightarrow s^n_{i,1,B} = 0 \longrightarrow x^n_{i,1} = 0$.
- State constraint $a \leq a^* \longrightarrow s^n_{i,J,F} = 0 \longrightarrow z^n_{i,J} = 0$.

# Matrix notation

- The HJB is a system of $2J$ linear equations which can be written in matrix notation as:

$$\frac{1}{\Delta}\left(\boldsymbol{v}^{n+1}-\boldsymbol{v}^{n}\right)+\rho\boldsymbol{v}^{n+1}=\boldsymbol{u}^{n}+\boldsymbol{A}^{n}\boldsymbol{v}^{n+1}$$

- This is equivalent to a discrete-time, discrete-space dynamic programming problem $\left(\frac{1}{\Delta}=0\right)$:

$$\boldsymbol{v}=\boldsymbol{u}+\beta\Pi\boldsymbol{v},$$

where $\Pi=\boldsymbol{I}+\frac{1}{(1-\rho)}\boldsymbol{A}$ and $\beta=(1-\rho)$.

# Matrix $\boldsymbol{A}$

- Matrix $\boldsymbol{A}$ is the discrete-space approximation of the infinitesimal generator $\mathcal{A}$.
- Advantage: this is a sparse matrix.

$$\boldsymbol{A}^n = \begin{bmatrix} y^n_{1,1} & z^n_{1,1} & 0 & \cdots & \lambda_1 & 0 & 0 & \cdots & 0 \\ x^n_{1,2} & y^n_{1,2} & z^n_{1,2} & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & x^n_{1,3} & y^n_{1,3} & z^n_{1,3} & \cdots & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & x^n_{1,J} & y^n_{1,J} & 0 & 0 & 0 & 0 & \lambda_1 \\ \lambda_2 & 0 & 0 & \cdots & y^n_{2,1} & z^n_{2,1} & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & x^n_{2,2} & y^n_{2,2} & z^n_{2,2} & 0 & \cdots \\ 0 & 0 & \lambda_2 & \cdots & 0 & x^n_{2,3} & y^n_{2,3} & z^n_{2,3} & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \lambda_2 & 0 & \cdots & 0 & x^n_{2,J} & y^n_{2,J} \end{bmatrix}$$

# How to solve it

- Given $\boldsymbol{u}^n = \begin{bmatrix} u(c_{1,1}^n) \\ \vdots \\ u(c_{1,J}^n) \\ u(c_{2,1}^n) \\ \vdots \\ u(c_{2,J}^n) \end{bmatrix}, \quad \boldsymbol{v}^{n+1} = \begin{bmatrix} v_{1,1}^{n+1} \\ \vdots \\ v_{1,J}^{n+1} \\ v_{2,1}^{n+1} \\ \vdots \\ v_{2,J}^{n+1} \end{bmatrix},$

  the system can in turn be written as:

$$\boldsymbol{B}^n \boldsymbol{v}^{n+1} = \boldsymbol{b}^n, \qquad \boldsymbol{B}^n = \left(\frac{1}{\Delta} + \rho\right)\boldsymbol{I} - \boldsymbol{A}^n, \qquad \boldsymbol{b}^n = \boldsymbol{u}^n + \frac{1}{\Delta}\boldsymbol{v}^n$$

# The algorithm

1. Begin with an initial guess $v_{i,j}^0 = \frac{u(z_i + ra_j)}{\rho}$.

2. Set $n = 0$.

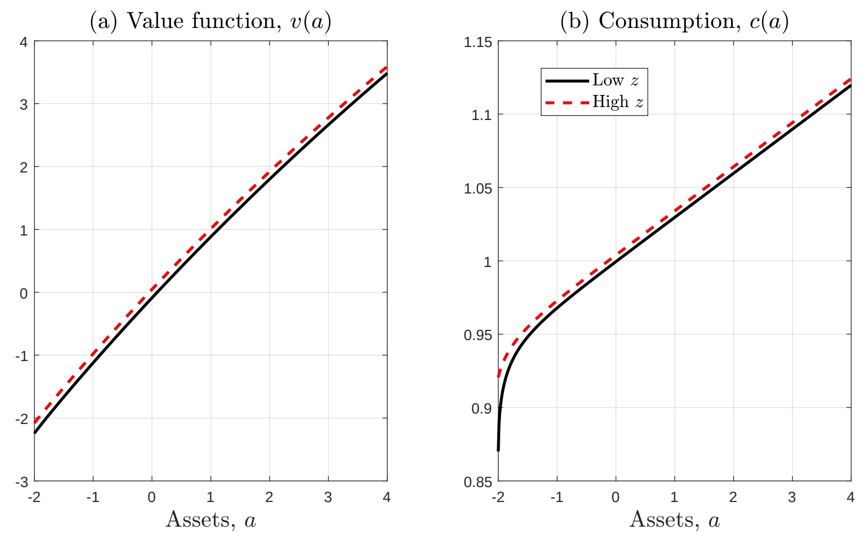
3. Then:

   3.1 Policy update: Compute $\partial_F v_{i.j}^n$, $\partial_B v_{i.j}^n$, and $c_{i,j}^n$.

   3.2 Value update: Compute $v_{i,j}^{n+1}$ solving the linear system of equations.

   3.3 Check: If $v_{i,j}^{n+1}$ is close enough to $v_{i,j}^n$, stop. If not, set $n := n + 1$ and go to 1.

# Results



(a) Value function, $v(a)$
4
3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3
4
Assets, $a$
(b) Consumption, $c(a)$
Low $z$
High $z$
1.15
1.1
1.05
1
0.95
0.9
0.85
-2
-1
0
1
2
3
4
Assets, $a$

# The case with diffusions

- Assume now that labor productivity evolves according to a Ornstein–Uhlenbeck process:

$$dz_t = \theta(\hat{z} - z_t)dt + \sigma dB_t,$$

on a bounded interval $[\underline{z}, \bar{z}]$ with $\underline{z} \geq 0$, where $B_t$ is a Brownian motion.

- The HJB is now:

$$\rho V(a,z) = \max_{c \geq 0} u(c) + s(a,z,c)\frac{\partial V}{\partial a} + \theta(\hat{z} - z)\frac{\partial V}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2}$$

# The new grid

- We approximate the value function $V(a, z)$ on a finite grid with steps $\Delta a$ and $\Delta z$ : $a \in \{a_1, ..., a_I\}$, $z \in \{z_1, ..., z_J\}$.

- We use now the notation $V_{i,j} := V(a_i, z_j)$, $i = 1, ..., I$; $j = 1, ..., J$.
- It does not matter if we consider forward or backward for the first derivative with respect to the exogenous state.

- Use central for the second derivative:

$$\frac{\partial V(a_i, z_j)}{\partial z} \approx \partial_z V_{i,j} := \frac{V_{i,j+1} - V_{i,j}}{\Delta z},$$
$$\frac{\partial^2 V(a_i, z_j)}{\partial z^2} \approx \partial_{zz} V_{i,j} := \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{(\Delta z)^2}$$

# HJB approximation

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} + \rho V_{i,j}^{n+1} = u(c_{i,j}^{n}) + V_{i-1,j}^{n+1}\varrho_{i,j} + V_{i,j}^{n+1}\beta_{i,j} + V_{i+1,j}^{n+1}\chi_{i,j} + V_{i,j-1}^{n+1}\xi + V_{i,j+1}^{n+1}\varsigma_{j},$$

$$\begin{aligned}
\varrho_{i,j} &= -\frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a}, \\
\beta_{i,j} &= -\frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a} + \frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a} - \frac{\theta(\hat{z} - z_j)}{\Delta z} - \frac{\sigma^2}{(\Delta z)^2}, \\
\chi_{i,j} &= \frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a}, \\
\xi &= \frac{\sigma^2}{2(\Delta z)^2}, \\
\varsigma_{j} &= \frac{\sigma^2}{2(\Delta z)^2} + \frac{\theta(\hat{z} - z_j)}{\Delta z}
\end{aligned}$$

# Boundary conditions

- The boundary conditions with respect to $z$ are:

$$\frac{\partial V(a,\underline{z})}{\partial z} = \frac{\partial V(a,\bar{z})}{\partial z} = 0,$$

as the process is reflected.

- At the boundaries in the $j$ dimension, the HJB becomes:

$$\begin{aligned}
\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} + \rho V_{i,j}^{n+1} &= u(c_{i,1}^{n}) + V_{i-1,j}^{n+1}\varrho_{i,1} + V_{i,1}^{n+1}\left(\beta_{i,1} + \xi\right) + V_{i+1,1}^{n+1}\chi_{i,1} + V_{i,2}^{n+1}\varsigma_{1}, \\
\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} + \rho V_{i,j}^{n+1} &= u(c_{i,J}^{n}) + V_{i-1,J}^{n+1}\varrho_{i,J} + V_{i,J}^{n+1}\left(\beta_{i,J} + \varsigma_{J}\right) + V_{i+1,J}^{n+1}\chi_{i,J} + V_{i,J-1}^{n+1}\xi_{J}
\end{aligned}$$

# The problem

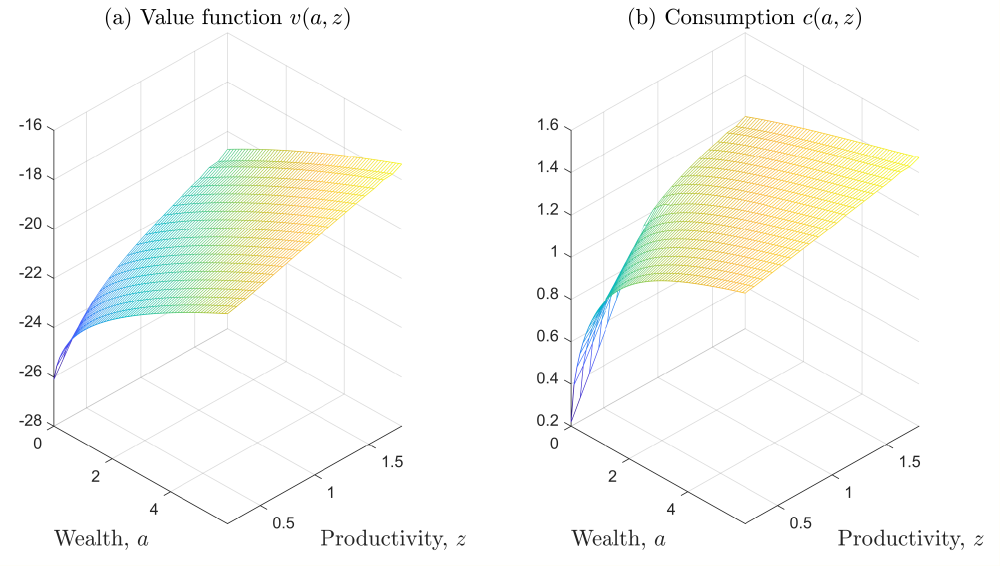
- In matrix notation as:

$$\frac{\boldsymbol{V}^{n+1}-\boldsymbol{V}^{n}}{\Delta}+\rho \boldsymbol{V}^{n+1}=\boldsymbol{u}^{n}+\boldsymbol{A}^{n}\boldsymbol{V}^{n+1},$$

where (sparsity again):

$$\boldsymbol{A}^n = \begin{bmatrix}
\beta_{1,1}+\xi & \chi_{1,1} & 0 & \cdots & 0 & \varsigma_1 & 0 & 0 & \cdots & 0 \\
\varrho_{2,1} & \beta_{2,1}+\xi & \chi_{2,1} & 0 & \cdots & 0 & \varsigma_1 & 0 & \cdots & 0 \\
0 & \varrho_{3,1} & \beta_{3,1}+\xi & \chi_{3,1} & 0 & \cdots & 0 & \varsigma_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \varrho_{I,1} & \beta_{I,1}+\xi & \chi_{I,1} & 0 & 0 & \cdots & 0 \\
\xi & 0 & \cdots & 0 & \varrho_{1,2} & \beta_{1,2} & \chi_{1,2} & 0 & \cdots & 0 \\
0 & \xi & \cdots & 0 & 0 & \varrho_{2,2} & \beta_{2,2} & \chi_{2,2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \varrho_{I-1,J} & \beta_{I-1,J}+\varsigma_J & \chi_{I-1,J} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \varrho_{I,J} & \beta_{I,I}+\varsigma_J
\end{bmatrix}$$

# Results



(a) Value function $v(a, z)$
-16
-18
-20
-22
-24
-26
-28
0
2
4
Wealth, $a$
0.5
1
1.5
Productivity, $z$
(b) Consumption $c(a, z)$
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
2
4
Wealth, $a$
0.5
1
1.5
Productivity, $z$

# Why does the finite difference method work?

- The finite difference method converges to the viscosity solution of the HJB as long as it satisfies three properties:
  1. Monotonicity.
  2. Stability.
  3. Consistency.
- The proposed method does satisfy them (proof too long, check Fleming and Soner, 2006).