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On the power of Dickey–Fuller tests against fractional alternatives *

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We examine the properties of Dickey–Fuller unit root tests under fractionally-integrated alternatives and find that these tests have quite low power.

1. Fractional integration versus unit roots

Nelson and Plosser (1982) and many other studies have failed to reject the null hypothesis of a single autoregressive unit root for many macroeconomic time series. Fractionally-integrated models have received recent attention because of their ability to provide a natural and flexible characterization of persistent processes, while nesting the unit root hypothesis as a special, and potentially restrictive, case. Our analysis will consider whether common unit root tests when applied to fractionally-integrated data are able reject the unit root null hypothesis.

The general ARFIMA (henceforth, AutoRegressive Fractionally-Integrated Moving Average) model can be written as

$$\Phi(L)(1-L)^d X_t = \Theta(L)\epsilon_t, \tag{1}$$

where d is any real number. [As usual, $\Theta(L)$ and $\Theta(L)$ are the autoregressive and moving-average polynomials in the lag operator (with roots outside the unit circle), and $\epsilon_t \sim (0, \sigma_\epsilon^2)$.] Operationally, the fractional difference operator $(1-L)^d$ can be expressed by a binomial expansion. The process is stationary and invertible if $d \in (-1/2, 1/2)$, and one can always transform a fractionally-integrated series of higher order ($d > 1/2$) into this range by taking a suitable number of integer differences. [1]

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[1] For a proof, see Hosking (1981). More extensive discussion and references may be found in Diebold and Rudebusch (1989a, 1989b) and Diebold and Nerlove (1990).

with non-zero d, the ARFIMA model displays 'long memory', that is, substantial dependence between observations τ periods apart, even for large τ. Indeed, as τ increases, the ARFIMA autocorrelations decline at a very slow hyperbolic rate; in contrast, ARMA autocorrelations decline at a quite rapid geometric rate.

In our earlier work [Diebold and Rudebusch (1989a)], we obtained a point estimate of d equal to 0.68 for the level of U.S. postwar quarterly real GNP per capita (equivalently, the fractional order for the first difference of GNP is equal to -0.32). Thus, the level of real output appears to have long memory, in the sense that the d estimate is significantly greater than zero. However, the long memory may not be associated with a unit autoregressive root; notably the d estimate is substantially less than one, and the unit root hypothesis can be rejected at conventional significance levels. More generally, regardless of the significance of the difference of the d estimate from unity, the provocative point estimates suggest the following thought experiment: If a series *were* best characterized by a fractional process, would a researcher be able to detect that fact by rejecting the hypothesis of a unit root using conventional Dickey–Fuller [Fuller (1976)] tests? With this motivation, we proceed to examine the power of Dickey-Fuller tests when the true data-generating process is fractionally integrated.

2. Analytical conjectures about power

Consider a random walk

$$(1-L)^d X_t = \epsilon_t, \qquad d=1, \tag{2}$$

with white noise innovation, $\epsilon_t \sim (0, \sigma_\epsilon^2)$. In a classic paper, White (1958) characterized the distribution of the least-squares estimator $\hat{\beta}$ in the first-order autoregressive [henceforth, AR(1)] model,

$$X_t = \beta X_{t-1} + \epsilon_t, \tag{3}$$

under the unit-root null hypothesis that $\beta = 1$, i.e., when the true data-generating process is (2).

Consider the natural generalization of (2) to the case of a pure fractionally-integrated process,

$$(1-L)^d X_t = \epsilon_t, \qquad 1/2 < d < 3/2. \tag{4}$$

Or equivalently,

$$(1-L)X_t = u_t, \quad \text{with} \quad (1-L)^{\tilde{d}} u_t = \epsilon_t, \tag{5}$$

where $\tilde{d} = d - 1$. Thus, $d \in (1/2, 3/2)$ corresponds to $\tilde{d} \in (-1/2, 1/2)$, and the natural analog of the White regression (3) is

$$X_t = \beta X_{t-1} + u_t, \quad \text{with} \quad (1-L)^{\tilde{d}} u_t = \epsilon_t, \tag{6}$$

where the unit root null hypothesis is again $\beta = 1$; however, although the regression innovation u_t is stationary and invertible, it is not white but rather fractionally integrated. White's results emerge for the special case of $\tilde{d} = 0$ ($d = 1$). In an important paper, Sowell (1990) shows that the asymptotic distribution theory depends crucially on $\tilde{d}$; in particular, the speed of convergence to the asymptotic distribution varies with $\tilde{d}$. As illustrated in fig. 1, if $\tilde{d} = 0$, the well-known [e.g., Fuller (1976)] result

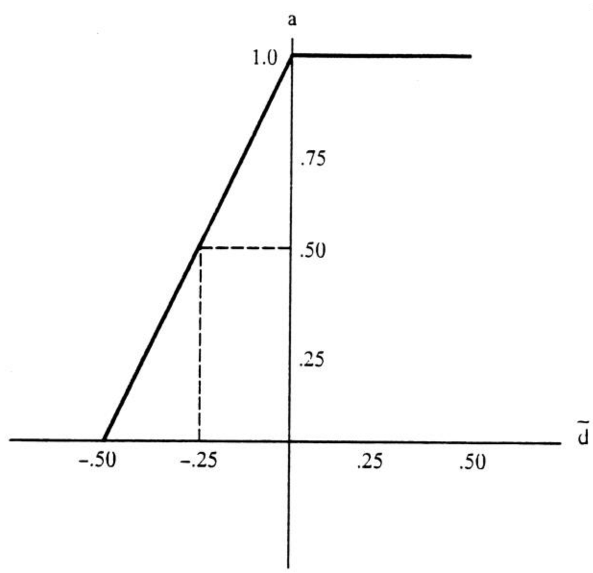

Fig. 1. Convergence speed, T^{-a}.

about convergence speed that $(\hat{\beta}-1)=O(T^{-1})$ holds (where T is sample size), and this result *continues* to hold for $\tilde{d}\in(0, 1/2)$. Conversely, if $\tilde{d}\in(-1/2, 0)$, then $(\hat{\beta}-1)=O(T^{-1-2\tilde{d}})$. Thus, convergence is faster or slower than $O(T^{1/2})$ as $\tilde{d}$ is greater or less than $-1/4$.

Sowell (1990) also shows that the asymptotic fractional unit root distribution theory may be severely misleading in all but *very* large samples. This is because the distribution of $\hat{\beta}$ depends on two underlying random variables, the convergence of one of which to its asymptotic distribution is very slow for plausible d values. The resulting finite-sample similarity of the integer unit root distribution and the fractional unit root distribution – in spite of their sharp asymptotic differences – leads Sowell to conjecture that conventional unit root tests may have low power against fractional alternatives.

3. A Monte Carlo experiment

We consider two data-generating processes: fractional noise,

$$(1-L)^{d}X_t=\epsilon_t, \tag{7}$$

with values of d equal to 0.3, 0.45, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, and 1.3; and the AR(1),

$$(1-bL)X_t=\epsilon_t, \tag{8}$$

with values of b equal to 0.7, 0.8, 0.9, 0.95, 0.98, 1.0, 1.02, 1.05, and 1.1. For both processes, parameter values equal to one correspond to the unit root null hypothesis. The innovation variance

Table 1
Monte-Carlo power of unit root tests ARFIMA(0, d, 0) alternative. [a]

	d									
	0.3	0.45	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3
$T=50$										
τ	1.00	0.65	0.51	0.32	0.17	0.07	0.05	0.12	0.27	0.46
ρ	1.00	0.62	0.50	0.31	0.16	0.07	0.05	0.07	0.12	0.18
$T=100$										
τ	1.00	0.88	0.71	0.48	0.25	0.09	0.04	0.14	0.33	0.54
ρ	1.00	0.87	0.71	0.48	0.25	0.09	0.04	0.08	0.15	0.21
$T=250$										
τ	1.00	0.99	0.90	0.70	0.39	0.13	0.05	0.17	0.40	0.62
ρ	1.00	0.99	0.90	0.70	0.39	0.14	0.05	0.09	0.18	0.25

[a] All tests are two-sided and at the 5% level. Standard errors of the estimates vary, but all are less than 0.0071.

σ_ϵ^2 is held fixed at 1.0, and sample sizes (denoted by T) of 50, 100, and 250 are examined, with 5000 replications (denoted by N) performed for each sample size.

Sample for the ARFIMA(0, d, 0) process (7) with $d=0.3$ and $d=0.45$ (stationary parameter configurations) are formed a follows. First, a vector, v, consisting of $TN(0, 1)$ deviates is generated using IMSL subroutine GGNML. Then the desired $T\times T$ data covariance matrix (Σ) is constructed. This is simply the Toeplitz matrix formed from the autocovariances, which are given by

$$\gamma_X(\tau)=\left[(\Gamma(1-2d)\Gamma(d+\tau))/(\Gamma(d)\Gamma(1-d)\Gamma(1-d+\tau))\right]\sigma_\epsilon^2, \tag{9}$$

where $\Gamma(\cdot)$ is the gamma function [see Hosking (1981)]. We next obtain the Choleski factorization of

Table 2
Monte-Carlo power of unit root tests AR(1) alternative. [a]

	b								
	0.7	0.8	0.9	0.95	0.98	1.0	1.02	1.05	1.1
$T=50$									
τ	0.90	0.58	0.18	0.08	0.04	0.05	0.21	0.70	0.97
ρ	0.89	0.55	0.17	0.07	0.04	0.05	0.12	0.70	0.97
$T=100$									
τ	1.00	0.99	0.58	0.18	0.06	0.05	0.58	0.97	1.00
ρ	1.00	0.99	0.56	0.17	0.06	0.05	0.52	0.98	1.00
$T=250$									
τ	1.00	1.00	1.00	0.75	0.17	0.05	0.97	1.00	1.00
ρ	1.00	1.00	1.00	0.75	0.17	0.05	0.98	1.00	1.00

[a] All tests are two-sided and at the 5% level. Standard errors of the estimates vary, but all are less than 0.0071.

Σ, $\Sigma = PP'$, where P is lower triangular, using IMSL subroutine LUDECP. Finally the sample, x, is generated as $x = Pv$; clearly, $\text{cov}(x) = PP' = \Sigma$. Construction of x in this way eliminates dependence on pre-sample startup values, which can be particularly problematic with long-memory models. For the non-stationary parameter configurations $d = 0.6$, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2 and 1.3, we generate fractional noise with a parameter $\tilde{d} = d - 1$, as above, which yields observations on $(1 - L)x_t$. Then, taking $x_0 = 0$, we construct the sample $\{x_1, \ldots, x_T\}$ as cumulated sums.

For the stationary realizations of the AR(1) process (8), we generate N(0, 1) variates and again transform by the Choleski factor of the covariance matrix. The Toeplitz covariance matrix Σ is formed from the autocovariances, which are given by

$$\gamma_X(\tau) = \left[\sigma_\epsilon^2/(1 - b^2)\right] b^\tau. \tag{10}$$

Data for the nonstationary parameter configurations corresponding to $b = 1.0$, 1.02, 1.05 and 1.1 are constructed directly from the recursion (8), using a startup value of $x_0 = 0$.

For each Monte Carlo replication $i = 1, \ldots, N$, we apply two unit root tests based on the regression,

$$X_t = \delta X_{t-1} + v_t. \tag{11}$$

We report the Dickey–Fuller t-statistic τ (for the null hypothesis that $\delta = 1$) and the Dickey-Fuller 'normalized bias' statistic ρ, which is equal to $T(\hat{\delta} - 1)$. All tests are two-sided and are performed at the 5 percent level using critical values from Fuller (1976).

The power estimates are computed as relative rejection frequencies and are presented in table 1 for ARFIMA(0, d, 0) alternatives and in table 2 for AR(1) alternatives. Some similarities are apparent for both the ARFIMA and AR alternatives. First, for a fixed parameter value d or b, power increases monotonically with sample size, T. Second, for a fixed sample size, power increases monotonically with both $|d - 1|$ and $|b - 1|$. Third, for fixed sample size, power is always asymmetric around the respective null hypotheses, $d = 1$ and $b = 1$, rising more quickly for parameter values greater than unity. Finally, for fixed sample size, the power of the τ and ρ tests is always approximately equal for $d < 1$ or $b < 1$, whereas divergences occur for other parameter configurations. Of great interest, however, is the *difference* across the alternatives: namely, power grows much more slowly with divergence of d from 1 than with divergence of b from 1.

4. Conclusion

Schwert (1989) has evaluated the ability of unit root tests to correctly determine the presence of an autoregressive unit root in a mixed ARIMA(p, 1, q) process. His results suggest that the empirical size of these tests when applied to a mixed process is often greater than their nominal size. For example, in the presence of a first-order moving-average term, the unit root tests often lead to the incorrect conclusion that the time series under investigation is stationary. We focus on power rather than size, but our results lead to the complementary conclusion that the power of the conventional unit root tests against fractionally-integrated alternatives is quite low. De Jong, Nankervis, Savin, and Whiteman (1989) and Rudebusch (1990) reach similar conclusions after examining the power functions of unit root tests against short-memory alternatives with dominant roots close (in the Euclidian sense) to unity. Here, we have effectively extended and amplified that argument by showing that fractional alternatives far (in the Euclidian sense) from the null are nevertheless close in terms of induced persistence, which translates into low power. Thus, the tests proposed by Dickey and Fuller can in the face of fractionally-integrated processes lead to an incorrect conclusion that a time series has a unit root. We believe that our work, along with the other recent size and power studies, provide a joint condemnation of the widespread mechanical application of unit root tests. A more appropriate testing procedure would carefully consider the correct specification of the complete stochastic process before drawing conclusions about the presence of a unit root.

References

DeJong, D.N., J.C. Nankervis, N.E. Savin and C.H. Whiteman, 1989, Integration versus trend stationarity in macroeconomic time series, Working paper 89-31 (University of Iowa, Iowa City, IA).

Diebold, F.X. and M. Nerlove, 1990, Unit roots in economic time series: A selective survey, in: Thomas B. Fomby and George F. Rhodes, eds., Advances in econometrics: Co-Integration, spurious regressions, and unit roots (JAI Press, Greenwich, CT) 3–69.

Diebold, F.X. and G.D. Rudebusch, 1989a, Long memory and persistence in aggregate output, Journal of Monetary Economics 24, 189–209.

Diebold, F.X. and G.D. Rudebusch, 1989b, Is consumption too smooth? Long memory and the Deaton paradox, FED series no. 57 (Federal Reserve Board, Washington, DC) forthcoming in Review of Economics and Statistics.

Fuller, W.A., 1976, Introduction to statistical time series (Wiley, New York).

Hosking, J.R.M., 1981, Fractional differencing, Biometrika 68, 165–176.

Nelson, C.R. and C.I. Plosser, 1982, Trends and random walks in macroeconomic time series: Some evidence and implications, Journal of Monetary Economics 10, 139–162.

Rudebusch, G.D., 1990, Trends and random walks in macroeconomic time series: A re-examination, Economic Activity Working Paper 105 (Federal Reserve Board, Washington, DC).

Schwert, G.W., 1989, Tests for unit roots: A Monte Carlo investigation, Journal of Business and Economic Statistics 7, 147–159.

Sowell, F.B., 1990, Fractional unit root distribution, Econometrica 58, 495–506.

White, J.S., 1958, The limiting distribution of the serial correlation coefficient in the explosive case, Annals of Mathematical Statistics 29, 1188–1197.