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TESTING FOR BUBBLES, REFLECTING BARRIERS AND OTHER ANOMALIES*

Francis X. DIEBOLD

Board of Governors of the Federal Reserve System, Washington, DC 20551, USA

1. Introduction

Tests of long-run dependence based on the variance-time function (the variances of k-aggregates or k-differences, as a function of k) have a long history in economics and finance. Earlier literature includes Working (1949), Osborne (1959), Sprenkle (1961), many of the contributions in Cootner (1964), as well as Poole (1967), *inter alia*. This literature, mostly concerned with long-run dependence in asset returns, makes use only of *point* estimates of the variance-time function. As Poole (1967) states:

> Systematic differences in these (variance) estimates as a function of the differencing interval may indicate the nature of any serial dependence...(but) there is no statistical test for testing the significance of the differences in the estimates of the one-period variances since the samples from which the estimates are computed are not independent.

More recent advances have been made by Young (1971), Mandelbrot (1972), Lo and MacKinlay (1987), Cochrane (1987a, b), and Poterba and Summers (1987), among others. Young uses an orthogonalizing transformation to reduce the dependence problem, while Mandelbrot advocates tests based on the rescaled range. Lo and MacKinlay provide a direct solution to Poole's dependence dilemma via application of a central limit theorem, which yields a normal asymptotic distribution for the statistics of interest. In independent work, Cochrane (1987a) also establishes the asymptotic distributions by showing that the relevant variance ratios may be expressed as functions of sample autocorrelations, and then applying the well-known asymptotic distribution of the sample autocorrelations. Poterba and Summers take a different approach, using bootstrap methods to obtain finite-sample standard errors.

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Each of the above approaches has associated costs and benefits. The Cochrane and Lo and MacKinlay asymptotic tests are convenient and easy to use, but they may deliver misleading inferences in samples of the size that typically arise in macroeconomics. Conversely, the bootstrap methods of Poterba and Summers are robust to non-normal innovations, but they are tedious and non-portable, and their small-sample properties have not been fully investigated. Furthermore, a very large number of bootstrap replications may be necessary for accurate estimation of tail probabilities.

In this paper, we fill a gap in the literature by tabulating the null distributions of existing test statistics based on scalar variance ratios. [Extensions of the results to joint tests, joint tests robust to innovation non-normality, and tests based on order statistics are contained in Diebold (1987).] The distributions are non-model-specific (apart from a necessary innovation distribution assumption) and should therefore be useful in applied work. The tests are developed in section 2. Section 3 contains some brief examples, and section 4 concludes the paper.

2. Hypothesis tests based on the variance-time function

Let $\{_1x_t\}_{t=0}^{T}$ denote a time series observed at some frequency f, which corresponds to the unit subscript. Under the normal random-walk hypothesis, we have $_1x_t = {_1x_{t-1}} + {_1\varepsilon_t}$ and $_1\varepsilon_t \sim \mathrm{N}(0, \sigma_1^2)$, so that the 'change' series $\{\Delta_1 x_t\}_{t=1}^{T}$ is white noise, also observed at frequency f, whose variance we denote by σ_1^2. Now consider the same change series observed at half the original frequency (e.g., if monthly differences were originally obtained, consider now the case of bi-monthly observations), whose first-differences we denote $\{\Delta_2 x_\tau\}_{\tau=1}^{F(T/2)}$, where the 'floor' operator F rounds down to the nearest integer. Because $\{\Delta_1 x_t\}$ is a flow variable, any element of $\{\Delta_2 x_\tau\}$ is the sum of two contiguous elements of $\{\Delta_1 x_t\}$. Thus, for example,

$$\Delta_2 x_1 = \Delta_1 x_1 + \Delta_1 x_2, \qquad \Delta_2 x_2 = \Delta_1 x_3 + \Delta_1 x_4, \tag{1}$$

and so forth. We refer to this newly created series as '2-aggregated'. Higher-ordered (i.e., k-aggregated, $k > 2$) aggregates are constructed in an analogous fashion. In this paper we consider exact finite-sample tests of random-walk behavior based on these k-aggregates.

Denote the variances (under the null) of the temporal k-aggregates $\{\Delta_1 x_.\}, \{\Delta_2 x_.\}, \ldots, \{\Delta_k x_.\}, \ldots, \{\Delta_K x_.\}$ by $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2, \ldots, \sigma_K^2$, respectively. Then, under the null,

$$\sigma_k^2 = k\sigma_1^2, \qquad k = 1, \ldots, K, \tag{2}$$

or

$$2\sigma_1^2/\sigma_2^2 = 3\sigma_1^2/\sigma_3^2 = \cdots = K\sigma_1^2/\sigma_K^2 = 1. \tag{3}$$

This fact is easily exploited to obtain formal hypothesis tests for random walk behavior.

Suppose (incorrectly) that the samples $\{\Delta_1 x_t\}_{t=1}^{T}$ and $\{\Delta_k x_{\tau_k}\}_{\tau_k=1}^{T_k}$ were independent, where $T_k \doteq F(T/k)$. Then a standard F-test would be appropriate since, under the null,

$$\hat{\sigma}_1^2 = \left(\sigma_1^2/T\right)\chi_T^2 \quad \text{and} \quad \hat{\sigma}_k^2 = \left(k\sigma_1^2/T_k\right)\chi_{T_k}^2, \tag{4}$$

where χ_j^2 denotes a chi-square random variable with j degrees of freedom and

$$\hat{\sigma}_k^2 \doteq 1/T_k \sum_{\tau_k=1}^{T_k} \Delta_k x_{\tau_k}^2, \qquad k = 1, \ldots, K. \tag{5}$$

Thus, the variance-ratio statistics $R(k)$, $k = 1, \ldots, K$, would have central F-distributions:

$$R(k) \doteq k\hat{\sigma}_1^2/\hat{\sigma}_k^2 = \left(\chi_T^2/T\right)/\left(\chi_{T_k}^2/T_k\right) \sim F_{T,T_k}. \tag{6}$$

Unfortunately, due to the fact that $\{\Delta_k x_{\tau_k}\}$ is a temporal aggregate of $\{\Delta_1 x_t\}$, the χ^2 random variables in the above formulae are not independent; hence, the variance ratio test statistics $R(k)$ do not have F-distributions. The fractiles of the $R(k)$ are easily calculated by Monte Carlo, however, and they are reported in Diebold (1987) for a range of primary sample sizes (T) and aggregation intervals (k).

We now proceed to consider some related test statistics of interest. First, for many applications we may want to relax the maintained zero-drift assumption. We denote the corresponding test statistics with estimated drift by

$$R_d(k) = k\hat{\sigma}_{1,d}^2/\hat{\sigma}_{k,d}^2, \tag{7}$$

where

$$\hat{\sigma}_{1,d}^2 = 1/T \sum_{t=1}^{T} \left(\Delta_1 x_t - \hat{\mu}\right)^2, \tag{8}$$

$$\hat{\sigma}_{k,d}^2 = 1/T_k \sum_{\tau_k=1}^{T_k} \left(\Delta_k x_{\tau_k} - k\hat{\mu}\right)^2, \tag{9}$$

$$\hat{\mu} = 1/T \sum_{t=1}^{T} \Delta_1 x_t. \tag{10}$$

Second, note that all of the above tests, which make use of the variances of k-aggregates of the first-differenced series, may equivalently be cast in terms of variances of non-overlapping k-differences of the level series. Thus, regardless of whether or not a drift is estimated, the power of the above tests may be increased by allowing the k-differences to overlap. We therefore consider the following:

$$R_{\mathrm{o}}(k) = k\hat{\sigma}_1^2/\hat{\sigma}_{k,\mathrm{o}}^2 \qquad \text{(zero-drift case)}, \tag{11}$$

$$R_{\mathrm{do}}(k) = k\hat{\sigma}_1^2/\hat{\sigma}_{k,\mathrm{do}}^2 \qquad \text{(estimated-drift case)}, \tag{12}$$

where

$$\hat{\sigma}_{k,\mathrm{o}}^2 = 1/(T-k+1)\sum_{t=k}^{T}(x_t - x_{t-k})^2, \tag{13}$$

$$\hat{\sigma}_{k,\mathrm{do}}^2 = 1/(T-k+1)\sum_{t=k}^{T}(x_t - x_{t-k} - k\hat{\mu})^2, \tag{14}$$

and the subscript 'o' denotes overlap. The R_{do} statistics are the most useful in practice, since we are usually not justified in assuming a zero drift, and power is increased by the use of overlapping differences. Fractiles of $R_{\mathrm{do}}(k)$, for various T and k values, appear in tables 1–5, which are based upon 25,000 replications. The fractiles are read across the first rows, and the sample sizes (T) are read down the first columns.

3. Examples: Reflecting barriers and bubbles

The random walk with reflecting barriers is a model of great importance in the physical sciences (e.g., Brownian motion in a closed container), and it appears in economics and finance as well. For example, many models of the

Table 1

Fractiles of $R_{\mathrm{do}}(2)$.

Fractiles:	0.005	0.025	0.050	0.100	0.900	0.950	0.975	0.995
64	0.765	0.813	0.841	0.873	1.213	1.282	1.342	1.493
96	0.795	0.839	0.864	0.892	1.164	1.216	1.267	1.369
128	0.819	0.860	0.880	0.904	1.137	1.179	1.219	1.302
160	0.834	0.870	0.890	0.912	1.118	1.155	1.189	1.270
192	0.846	0.878	0.897	0.919	1.107	1.140	1.171	1.237
256	0.861	0.892	0.909	0.928	1.091	1.119	1.144	1.194
512	0.900	0.922	0.934	0.948	1.061	1.080	1.096	1.129
1024	0.926	0.943	0.952	0.962	1.042	1.055	1.066	1.088
2048	0.946	0.959	0.965	0.973	1.030	1.038	1.046	1.061
4096	0.962	0.971	0.975	0.980	1.021	1.027	1.032	1.042

Table 2

Fractiles of $R_{do}(4)$.

Fractiles:	0.005	0.025	0.050	0.100	0.900	0.950	0.975	0.995
64	0.612	0.687	0.733	0.795	1.485	1.648	1.802	2.144
96	0.660	0.728	0.769	0.821	1.355	1.473	1.587	1.819
128	0.696	0.758	0.792	0.836	1.292	1.380	1.469	1.667
160	0.718	0.774	0.808	0.849	1.249	1.324	1.400	1.570
192	0.733	0.788	0.820	0.861	1.226	1.293	1.359	1.495
256	0.759	0.815	0.841	0.876	1.184	1.242	1.295	1.410
512	0.819	0.862	0.882	0.908	1.123	1.160	1.194	1.256
1024	0.866	0.896	0.913	0.932	1.082	1.106	1.126	1.175
2048	0.904	0.926	0.937	0.951	1.057	1.073	1.088	1.117
4096	0.929	0.945	0.954	0.964	1.039	1.050	1.060	1.081

Table 3

Fractiles of $R_{do}(8)$.

Fractiles:	0.005	0.025	0.050	0.100	0.900	0.950	0.975	0.995
64	0.474	0.584	0.647	0.735	2.042	2.394	2.719	3.535
96	0.533	0.628	0.684	0.758	1.708	1.943	2.159	2.700
128	0.572	0.660	0.708	0.777	1.564	1.745	1.912	2.317
160	0.601	0.681	0.726	0.788	1.468	1.621	1.760	2.090
192	0.627	0.701	0.743	0.802	1.408	1.533	1.654	1.958
256	0.660	0.731	0.773	0.823	1.333	1.434	1.530	1.744
512	0.739	0.796	0.826	0.864	1.210	1.277	1.335	1.449
1024	0.802	0.845	0.868	0.898	1.140	1.178	1.212	1.287
2048	0.855	0.887	0.904	0.925	1.094	1.122	1.146	1.196
4096	0.890	0.915	0.928	0.944	1.064	1.082	1.099	1.132

Table 4

Fractiles of $R_{do}(16)$.

Fractiles:	0.005	0.025	0.050	0.100	0.900	0.950	0.975	0.995
64	0.375	0.511	0.597	0.726	3.443	4.218	5.017	7.104
96	0.424	0.540	0.613	0.716	2.516	3.018	3.537	4.690
128	0.459	0.564	0.631	0.725	2.121	2.500	2.862	3.740
160	0.487	0.589	0.654	0.733	1.914	2.209	2.482	3.131
192	0.518	0.615	0.669	0.747	1.767	2.011	2.256	2.759
256	0.557	0.645	0.699	0.766	1.598	1.786	1.980	2.380
512	0.641	0.716	0.760	0.815	1.354	1.462	1.564	1.777
1024	0.722	0.782	0.816	0.857	1.226	1.287	1.348	1.471
2048	0.795	0.839	0.863	0.892	1.148	1.191	1.229	1.304
4096	0.845	0.878	0.898	0.920	1.100	1.128	1.152	1.203

Table 5

Fractiles of $R_{\mathrm{do}}(32)$.

Fractiles:	0.005	0.025	0.050	0.100	0.900	0.950	0.975	0.995
64	0.468	0.644	0.768	0.964	7.259	9.380	11.755	16.862
96	0.364	0.501	0.605	0.758	4.609	5.891	7.172	10.090
128	0.371	0.500	0.587	0.716	3.506	4.331	5.216	7.373
160	0.387	0.510	0.596	0.709	2.923	3.575	4.294	5.862
192	0.413	0.528	0.601	0.714	2.551	3.076	3.634	4.834
256	0.449	0.558	0.626	0.721	2.152	2.543	2.914	3.867
512	0.545	0.633	0.689	0.762	1.621	1.814	2.005	2.473
1024	0.639	0.713	0.755	0.811	1.372	1.480	1.577	1.791
2048	0.727	0.779	0.812	0.855	1.233	1.302	1.360	1.498
4096	0.788	0.834	0.859	0.889	1.155	1.199	1.238	1.316

stock market lead to random-walk returns subject to reflecting barriers [e.g., Cootner (1964)]. More recent work in international finance, such as Pippinger (1986), shows that both older arbitrage-based purchasing power parity models of real exchange rate behavior and the recent efficient-markets real exchange rate model of Roll (1979) imply random-walk equilibrium real exchange rate behavior, subject to reflecting barriers. Thus, real exchange rates may wander with martingale-like persistence, but they never deviate 'too far' from their equilibrium values.

Consider the simple random walk:

$$x_t = x_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon^2). \tag{15}$$

As discussed extensively above, the variance of k-differences (whether overlapping or non-overlapping) is linear in k:

$$\mathrm{var}(x_t - x_{t-k}) = \mathrm{var}(\varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_{t-k}) = k\sigma_\varepsilon^2. \tag{16}$$

Suppose now that we introduce a symmetric reflecting barrier:

$$\begin{aligned} x_t &= x_{t-1} + \varepsilon_t \quad &&\text{if} \quad \mathrm{abs}(x_{t-1} + \varepsilon_t) \le B, \\ x_t &= B &&\text{if} \quad (x_{t-1} + \varepsilon_t) > B, \\ x_t &= -B &&\text{if} \quad (x_{t-1} + \varepsilon_t) < -B. \end{aligned} \tag{17}$$

Then, if the reflecting barrier is binding, the variance of the k-differences grows slower than linearly, i.e., $\mathrm{var}(x_t - x_{t-k}) < k\sigma_\varepsilon^2$. Furthermore, the variance of k-differences is bounded above as $k \to \infty$.

Suppose instead that we have a bubble process:

$$\begin{aligned} x_t &= 1.02x_{t-1} + \varepsilon_t, \qquad && \text{w.p.} = 0.95, \\ x_t &= \varepsilon_t, && \text{w.p.} = 0.05. \end{aligned} \tag{18}$$

Such a model arises naturally in linear rational expectations models of asset markets, such as Blanchard and Watson (1982), in which 'no arbitrage' conditions yield the forward solution $p_t^* = f_t + b_t$, where $f_t = \sum_{i=0}^{\infty}\theta^{i+1}\mathrm{E}(x_{t+i}/\Omega_t)$ is the fundamental solution [abs$(\theta) < 1$] and b_t is the bubble solution. Under rational expectations, it must be the case that $\mathrm{E}(b_{t+1}/\Omega_t) = \theta^{-1}b_t$. This restriction is satisfied by

$$\begin{aligned} b_t &= (\pi\theta)^{-1}b_{t-1} + \varepsilon_t, \qquad && \text{w.p. } \pi, \\ b_t &= \varepsilon_t, && \text{w.p. } (1-\pi), \end{aligned} \tag{19}$$

where $\mathrm{E}(\varepsilon_t/\Omega_{t-1}) = 0$. Periodic bursting of the bubble (given a long enough sample) leads to asymptotic flattening of the variance-time function.

4. Concluding remarks

We have developed exact finite-sample tests for random-walk behavior based upon the variance-time function. The fact that tests based on the variance-time function are not directed against any particular alternative may actually enhance their usefulness as a diagnostic tool. Similarly, plots of the variance-time function may serve as a useful graphical aid in the determination of underlying dynamics. In this vein, shapes of the variance-time function under two particular deviations from random-walk behavior were briefly examined.

In future work, we plan to explore the power of tests of dependence based on the variance-time function under alternatives such as those sketched above and the 'long-memory' models studied by Diebold and Rudebusch (1988).

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