
TESTING FOR BUBBLES, REFLECTING BARRIERS AND OTHER ANOMALIES*

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I. Introduction

Tests of long-run dependence based on the variance-time function (the variances of k-aggregates or k-differences, as a function of k) have a long history in economics and finance. Earlier literature includes Working (1949), Osborne (1959), Srenkle (1961), many of the contributions in Cootner (1964), as well as Poole (1967), *inter alia*. This literature, mostly concerned with long-run dependence in asset returns, makes use only of point estimates of the variance-time function. As Poole (1967) states:

Systematic differences in these (variance) estimates as a function of the differencing interval may indicate the nature of any serial dependence...(but) there is no statistical test for testing the significance of the differences in the estimates of the one-period variances since the samples from which the estimates are computed are not independent.

More recent advances have been made by Young (1971), Mandelbrot (1972), Lo and MacKinlay (1987), Cochrane (1987a,b), and Poterba and Summers (1987), among others. Young uses an orthogonalizing transformation to reduce the dependence problem, while Mandelbrot advocates tests based on the rescaled range. Lo and MacKinlay provide a direct solution to Poole’s dependence dilemma via application of a central limit theorem, which yields a normal asymptotic distribution for the statistics of interest. In independent work, Cochrane (1987a) also establishes the asymptotic distributions by showing that the relevant variance ratios may be expressed as functions of sample autocorrelations, and then applying the well-known asymptotic distribution of the sample autocorrelations. Poterba and Summers take a different approach, using bootstrap methods to obtain finite-sample standard errors.

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Each of the above approaches has associated costs and benefits. The
Cochrane and Lo and MacKinlay asymptotic tests are convenient and easy
to use, but they may deliver misleading inferences in samples of the size
that typically arise in macroeconomics. Conversely, the bootstrap methods
of Poterba and Summers are robust to non-normal innovations, but they
are tedious and non-portable, and their small-sample properties have not
been fully investigated. Furthermore, a very large number of bootstrap
replications may be necessary for accurate estimation of tail probabilities.

In this paper, we fill a gap in the literature by tabulating the null
distributions of existing test statistics based on scalar variance ratios. [Exten-
sions of the results to joint tests, joint tests robust to innovation non-normality,
and tests based on order statistics are contained in Diebold (1987).] The
distributions are non-model-specific (apart from a necessary innovation distribution
assumption) and should therefore be useful in applied work. The tests are
developed in section 2. Section 3 contains some brief examples, and section 4
concludes the paper.

2. Hypothesis tests based on the variance-time function

Let \(\{\xi_{t}\}_{t=0}^{T}\) denote a time series observed at some frequency \(f\),
which corresponds to the unit subcript. Under the normal random-walk hypothesis,
we have \(\xi_{t} = \xi_{t-1} + \epsilon_{t}\) and \(\epsilon_{t} \sim N(0, \sigma_{\epsilon}^{2})\),
so that the ‘change’ series \(\{\Delta_{1}\xi_{t}\}_{t=1}^{T}\) is white noise,
also observed at frequency \(f\), whose variance we denote by \(\sigma_{\epsilon}^{2}\).
Now consider the same change series observed at half the
original frequency (e.g., if monthly differences were originally obtained,
consider now the case of bi-monthly observations), whose first-differences we
denote \(\{\Delta_{2}\xi_{t}\}_{t=1}^{T/2}\), where the ‘floor’ operator \(F\) rounds down to the nearest
integer. Because \(\{\Delta_{2}\xi_{t}\}\) is a flow variable, any element of \(\{\Delta_{2}\xi_{t}\}\) is the sum
of two contiguous elements of \(\{\Delta_{1}\xi_{t}\}\). Thus, for example,

\[
\Delta_{2}\xi_{1} = \Delta_{1}\xi_{1} + \Delta_{1}\xi_{2}, \quad \Delta_{2}\xi_{2} = \Delta_{1}\xi_{3} + \Delta_{1}\xi_{4},
\]

(1)

and so forth. We refer to this newly created series as ‘2-aggregated’. Higher-
ordered (i.e., \(k\)-aggregated, \(k > 2\)) aggregates are constructed in an analogous
fashion. In this paper we consider exact finite-sample tests of random-walk
behavior based on these \(k\)-aggregates.

Denote the variances (under the null) of the temporal \(k\)-aggregates
\(\{\Delta_{1}\xi_{t}\}, \{\Delta_{2}\xi_{t}\}, \ldots, \{\Delta_{k}\xi_{t}\}\) by \(\sigma_{1}^{2}, \sigma_{2}^{2}, \ldots, \sigma_{k}^{2}\), respectively.
Then, under the null,

\[
\sigma_{k}^{2} = k\sigma_{1}^{2}, \quad k = 1, \ldots, K.
\]

(2)

or

\[
2\sigma_{1}^{2}/\sigma_{k}^{2} = 3\sigma_{1}^{2}/\sigma_{k}^{2} = \ldots = K\sigma_{1}^{2}/\sigma_{k}^{2} = 1.
\]

(3)

This fact is easily exploited to obtain formal hypothesis tests for random walk
behavior.

Suppose (incorrectly) that the samples \(\{\Delta_{1}\xi_{t}\}_{t=1}^{T/2}\) and
\(\{\Delta_{k}\xi_{t}\}_{t=1}^{T/2}\) were independent, where \(T_{k} = F(T/k)\). Then a standard
\(F\)-test would be appropriate since, under the null,

\[
\hat{\delta}_{k}^{2} = (\sigma_{1}^{2}/T)\chi_{F_{k}}^{2} \quad \text{and} \quad \hat{\delta}_{k}^{2} = (k\sigma_{1}^{2}/T_{k})\chi_{T_{k}^{2}}^{2},
\]

(4)

where \(\chi_{j}^{2}\) denotes a chi-square random variable with \(j\) degrees of freedom and

\[
\hat{\delta}_{k}^{2} = 1/T_{k}\sum_{t=1}^{T_{k}} (\Delta_{k}\xi_{t})^{2}, \quad k = 1, \ldots, K.
\]

(5)

Thus, the variance-ratio statistics \(R(k), k = 1, \ldots, K\), would have central
\(F\)-distributions:

\[
R(k) \sim k\hat{\delta}_{1}^{2}/\hat{\delta}_{k}^{2} = (\chi_{F_{k}}^{2}/T)/(\chi_{T_{k}^{2}}^{2}/T_{k}) \sim F_{T_{k}, T_{k}}.
\]

(6)

Unfortunately, due to the fact that \(\{\Delta_{k}\xi_{t}\}\) is a temporal aggregate of \(\{\Delta_{1}\xi_{t}\}\),
the \(\chi_{j}^{2}\) random variables in the above formulae are not independent; hence,
the variance ratio test statistics \(R(k)\) do not have \(F\)-distributions. The fractiles
of the \(R(k)\) are easily calculated by Monte Carlo, however, and they are
reported in Diebold (1987) for a range of primary sample sizes \(T\) and
aggregation intervals \(k\).

We now proceed to consider some related test statistics of interest. First, for
many applications we may want to relax the maintained zero-drift assumption.
We denote the corresponding test statistics with estimated drift by

\[
R_{d}(k) = k\hat{\delta}_{1, d}^{2}/\hat{\delta}_{k, d}^{2},
\]

(7)

where

\[
\hat{\delta}_{1, d}^{2} = 1/T\sum_{t=1}^{T} (\Delta_{1}\xi_{t} - \hat{\mu})^{2},
\]

(8)

\[
\hat{\delta}_{k, d}^{2} = 1/T_{k}\sum_{t=1}^{T_{k}} (\Delta_{k}\xi_{t} - k\hat{\mu})^{2},
\]

(9)

\[
\hat{\mu} = 1/T\sum_{t=1}^{T} \Delta_{1}\xi_{t}.
\]

(10)
Second, note that all of the above tests, which make use of the variances of k-aggregates of the first-differenced series, may equivalently be cast in terms of variances of non-overlapping k-differences of the level series. Thus, regardless of whether or not a drift is estimated, the power of the above tests may be increased by allowing the k-differences to overlap. We therefore consider the following:

\[ R_o(k) = k \hat{a}_1^2 / \hat{a}_{k,o}^2 \quad \text{(zero-drift case),} \]  
\[ R_{do}(k) = k \hat{a}_1^2 / \hat{a}_{k,do}^2 \quad \text{(estimated-drift case),} \]  

where

\[ \hat{a}_{k,o}^2 = 1 / (T - k + 1) \sum_{i=k}^{T} (x_i - x_{i-k})^2, \]  
\[ \hat{a}_{k,do}^2 = 1 / (T - k + 1) \sum_{i=k}^{T} (x_i - x_{i-k} - k\hat{\mu})^2, \]  

and the subscript 'o' denotes overlap. The \( R_{do} \) statistics are the most useful in practice, since we are usually not justified in assuming a zero drift, and power is increased by the use of overlapping differences. Fractiles of \( R_{do}(k) \), for various \( T \) and \( k \) values, appear in tables 1–5, which are based upon 25,000 replications. The fractiles are read across the first rows, and the sample sizes \( (T) \) are read down the first columns.

3. Examples: Reflecting barriers and bubbles

The random walk with reflecting barriers is a model of great importance in the physical sciences (e.g., Brownian motion in a closed container), and it appears in economics and finance as well. For example, many models of the

|Table 2| Fractiles of \( R_{do}(4) \). |
|---|---|---|---|---|---|---|---|
|Fractiles: 0.005| 0.025| 0.050| 0.100| 0.900| 0.950| 0.975| 0.995 |
|64| 0.612| 0.638| 0.737| 0.795| 1.485| 1.648| 1.802| 2.144 |
|96| 0.660| 0.728| 0.769| 0.821| 1.355| 1.473| 1.587| 1.819 |
|128| 0.697| 0.758| 0.792| 0.836| 1.292| 1.380| 1.469| 1.667 |
|160| 0.718| 0.774| 0.808| 0.849| 1.249| 1.324| 1.400| 1.570 |
|192| 0.733| 0.788| 0.820| 0.861| 1.226| 1.293| 1.359| 1.495 |
|256| 0.759| 0.811| 0.841| 0.876| 1.184| 1.242| 1.295| 1.410 |
|512| 0.819| 0.862| 0.882| 0.908| 1.123| 1.186| 1.218| 1.256 |
|1024| 0.866| 0.906| 0.913| 0.932| 1.082| 1.106| 1.126| 1.175 |
|2048| 0.904| 0.926| 0.937| 0.951| 1.037| 1.073| 1.088| 1.117 |
|4096| 0.929| 0.945| 0.954| 0.964| 1.039| 1.050| 1.060| 1.081 |

|Table 3| Fractiles of \( R_{do}(8) \). |
|---|---|---|---|---|---|---|---|
|Fractiles: 0.005| 0.025| 0.050| 0.100| 0.900| 0.950| 0.975| 0.995 |
|64| 0.474| 0.584| 0.647| 0.735| 2.042| 2.394| 2.719| 3.535 |
|96| 0.533| 0.628| 0.684| 0.758| 1.708| 1.943| 2.159| 2.700 |
|128| 0.572| 0.660| 0.708| 0.777| 1.564| 1.745| 1.912| 2.317 |
|160| 0.601| 0.681| 0.726| 0.788| 1.468| 1.621| 1.760| 2.090 |
|192| 0.627| 0.701| 0.743| 0.802| 1.408| 1.533| 1.654| 1.958 |
|256| 0.660| 0.731| 0.773| 0.823| 1.333| 1.434| 1.530| 1.744 |
|512| 0.739| 0.796| 0.826| 0.864| 1.210| 1.277| 1.335| 1.449 |
|1024| 0.802| 0.845| 0.868| 0.898| 1.140| 1.178| 1.212| 1.287 |
|2048| 0.855| 0.887| 0.904| 0.925| 1.094| 1.122| 1.146| 1.196 |
|4096| 0.890| 0.915| 0.928| 0.944| 1.064| 1.082| 1.099| 1.132 |

|Table 4| Fractiles of \( R_{do}(16) \). |
|---|---|---|---|---|---|---|---|
|Fractiles: 0.005| 0.025| 0.050| 0.100| 0.900| 0.950| 0.975| 0.995 |
|64| 0.375| 0.511| 0.597| 0.726| 1.754| 2.143| 4.218| 5.017| 7.104 |
|96| 0.424| 0.540| 0.613| 0.716| 1.627| 1.967| 3.018| 3.537| 4.690 |
|128| 0.459| 0.564| 0.631| 0.725| 1.521| 1.800| 2.862| 3.748| 4.855 |
|160| 0.487| 0.589| 0.654| 0.733| 1.414| 1.649| 2.468| 3.274| 3.931 |
|192| 0.518| 0.615| 0.699| 0.747| 1.317| 1.576| 2.209| 2.842| 3.520 |
|256| 0.557| 0.645| 0.709| 0.766| 1.234| 1.462| 1.986| 2.380| 2.908 |
|512| 0.614| 0.716| 0.760| 0.815| 1.134| 1.362| 1.812| 2.387| 2.977 |
|1024| 0.672| 0.782| 0.816| 0.857| 1.026| 1.266| 1.734| 2.300| 2.873 |
|2048| 0.795| 0.839| 0.863| 0.892| 1.148| 1.191| 1.229| 1.304| 1.304 |
|4096| 0.845| 0.878| 0.898| 0.920| 1.099| 1.128| 1.152| 1.203| 1.203 |
Table 5
Fractiles of $R_{aw}(32)$.  

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<th>Fractile</th>
<th>0.005</th>
<th>0.025</th>
<th>0.050</th>
<th>0.100</th>
<th>0.300</th>
<th>0.500</th>
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<th>0.950</th>
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<td>3.930</td>
<td>11.755</td>
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<tr>
<td>96</td>
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<td>0.501</td>
<td>0.605</td>
<td>0.758</td>
<td>0.869</td>
<td>1.011</td>
<td>2.591</td>
<td>7.172</td>
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<tr>
<td>128</td>
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<td>0.500</td>
<td>0.587</td>
<td>0.716</td>
<td>0.936</td>
<td>1.248</td>
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<tr>
<td>160</td>
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<td>192</td>
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<td>0.714</td>
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<td>256</td>
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<td>1.199</td>
<td>1.238</td>
<td>1.316</td>
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</table>

Suppose instead that we have a bubble process:

\[ x_t = 1.02x_{t-1} + \epsilon_t, \quad \text{w.p.} \ 0.95, \]
\[ x_t = \epsilon_t, \quad \text{w.p.} \ 0.05. \quad (18) \]

Such a model arises naturally in linear rational expectations models of asset markets, such as Blanchard and Watson (1982), in which 'no arbitrage' conditions yield the forward solution $p_t^* = f_t + b_t$, where $f_t = \sum_{i=0}^{\infty} \theta^{i+1} E(x_{t+i}/\Omega_t)$ is the fundamental solution [abs$(\theta) < 1$] and $b_t$ is the bubble solution. Under rational expectations, it must be the case that $E(b_{t+1}/\Omega_t) = \theta^{-1}b_t$. This restriction is satisfied by

\[ b_t = (\pi \theta)^{-1} b_{t-1} + \epsilon_t, \quad \text{w.p.} \ \pi, \]
\[ b_t = \epsilon_t, \quad \text{w.p.} \ (1-\pi), \quad (19) \]

where $E(\epsilon_t/\Omega_{t-1}) = 0$. Periodic bursting of the bubble (given a long enough sample) leads to asymptotic flattening of the variance-time function.

4. Concluding remarks

We have developed exact finite-sample tests for random-walk behavior based upon the variance-time function. The fact that tests based on the variance-time function are not directed against any particular alternative may actually enhance their usefulness as a diagnostic tool. Similarly, plots of the variance-time function may serve as a useful graphical aid in the determination of underlying dynamics. In this vein, shapes of the variance-time function under two particular deviations from random-walk behavior were briefly examined.

In future work, we plan to explore the power of tests of dependence based on the variance-time function under alternatives such as those sketched above and the 'long-memory' models studied by Diebold and Rudebusch (1988).

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CYCLICAL AND SECULAR TRADE ELASTICITIES
An Application to LDC Exports*

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1. Introduction

This paper estimates income and price elasticities of non-oil imports of major industrial countries from developing countries. Besides their traditional role in analyses of international linkages and trade policies, these elasticities are becoming increasingly important because of their role in formulating policy responses to the existing debt crisis. However, an examination of the more influential studies reveals a lack of consensus about their magnitudes. For example, existing estimates of the income elasticity for non-oil imports of industrial countries from developing countries range from 1.3 (Goldstein and Khan (1982)) to 4.7 (Dornbusch (1985)).

A review of several studies (Marquez and McNeilly (1986)) reveals that several factors might be responsible for such a wide range of estimates. First, the own-price elasticity of import demand is commonly set to zero, an assumption that might bias the income elasticity. Second, many empirical studies use multilateral rather than bilateral trade data for parameter estimation. This use of multilateral trade data introduces systematic errors in the measurement of bilateral imports, which results in biased elasticity estimates. Third, existing analyses use non-oil imports of all industrial countries for parameter estimation and rarely recognize that import elasticities differ across commodities and that the commodity mix varies across importing countries. As a result, reliance on existing elasticity estimates is likely to produce systematic errors if used to forecast imports of an individual country. These forecast errors would be more serious the greater the dispersion across countries in either the commodity mix or the associated income and price elasticities.

*An enlarged version of this paper, available on request, provides greater discussion of the literature, the results, and the implications. This paper represents the views of the author and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or other members of its staff.