Dynamic Analysis of Multivariate Time Series Using Conditional Wavelet Graphs

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Introduction

Aims

- Analyze local properties of nonstationary multivariate time series using wavelets (cross correlations)
- Introduce a novel graphical model defined on basis of partial wavelet coherences (PWC)
- Select the graphical model based on observed data: estimate PWC and test for statistical significance
Related Literature

Partial correlation graph for time series
- generalize classical concentration graphs to time series
- restricted to linear dependencies
- accounts for the non-contemporaneous influences (lags)

Barigozzi and Brownless (2014)
Graphical Models

A simple graph $G = (V, E)$ consists of:
- a set of vertices $V = \{v_1, \ldots, v_k\} < \infty$
- a set of edges $E \subseteq V \times V$, $e_{ij} = (v_i, v_j)$
- undirected edges $e_{ij} \in E(G) \iff e_{ji} \in E(G)$
- no graph loops or multiple edges

Usually, $v_i \in V$ represents a random variable/process.

A multigraph consists of multiple or parallel edges b/w $v_i$ and $v_j$.

We derive a loopless undirected multigraph model from which simple graphs can be obtained as subgraphs.
Undirected Graphical Models

In this talk we focus on undirected graphs where 

\[(v_i, v_j) \in E(G) \iff (v_j, v_i) \in E(G)\].

(a) MRF

(b) Multigraph

Quantile Graphical modelling of Point Processes

Wavelet Graph
Graphical Models for Time Series

k-dimensional multivariate time series $X_V(t)$
- $X_V(t) = \{X_i(t)\}_{i \in V}, t \in \mathbb{Z}, V = \{1, \ldots, k\}$
- $X_{V\setminus\{i,j\}}(t) = \{X_i(t)\}_{i \in V\setminus\{i,j\}}$.

The time series graph of a process $X_V$
- vertex $v_i$ refers to the $X_i$ component processes of $X_V$, $V = k \times \mathbb{Z}$

Linear dependence graphs
- edge $e_{ij}$ is missing if the components $X_i$ and $X_j$ are uncorrelated (given all the other components), i.e. $X_i \perp \perp X_j(\mid X_{V\setminus\{i,j\}})$ orthogonality (conditional)

Remark: For Gaussian time series - conditional independence.

Wavelet Graph
Partial Correlation Graph for Time Series

Definition: The partial correlation graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ for a stationary process $X_V$ is given by

$e_{ij} \notin \mathcal{E} \iff X_i \perp \perp X_j \mid X_V \setminus \{i,j\}$

$\iff \text{cov}(\varepsilon_i|_{V \setminus \{i,j\}}(t), \varepsilon_j|_{V \setminus \{i,j\}}(t + u)), \forall u \in \mathbb{Z}$

$\varepsilon_{i|V \setminus \{i,j\}} := X_i(t) - \mu_{i, opt} - \sum_{u=-\infty}^{+\infty} d_{i, opt}^{opt}(u) X_{V \setminus \{i,j\}}(t - u)$

$(\mu_{i, opt}, d_{i, opt}^{opt}) = \arg \min_{\mu_i, d_i} \mathbb{E}(X_i(t) - \mu_i - \sum_{u=-\infty}^{+\infty} d_i(u) X_{V \setminus \{i,j\}}(t - u))^2$
Correlation and Partial Correlation Graphs

Example: Dahlhaus (2000)
Let $X_1(t) = a_1 X_1(t - 1) + \varepsilon_1(t)$,

$$X_j(t) = a_j X_j(t - 1) + b_j X_{j-1}(t - t_j) + \varepsilon_j(t)$$

$t_j \in \mathbb{N}$ and $\varepsilon_j(t) \sim N(0, \sigma)$ iid.

All processes are correlated, i.e. the (simple) correlation graph is complete, while the conditional correlation graph is given below.

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1 ---- 2 ---- 3 ---- 4
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Generalization: If in the partial correlation graph there exist a path between two vertices, then the component processes associated with them are correlated (and vice-versa).
Frequency Domain Formulation

Partial cross-spectrum b/w $X_i$ and $X_j$ at frequency $\omega \in [-\pi, \pi]$

$$f_{ij|\mathcal{V}\backslash\{i,j\}}(\omega) = \frac{1}{2\pi} \sum_{t=-\infty}^{+\infty} \left[ \sum_{u=-\infty}^{+\infty} \varepsilon_i|\mathcal{V}\backslash\{i,j\}(t)\varepsilon_j|\mathcal{V}\backslash\{i,j\}(t+u) \right] e^{-i\omega t}$$

$$= \frac{1}{2\pi} \sum_{u=-\infty}^{+\infty} \text{cov}(\varepsilon_i|\mathcal{V}\backslash\{i,j\}(t), \varepsilon_j|\mathcal{V}\backslash\{i,j\}(t+u))e^{-i\omega t}$$

- is the Fourier transform of the cross-correlation function
- is a measure of covariance b/w $\varepsilon_i|\mathcal{V}\backslash\{i,j\}$ and $\varepsilon_j|\mathcal{V}\backslash\{i,j\}$

$$\rightarrow X_i \perp \perp X_j \mid X_{\mathcal{V}\backslash\{i,j\}} \Leftrightarrow f_{ij|\mathcal{V}\backslash\{i,j\}}(\omega) = 0, \forall \omega$$
Partial Spectral Coherence

Observation: The estimation of residuals $\varepsilon_{i|V\{i,j\}}(t)$ is computationally intensive.

Alternative: If the spectral matrix $f_V(\omega) = \{f_{ij}(\omega)\}_{i,j \in V}$ is regular and $g(\omega) := f(\omega)^{-1}$ then the partial spectral coherence matrix is $R(\omega) = -\text{diag}(g(\omega))^{-1/2}g(\omega)\text{diag}(g(\omega))^{-1/2}$, whose elements can be shown to satisfy

$$ R_{ij|V\{i,j\}}(\omega) = \frac{f_{ij|V\{i,j\}}(\omega)}{\left[f_{ii|V\{i,j\}}(\omega)f_{jj|V\{i,j\}}(\omega)\right]^{1/2}}. $$

$\rightarrow X_i \perp \perp X_j | X_{V\{i,j\}} \iff R_{ij|V\{i,j\}}(\omega) = 0, \forall \omega \iff g(\omega) = f(\omega)^{-1}, \forall \omega$
Localized Partial Correlation Graph

For (possibly) non-ergodic and non-stationary multivariate time series wavelet-based methods

- allow time varying analysis of spectral behavior
- characterize dependence in time-frequency domain
- similar to applying linear filters locally \((\mu_{i,t}^{opt}, d_{i,t}^{opt})\) to obtain the errors \(\varepsilon_{i|V\backslash\{i,j\}}(t)\)
- Similar to local covariance functions, local cross-spectra and local coherence

**Remark:** If the time series are stationary, their spectral behavior will be constant over time.
Wavelets

- "Mother wavelet" $\psi \in L_2(\mathbb{R})$ s.t.
  $$\int_{-\infty}^{\infty} \psi(t)dt = 0$$  admissibility condition
  $$\int_{-\infty}^{\infty} \psi^2(t)dt = \|\psi\|^2 = 1 'unit' energy property.$$

- Families of basis functions $\psi_{\tau,s}(t)$
  $$\psi_{\tau,s}(t) = \frac{1}{\sqrt{s}} \psi \left( \frac{t-\tau}{s} \right), \ s \in \mathbb{R}^+, \ \tau \in \mathbb{R}$$  (1)
  $\tau$ location and $s$ scale (pseudo-frequency); $\|\psi_{\tau,s}\| = 1$

*Note: We will consider complex wavelets further on.*
Example: Morlet Wavelet

Morlet wavelet under translation and dilation
Wavelet Transform

Wavelet coefficients w.r.t. $X_i$

$$W_i(\tau, s) = \langle X_i, \psi_{\tau,s} \rangle$$

$$= \frac{1}{\sqrt{s}} \sum_{-\infty}^{+\infty} X_i(t) \overline{\psi_{\tau,s}(t)}$$

(·) stands for the complex conjugate. Additionally, a frequency domain representation of $W_i(\tau, s)$ follows as

$$W_i(\omega) = \frac{\sqrt{|s|}}{2\pi} \sum_{t=-\infty}^{\infty} X_i(t) f_{\psi_{s,\tau}}(st) e^{i\omega t},$$

where $f_{\psi_{s,\tau}}$ is the Fourier transform of the wavelet function $\psi_{\tau,s}(t)$. 
'Adaptive' Window

Heisenberg time-frequency boxes of two wavelet basis
Wavelet Graphs

Scaleogram

60Sec continuous Wavelet bior2.8

Arousal-valence scale for the EEG signal, Sorkhabi (2014)
Parseval’s Relation: Extension to Wavelets

*Recall*: The inner product of two time series equals the inner product of their Fourier transform.

- $X_i(t)$ can be recovered from the wavelet transform

$$X_i(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{s^2} W_i(\tau, s) \psi_{\tau, s}(t) d\tau ds$$

- For two processes $X_i(t)$ and $X_j(t)$, the energy in the time domain is preserved in the time-frequency domain

$$\langle X_i X_j \rangle = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{s^2} |W_i(\tau, s)\overline{W_i(\tau, s)}| d\tau ds,$$

for a finite constant $C_\psi$ satisfying

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\psi(\omega)|^2}{|\omega|} d\omega < \infty.$$
Partial Cross Wavelet

- Cross-wavelet coefficients - can be interpreted as a localized measure of correlation between two time series

\[ W_{ij}(\tau, s) = W_i(\tau, s) \overline{W_j}(\tau, s) \]

- Partial cross-wavelet

\[ W_{ij\setminus\{i,j\}}(\tau, s) = W_{ij}(\tau, s) \]

\[ - W_{i\setminus\{i,j\}}(\tau, s) W_{j\setminus\{i,j\}}(\tau, s)^{-1} W_{j\setminus\{i,j\}}(\tau, s) \]

It extends a result for partial cross-spectrum (Brillinger, 1981) and involves inversion of \((k - 2) \times (k - 2)\) dimensional matrix; alternatively solve via recursion formula.
Partial Wavelet Coherence

- Partial wavelet coherence

\[
R_{ij|\mathcal{V}\backslash\{i,j\}}(\tau, s) = \frac{|W_{ij|\mathcal{V}\backslash\{i,j\}}(\tau, s)|}{|W_{ii|\mathcal{V}\backslash\{i,j\}}(\tau, s)W_{jj|\mathcal{V}\backslash\{i,j\}}(\tau, s)|^{\frac{1}{2}}}
\]

\[
0 \leq |R_{ij|\mathcal{V}\backslash\{i,j\}}(\tau, s)|^2 \leq 1, \text{ interpreted as a localized correlation in the time-frequency domain}
\]

**Remark.** \(X_i \perp\!\!\!\!\perp X_j \mid X_{\mathcal{V}\backslash\{i,j\}} \iff R_{ij|\mathcal{V}\backslash\{i,j\}}(\tau, s) = 0, \forall s, \tau \iff |W_{ij|\mathcal{V}\backslash\{i,j\}}(\tau, s)| = 0, \forall s, \tau\)
Wavelet Dependence Graph

For $X_V(t)$ a multivariate stochastic process evolving in discrete time a wavelet dependence graph is an undirected multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which any $v_i \in \mathcal{V}(\mathcal{G})$ encodes the $i$-th component $X_i(t)$ of $X_V(t)$ s.t.

$$X_{i,s} \perp \perp X_{j,s} \mid X_{V\setminus\{i,j\},s} \iff e_{ij,s} \notin \mathcal{E}_s(\mathcal{G}) \iff R_{ij\mid \mathcal{V}\setminus\{i,j\}}(\tau, s) = 0, \forall \tau$$

at fixed scale $s$, where $\mathcal{E}_s(\mathcal{G})$ is a scale-specific subset and it holds that $\mathcal{E}(\mathcal{G}) = \bigcup \mathcal{E}_s(\mathcal{G})$.

**Remark:** A partial correlation (wavelet) graph can be obtained from the multigraph by replacing any multiedge by at most one edge.
Outlook

Graph estimation

- noisy observation: shrinkage/smoothing of the wavelet coefficients, LASSO
- distributional assumptions for testing: Gaussian errors, Monte-Carlo methods

Extensions

- Directed graphs - Granger causality
- Dynamic graphs
- Simulation, real data
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