Machine Learning Overview
(Inference For High Dimensional Regression)

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OUTLINE

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-conditioning
-uniform
-sample splitting
-subsampling
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- A few comments on causal inference.

(Susan and Victor will give talks about this.)

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I am skeptical of assumptions, especially in high-dimensional settings. (But I seem to be an outlier.)

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Economics	very heavy	prediction, inference and causation

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(In contrast, causal inference is a semiparametric problem and bias is worse than variance.)

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- Random Forests: best off-the-shelf nonparametric prediction method
- Deep learning: huge breakthrough or snake oil?

The Lasso for Linear Regression

Recall that the lasso estimator is

$$\widehat{\beta} = \underset{\beta}{\operatorname{argmin}} \left(\frac{1}{n} \sum_i (Y_i - \beta^T X_i)^2 + \lambda ||\beta||_1 \right)$$

where $||\beta||_1 = \sum_j |\beta_j|$.

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3. How do we do inference?

Random Forests

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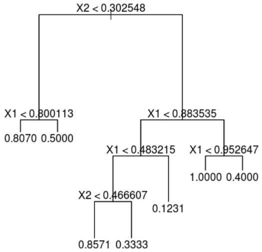
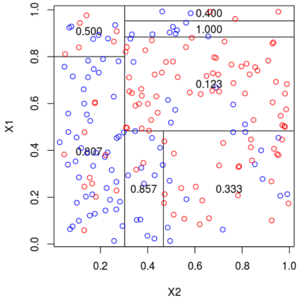
Same for classification (binary regression).

Tree





source: https://www.r-bloggers.com/regression-tree-using-ginis-index/

Forest

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One of the best general purpose prediction methods.

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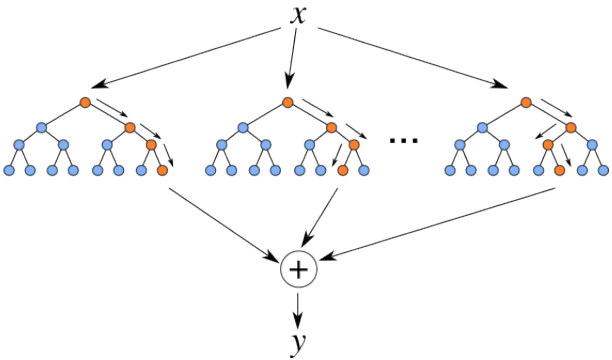

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http://kazoo04.hatenablog.com/entry/2013/12/04/175402

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$$B_L = \{\beta : \ ||\beta||_1 \leq L\}.$$

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Let $\widehat{\beta}$ be the lasso estimator. With probabilty at least $1 - \delta$,

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No assumptions! (Not even linearity).

INFERENCE?

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- Probably a bogus assumption.

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Note: β_S is a random parameter. (And it is not smooth.)

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Does not depend on linearity or model correctness. Interpretable.

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Lei, G'Sell, Rinaldo, Tibshirani, Wasserman (2016a,b), Hooker and Mentch (2016).

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The forgotten methods: unsupervised dimension reduction (variable clustering, PCA, non-linear dimension reduction, etc).

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We really want: Robust and Honest:

$$\liminf_{n \to \infty} \inf_{w \in W_n} \inf_{P \in \mathrm{BIG}} P^n(\theta \in C) \geq 1 - \alpha$$

where $\mathcal{W}_n =$ all model selection rules.

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But, it requires: linear model correct, incoherence, sparsity, constant variance, very carefully chosen tuning parameter.

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Carefully chosen event E_n.

By sufficiency:

$$\sqrt{n}(\widehat{\beta}(j) - \beta(j)) \;\Big|\; E_n$$

has a distribution (truncated Normal) indexed by one parameter.
Test and invert.

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Target is the projection parameter.

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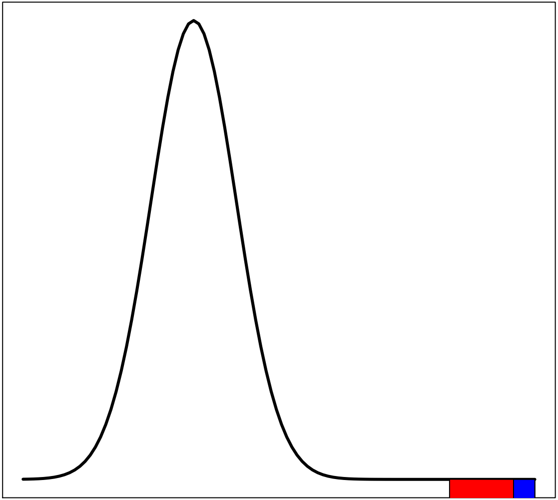
Tail Ratios



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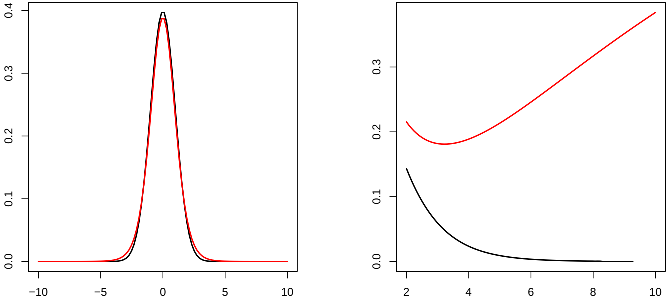
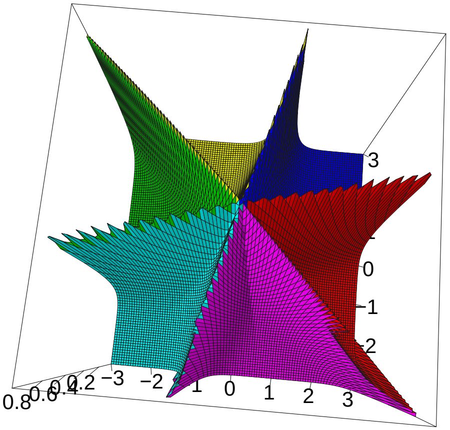


The Pivot



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T does not converge to $\mathrm{Unif}(0,1)$.

In fact, we create an example where

$$T \to 0$$

with probability at least $1/e$.

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Disadvantages: cannot estimate F_n unless we make extra assumptions.

Sample Splitting

Hartigan (1969), Moran (1973), Barnard (1974), Cox (1975), Mosteller and Tukey (1977, p 37), Picard and Berk (1990), Miller (1990, p13) and Faraway (1995), G'Sell, Lei, Rinaldo, Tibshirani, Wasserman (2016).

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Barnard:
" ... the simple idea of splitting a sample in two and then developing the hypothesis on the basis of one part and testing it on the remainder may perhaps be said to be one of the most seriously neglected ideas in statistics ..."

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Split data into $\mathcal{D}_1$ and $\mathcal{D}_2$.

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Sample Splitting + LOCO

G'Sell, Lei, Rinaldo, Tibshirani and Wasserma (2016).
Define variable importance directly:

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Truly distribution free.

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Wager and Athey (2015) have a different approach for random forests for treatment effects.

Conformalization (arXiv:1604.04173)

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Construct $C(x)$ such that

$$\mathbb{P}(Y \in C(X)) \geq 1 - \alpha \qquad \text{for all } P.$$

Basic idea

Observe

$$Y_1, \ldots, Y_n$$

Predict new Y_{n+1}.

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5. Invert:
   $$C_n(y) = \{y : \ p(y) \geq \alpha\}.$$

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Vovk and his colleagues have many papers with different versions and interesting applications.

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- Without any assumptions: $\mathbb{P}(Y \in C_n(X)) \geq 1 - \alpha$.

Conformalization

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R package conformalInference
https://github.com/ryantibs/conformal

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$$\mu(x, w) = \mathbb{E}[Y|X = x, W = w]$$

ML vs Statistics vs Economics

In ML we want low prediction error

$$\mathbb{E}[(Y - \widehat{\mu}(X))^2].$$

Balance bias and variance.

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In economics, we may want to estimate the causal effect

$$\theta = \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)].$$

Not identifiable unless we assume no unmeasured confounding:

$$W \perp\!\!\!\perp (Y(1), Y(0)) \;\Big|\; X.$$

In that case

$$\theta = \int [\mu(x,1) - \mu(x,0)] dP(x)$$

The Causal Effect

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In principle:

$$\widehat{\theta} = \frac{1}{n} \sum_i [\widehat{\mu}(X_i, 1) - \widehat{\mu}(X_i, 0)]$$

but:

1. $\mu(x, w)$ may be high-dimensional.
2. This is a semiparametric problem not a prediction problem so we don't want to balance bias and variance.
3. We also want a confidence interval.
4. We want to assess the sensitivity to the "no confounding" assumption.

(See Susan and Victor's talks.)

Sensitivity Analysis

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Let

$$q(1, x) = \mathbb{E}[Y(1)|W = 1, X = x] - \mathbb{E}[Y(1)|W = 0, X = x]$$

$$q(0, x) = \mathbb{E}[Y(0)|W = 0, X = x] - \mathbb{E}[Y(0)|W = 1, X = x].$$

These are 0 if there is no unmeasured confounding.

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Replace Y_i's with $\widetilde{Y}_i$'s. Repeat for every value of a.

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But in practice, we need to estimate $\pi(w|x)$ (another high-dimensional regression).

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