Identifying and Estimating Social Connections from Outcome Data

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Networks are Everywhere

- Social and economic networks mediate many aspects of individual choice and outcomes:
  - *Development:* technology adoption, insurance.
  - *Peer Effects:* learning, delinquency, consumption.
  - *IO:* buyer-supplier networks, strategic interactions.
  - *Macro, Finance and Trade:* contagion, gravity equations.
  - Many more examples: Jackson [2010], de Paula [2015].

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  (spatial model with *unobserved* neighbourhood matrix)

- ... under standard network "intransitivity" hypothesis ...

- ... and explore estimation strategies.
  - N individuals $\Rightarrow O(N^2)$ parameters to estimate.
  - High-dimensional model techniques.
  - Consistency and asymptotic distribution.

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The Model

- Many interdependent outcomes are mediated by connections ("networks").
- A popular representation follows the "linear-in-means" specification suggested in Manski [1993]. For example,

$$y_{it} = \alpha_t + \rho_0 \sum_{j=1}^{N} W_{0,ij} y_{jt} + \beta_0 x_{it} + \gamma_0 \sum_{j=1}^{N} W_{0,ij} x_{jt} + \epsilon_{it}$$

$$\Leftrightarrow$$

$$\mathbf{y}_{t,N\times 1} = \alpha_t \mathbf{1}_{N\times 1} + \rho_0 W_{0,N\times N} \mathbf{y}_{t,N\times 1} + \beta_0 \mathbf{x}_{t,N\times 1} + \gamma_0 W_{0,N\times N} \mathbf{x}_{t,N\times 1} + \epsilon_{t,N\times 1}$$

  with $\mathbb{E}(\epsilon_{it} | \mathbf{x}_t, W_0, \alpha_t) = 0$.
- Customary to assume $W_0 \mathbf{1} = \mathbf{1}$ and stationarity ($|\rho_0| < 1$).

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(Some) Literature

1. **Spatial Econometrics**, conditional on W_0.
   - Kelejian and Prucha [1998, 1999], Lee [2004], Lee, Liu and Lin [2010] and Anselin [2010].
2. **Identification.**
   - ...conditional on W_0: Manski [1993], Bramoullé et al. [2009], De Giorgi et al. [2009];
   - ...not conditional on W_0: Rose [2015], see also Blume, Brock, Durlauf and Jayaraman [2015].
3. **Estimating** W_0.
   - Lam and Souza [various].
   - Manresa [2013], Ahrens and Bhattacharjee [2014], Rose [2015], Gautier and Rose [in preparation].

- Manski [1993] categorises "social effects" as:
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…and the “reflection problem.”
($W_{0,ij} = (N-1)^{-1}$ if $i \neq j$, $W_{0,ii} = 0$)

- Potential avenue: “exclusion restrictions” in W_0.

If $\rho_0\beta_0 + \gamma_0 \neq 0$ and $\mathbf{I}$, W_0, W_0^2 are linearly independent, $(\alpha_0, \rho_0, \beta_0, \gamma_0)$ is point-identified.
(Bramoullé, Djebbari and Fortin [2009])

- Linear independence valid generally. In fact,

$\sum_{j=1}^{N} W_{0,ij} = 1$ and $\mathbf{I}$, W_0, W_0^2 linearly dependent $\Rightarrow$ W_0 block diagonal with blocks of the same size and nonzero entries are $(N_l - 1)^{-1}$.
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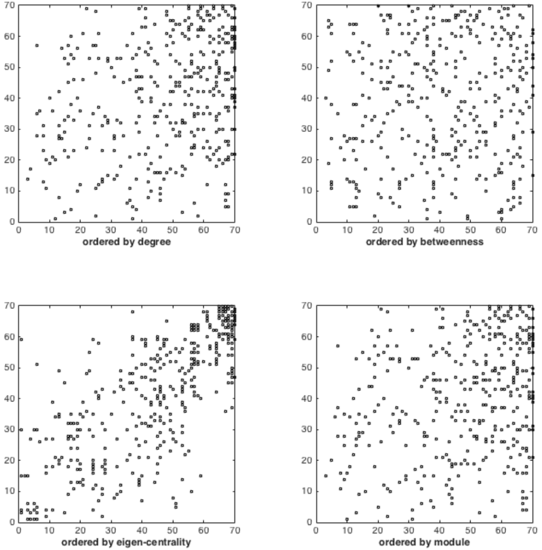
Figure: High School Friendship Network



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- The model has reduced-form

$$\mathbf{y}_t = \Pi_0 \mathbf{x}_t + \mathbf{v}_t$$

where

$$\Pi_0 = (\mathbf{I} - \rho_0 W_0)^{-1}(\beta_0 \mathbf{I} + \gamma_0 W_0)$$

- If $(\rho_0, \beta_0, \gamma_0)$ were known, W_0 would be identified:

$$W_0 = (\Pi_0 - \beta_0 \mathbf{I})(\rho_0 \Pi_0 + \gamma_0 \mathbf{I})^{-1}$$

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- Further assumptions are necessary to identify $\theta_0 = (\rho_0, \beta_0, \gamma_0, W_0)$.

- Take, for example, θ_0 and θ such that $\beta_0 = \beta = 1$, $\rho_0 = 0.5$, $\rho = 1.5$, $\gamma_0 = 0.5$, $\gamma = -2.5$,

$$W_0 = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix}.$$

- Then $(I - \rho_0 W_0)^{-1}(\beta_0 I + \rho_0 W_0) = (I - \rho W)^{-1}(\beta I + \rho W)$.

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- (Notice that I, W_0 and W_0^2 are LI and so are I, W and W^2!)

But . . .

If the spectral radius of $\rho_0 W_0$ is less than one, then an eigenvector of Π_0 is also an eigenvector of W_0.

Take the reduced-form parameter matrix:

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\begin{aligned}
\Pi_0 &= (I + \rho_0 W_0 + \rho_0^2 W_0^2 + \cdots)(\beta_0 \mathbf{I} + \gamma_0 W_0) \\
&= \beta_0 \mathbf{I} + (\rho_0 \beta_0 + \gamma_0) W_0 + \rho_0(\rho_0 \beta_0 + \gamma_0) W_0^2 + \cdots
\end{aligned}
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Postmultiplying by v_j, an eigenvector of W_0,

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\Pi_0 v_j = \frac{\beta_0 + \gamma_0 \lambda_{j,0}}{1 - \rho_0 \lambda_{j,0}} v_j
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Local Identification

- Can the model identify $\theta_0 = (\rho_0, \beta_0, \gamma_0, W_0)$?

- Assume:

(A1) $(W_0)_{ii} = 0, i = 1, \ldots, N$ (no self-links);

(A2) $\sum_{j=1}^{N} |(W_0)_{ij}| \leq 1$ for every $i = 1, \ldots, N$ and $|\rho_0| < 1$;

(A3) There is i such that $\sum_{j=1}^{N} (W_0)_{ij} = 1$ (normalization);

(A4) There are l and k such that $(W_0^2)_{ll} \neq (W_0^2)_{kk}$ ($\Rightarrow \mathbf{I}, W_0, W_0^2$ LI as in Bramoullé, Djebbari and Fortin [2009]);

(A5) $\beta_0 \rho_0 + \gamma_0 \neq 0$ (social effects do not cancel).

- *Under (A1)-(A5)* $(\rho_0, \beta_0, \gamma_0, W_0)$ *is locally identified.*

  (Application of Rothenberg [1971].)

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Global Identification

- Under (possibly strong) conditions it is straightforward to obtain global identification.

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Global Identification

- It is nevertheless possible to strengthen local identification conclusions obtained previously.

- *Assume (A1)-(A5).* $\{\theta : \Pi(\theta) = \Pi(\theta_0)\}$ *is finite.*
  (This obtains as $\Pi(\theta)$ is a proper mapping.)

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A Few Remarks

- $\mathbf{v}_j$ is an eigenvector of Π and W_0: eigencentralities are identified even when W_0 is not.
- Row-sum normalization of W_0 implies that row-sum of Π is constant: testable hypothesis.
- We also allow for network specific effects.
- Analysis extends to multivariate $\mathbf{x}_{i,t}$. The reduced-form model is

$$\mathbf{y}_t = \sum_{s=1}^{k} \Pi_s \mathbf{x}_{t,s} + \mathbf{v}_t$$

where $\mathbf{x}_{t,s}$ refers to the s-th column of $\mathbf{x}_t$ and

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Estimation Strategies

- Π has N^2 parameters, and possibly $NT \ll N^2$.
- Feasible if W or Π are sparse.
  (e.g., Atalay et al. [2011] $<1\%$; Carvalho [2014] $\approx 3\%$; AddHealth $\approx 2\%$).
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- So $[\Pi]_{ij} = 0$ iff there are no paths between i and j in W.

$\Rightarrow$ Sparsity of Π translates into a large number of (i,j) unconnected pairs in W.

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$$y_i = x\pi_i + v_i$$

stacking all observations for individual i at $t = 1, \ldots, T$.

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$$\tilde{\pi}_i = \arg\min_{\pi_i \in \mathbb{R}^N} \frac{1}{T} \|y_i - x\pi_i\|_2 + 2\lambda_T \|\pi_i\|_1$$

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  - $\mathbf{x}_t \perp \epsilon_t \Rightarrow$ moment conditions.

$$\tilde{\theta} = (1 + \lambda_2/T) \cdot \arg\min_{\theta \in \mathbb{R}^p} \left\{ g(\theta)^\top M_T g(\theta) + \lambda_1 \sum_{i,j=1}^{n} |w_{i,j}| + \lambda_2 \sum_{i,j=1}^{n} |w_{i,j}|^2 \right\}$$

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Simulations

- $\rho_0 = 0.3, \beta_0 = 0.4, \gamma_0 = 0.5$.
- 1,000 simulations.
- $N = 15, 30, 50$, $T = 50, 100, 150$.
- Estimators: EN, AL, SCAD, OLS.

- Table 1: N links; Table 2: $2N$ links; Table 3: various.
- Table 4: High School Friendship (Coleman [1964]).

Figure: High School Friendship Network

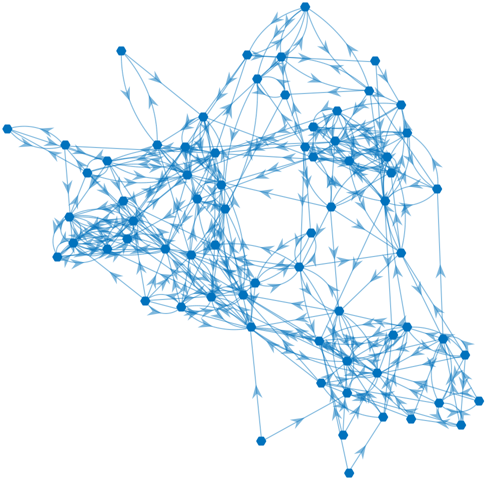
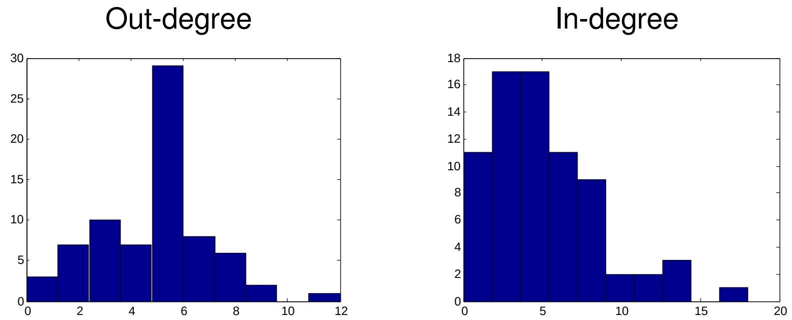


Figure: High School Friendship Network Degree Distribution



Simulations: p=1

	$\emptyset$	EN	AL	SC	OLS	$\emptyset$	EN	AL	SC	OLS	$\emptyset$	EN	AL	SC	OLS
	n = 15 , T = 50					n = 15 , T = 100					n = 15 , T = 150				
$mse(\hat{\Pi})$	0.000	1.656	0.659	0.490	2.929	0.000	0.623	0.304	0.288	1.188	0.000	0.281	0.275	0.221	0.746
	(0.000)	(1.947)	(0.643)	(0.437)	(0.398)	(0.000)	(0.803)	(0.294)	(0.425)	(0.136)	(0.000)	(0.268)	(0.172)	(0.195)	(0.077)
$mse(\hat{W})$	0.000	0.317	1.161	1.003	4.202	0.000	0.040	0.617	0.490	2.493	0.000	0.006	0.680	0.476	1.984
	(0.000)	(0.447)	(1.172)	(0.880)	(0.685)	(0.000)	(0.094)	(0.626)	(0.476)	(0.619)	(0.000)	(0.020)	(0.483)	(0.253)	(0.638)
% true 0s	1.000	0.980	0.934	0.943	0.004	1.000	0.995	0.936	0.969	0.006	1.000	0.999	0.915	0.962	0.007
	(0.000)	(0.023)	(0.072)	(0.047)	(0.005)	(0.000)	(0.008)	(0.066)	(0.028)	(0.006)	(0.000)	(0.003)	(0.059)	(0.019)	(0.006)
% true 1s	1.000	0.993	0.983	0.989	1.000	1.000	1.000	1.000	0.993	1.000	1.000	1.000	1.000	1.000	1.000
	(0.000)	(0.023)	(0.039)	(0.034)	(0.002)	(0.000)	(0.005)	(0.004)	(0.035)	(0.000)	(0.000)	(0.000)	(0.000)	(0.007)	(0.000)
$\hat{\rho}-\rho_0$	0.000	−0.050	−0.267	−0.254	−0.217	0.000	−0.011	−0.276	−0.265	−0.227	0.000	−0.008	−0.274	−0.252	−0.209
	(0.000)	(0.075)	(0.042)	(0.063)	(0.081)	(0.000)	(0.041)	(0.024)	(0.069)	(0.091)	(0.000)	(0.039)	(0.035)	(0.085)	(0.111)
$\hat{\beta}-\beta_0$	0.000	0.003	−0.111	−0.017	−0.003	0.000	0.001	−0.040	−0.011	0.017	0.000	0.001	−0.034	0.017	0.015
	(0.000)	(0.030)	(0.125)	(0.069)	(0.077)	(0.000)	(0.020)	(0.054)	(0.093)	(0.050)	(0.000)	(0.022)	(0.040)	(0.036)	(0.041)
$\hat{\gamma}-\gamma_0$	0.000	0.051	0.151	0.257	0.448	0.000	0.010	0.156	0.169	0.367	0.000	0.005	0.226	0.221	0.316
	(0.000)	(0.079)	(0.173)	(0.212)	(0.092)	(0.000)	(0.033)	(0.164)	(0.149)	(0.155)	(0.000)	(0.035)	(0.156)	(0.105)	(0.186)
	n = 30 , T = 50					n = 30 , T = 100					n = 30 , T = 150				
$mse(\hat{\Pi})$	0.000	2.840	0.560	0.400	5.190	0.000	1.817	0.341	0.543	1.450	0.000	1.486	0.432	0.455	0.841
	(0.000)	(3.122)	(0.391)	(0.325)	(0.600)	(0.000)	(1.806)	(0.733)	(0.790)	(0.105)	(0.000)	(0.833)	(0.605)	(0.611)	(0.051)
$mse(\hat{W})$	0.000	0.312	0.961	0.750	5.535	0.000	0.033	0.332	0.361	2.040	0.000	0.010	0.340	0.320	1.482
	(0.000)	(0.373)	(0.740)	(0.553)	(0.732)	(0.000)	(0.043)	(0.344)	(0.417)	(0.215)	(0.000)	(0.012)	(0.259)	(0.260)	(0.244)
% true 0s	1.000	0.976	0.962	0.959	0.004	1.000	0.995	0.972	0.997	0.006	1.000	0.997	0.968	0.973	0.008
	(0.000)	(0.027)	(0.039)	(0.033)	(0.002)	(0.000)	(0.006)	(0.029)	(0.007)	(0.003)	(0.000)	(0.002)	(0.024)	(0.024)	(0.003)
% true 1s	1.000	0.990	0.942	0.963	1.000	1.000	1.000	1.000	0.949	1.000	1.000	1.000	1.000	0.996	1.000
	(0.000)	(0.023)	(0.058)	(0.083)	(0.002)	(0.000)	(0.003)	(0.003)	(0.071)	(0.000)	(0.000)	(0.000)	(0.000)	(0.021)	(0.000)
$\hat{\rho}-\rho_0$	0.000	−0.084	−0.260	−0.248	−0.046	0.000	−0.013	−0.278	−0.267	−0.232	0.000	−0.006	−0.278	−0.280	−0.246
	(0.000)	(0.073)	(0.057)	(0.057)	(0.083)	(0.000)	(0.028)	(0.014)	(0.078)	(0.060)	(0.000)	(0.023)	(0.013)	(0.017)	(0.062)
$\hat{\beta}-\beta_0$	0.000	0.003	−0.215	−0.068	−0.046	0.000	0.001	−0.058	−0.170	0.005	0.000	−0.000	−0.055	−0.068	0.009
	(0.000)	(0.019)	(0.159)	(0.111)	(0.086)	(0.000)	(0.014)	(0.065)	(0.186)	(0.043)	(0.000)	(0.014)	(0.047)	(0.087)	(0.032)
$\hat{\gamma}-\gamma_0$	0.000	0.134	0.187	0.304	0.500	0.000	0.020	0.149	0.006	0.474	0.000	0.009	0.203	0.176	0.431
	(0.000)	(0.161)	(0.185)	(0.249)	(0.005)	(0.000)	(0.029)	(0.163)	(0.067)	(0.061)	(0.000)	(0.021)	(0.159)	(0.164)	(0.105)
	n = 50 , T = 50					n = 50 , T = 100					n = 50 , T = 150				
$mse(\hat{\Pi})$	0.000	0.191	0.096	0.184	–	0.000	0.419	0.165	0.394	2.042	0.000	2.149	0.159	0.468	0.577
	(0.000)	(0.458)	(0.245)	(0.216)	–	(0.000)	(0.823)	(0.487)	(0.267)	(0.119)	(0.000)	(0.318)	(0.014)	(0.219)	(0.295)
$mse(\hat{W})$	0.000	0.101	0.136	0.355	–	0.000	0.025	0.073	0.342	2.252	0.000	0.022	0.319	0.279	0.491
	(0.000)	(0.243)	(0.335)	(0.400)	–	(0.000)	(0.050)	(0.161)	(0.243)	(0.128)	(0.000)	(0.010)	(0.073)	(0.113)	(0.437)
% true 0s	1.000	0.991	0.995	0.978	–	1.000	0.996	0.994	1.000	0.006	1.000	0.994	0.968	1.000	0.800
	(0.000)	(0.021)	(0.014)	(0.027)	–	(0.000)	(0.008)	(0.011)	(0.002)	(0.002)	(0.000)	(0.002)	(0.008)	(0.002)	(0.397)
% true 1s	1.000	0.998	0.983	0.973	–	1.000	1.000	0.999	0.898	1.000	1.000	1.000	1.000	0.936	0.949
	(0.000)	(0.009)	(0.047)	(0.069)	–	(0.000)	(0.002)	(0.003)	(0.080)	(0.000)	(0.000)	(0.001)	(0.000)	(0.037)	(0.042)
$\hat{\rho}-\rho_0$	0.000	−0.116	−0.261	−0.261	–	0.000	−0.017	−0.277	−0.263	−0.121	0.000	−0.004	−0.278	−0.273	−0.222
	(0.000)	(0.056)	(0.046)	(0.044)	–	(0.000)	(0.024)	(0.015)	(0.079)	(0.061)	(0.000)	(0.017)	(0.013)	(0.070)	(0.055)
$\hat{\beta}-\beta_0$	0.000	0.001	−0.049	−0.103	–	0.000	−0.000	−0.023	−0.283	−0.018	0.000	0.001	−0.089	−0.367	−0.293
	(0.000)	(0.009)	(0.126)	(0.124)	–	(0.000)	(0.007)	(0.059)	(0.178)	(0.040)	(0.000)	(0.013)	(0.028)	(0.050)	(0.155)
$\hat{\gamma}-\gamma_0$	0.000	0.080	0.046	0.166	–	0.000	0.024	0.050	−0.029	0.500	0.000	0.028	0.311	−0.048	0.060
	(0.000)	(0.191)	(0.129)	(0.209)	–	(0.000)	(0.048)	(0.112)	(0.041)	(0.003)	(0.000)	(0.018)	(0.084)	(0.032)	(0.219)

Simulations: p=2

	$\emptyset$	EN	AL	SC	OLS	$\emptyset$	EN	AL	SC	OLS	$\emptyset$	EN	AL	SC	OLS
	n = 15 , T = 50					n = 15 , T = 100					n = 15 , T = 150				
$mse(\hat{\Pi})$	0.000	1.622	0.663	0.546	2.948	0.000	1.004	0.299	0.388	1.192	0.000	1.095	0.462	0.588	0.747
	(0.000)	(2.342)	(0.959)	(0.879)	(0.409)	(0.000)	(1.477)	(0.442)	(0.716)	(0.133)	(0.000)	(1.121)	(0.630)	(0.830)	(0.078)
$mse(\hat{W})$	0.000	0.718	1.278	0.924	4.046	0.000	0.364	0.800	0.678	2.321	0.000	0.324	0.786	0.709	1.738
	(0.000)	(1.058)	(1.707)	(1.232)	(0.862)	(0.000)	(0.541)	(1.283)	(1.108)	(0.872)	(0.000)	(0.331)	(1.018)	(0.921)	(0.787)
% true 0s	1.000	0.978	0.968	0.962	0.004	1.000	0.985	0.941	0.979	0.006	1.000	0.979	0.897	0.969	0.007
	(0.000)	(0.033)	(0.049)	(0.054)	(0.005)	(0.000)	(0.023)	(0.096)	(0.060)	(0.006)	(0.000)	(0.022)	(0.127)	(0.079)	(0.006)
% true 1s	1.000	0.873	0.791	0.862	0.999	1.000	0.920	0.893	0.864	1.000	1.000	0.922	0.899	0.847	1.000
	(0.000)	(0.188)	(0.285)	(0.197)	(0.006)	(0.000)	(0.121)	(0.162)	(0.246)	(0.004)	(0.000)	(0.086)	(0.111)	(0.229)	(0.003)
$\hat{\rho}-\rho_0$	0.000	−0.149	−0.244	−0.261	−0.234	0.000	−0.083	−0.239	−0.228	−0.234	0.000	−0.056	−0.248	−0.214	−0.222
	(0.000)	(0.111)	(0.067)	(0.086)	(0.072)	(0.000)	(0.084)	(0.096)	(0.163)	(0.088)	(0.000)	(0.061)	(0.076)	(0.211)	(0.109)
$\hat{\beta}-\beta_0$	0.000	0.004	−0.128	−0.042	−0.037	0.000	0.002	−0.058	−0.045	−0.004	0.000	0.003	−0.041	−0.037	0.002
	(0.000)	(0.027)	(0.174)	(0.096)	(0.081)	(0.000)	(0.018)	(0.094)	(0.120)	(0.046)	(0.000)	(0.018)	(0.059)	(0.113)	(0.038)
$\hat{\gamma}-\gamma_0$	0.000	0.034	0.019	0.114	0.417	0.000	0.013	0.001	0.030	0.328	0.000	0.021	0.009	0.034	0.291
	(0.000)	(0.089)	(0.103)	(0.170)	(0.108)	(0.000)	(0.050)	(0.096)	(0.118)	(0.168)	(0.000)	(0.048)	(0.109)	(0.118)	(0.183)
	n = 30 , T = 50					n = 30 , T = 100					n = 30 , T = 150				
$mse(\hat{\Pi})$	0.000	2.829	0.490	0.421	5.212	0.000	2.161	0.246	0.392	1.445	0.000	2.954	0.965	1.290	0.840
	(0.000)	(3.130)	(0.514)	(0.477)	(0.613)	(0.000)	(2.389)	(0.340)	(0.484)	(0.095)	(0.000)	(1.406)	(1.378)	(1.252)	(0.050)
$mse(\hat{W})$	0.000	0.500	0.890	0.749	5.227	0.000	0.257	0.611	0.731	2.017	0.000	0.279	0.785	1.009	1.470
	(0.000)	(0.562)	(0.881)	(0.731)	(0.733)	(0.000)	(0.298)	(0.756)	(0.849)	(0.433)	(0.000)	(0.146)	(0.889)	(0.762)	(0.538)
% true 0s	1.000	0.970	0.986	0.969	0.004	1.000	0.982	0.971	0.995	0.006	1.000	0.971	0.932	0.988	0.008
	(0.000)	(0.034)	(0.017)	(0.035)	(0.002)	(0.000)	(0.020)	(0.039)	(0.012)	(0.003)	(0.000)	(0.014)	(0.063)	(0.023)	(0.003)
% true 1s	1.000	0.829	0.653	0.761	0.999	1.000	0.889	0.803	0.710	0.999	1.000	0.868	0.784	0.559	1.000
	(0.000)	(0.203)	(0.351)	(0.269)	(0.005)	(0.000)	(0.134)	(0.235)	(0.347)	(0.003)	(0.000)	(0.080)	(0.138)	(0.335)	(0.003)
$\hat{\rho}-\rho_0$	0.000	−0.202	−0.248	−0.263	−0.073	0.000	−0.107	−0.242	−0.240	−0.246	0.000	−0.072	−0.238	−0.210	−0.243
	(0.000)	(0.091)	(0.044)	(0.054)	(0.090)	(0.000)	(0.070)	(0.081)	(0.083)	(0.062)	(0.000)	(0.048)	(0.101)	(0.124)	(0.078)
$\hat{\beta}-\beta_0$	0.000	0.004	−0.208	−0.092	−0.063	0.000	0.002	−0.111	−0.156	−0.022	0.000	0.003	−0.087	−0.224	−0.008
	(0.000)	(0.020)	(0.196)	(0.129)	(0.091)	(0.000)	(0.015)	(0.129)	(0.188)	(0.045)	(0.000)	(0.016)	(0.082)	(0.191)	(0.032)
$\hat{\gamma}-\gamma_0$	0.000	0.159	0.024	0.172	0.499	0.000	0.056	0.003	−0.031	0.421	0.000	0.075	0.011	−0.061	0.358
	(0.000)	(0.191)	(0.117)	(0.219)	(0.009)	(0.000)	(0.073)	(0.117)	(0.105)	(0.103)	(0.000)	(0.052)	(0.155)	(0.134)	(0.148)
	n = 50 , T = 50					n = 50 , T = 100					n = 50 , T = 150				
$mse(\hat{\Pi})$	0.000	0.357	0.152	0.151	–	0.000	0.428	0.129	0.218	2.042	0.000	0.568	0.105	0.235	1.010
	(0.000)	(0.635)	(0.258)	(0.260)	–	(0.000)	(0.561)	(0.179)	(0.286)	(0.118)	(0.000)	(0.694)	(0.191)	(0.309)	(0.042)
$mse(\hat{W})$	0.000	0.178	0.254	0.269	–	0.000	0.147	0.281	0.357	2.135	0.000	0.087	0.268	0.368	1.319
	(0.000)	(0.313)	(0.439)	(0.464)	–	(0.000)	(0.193)	(0.383)	(0.479)	(0.145)	(0.000)	(0.112)	(0.432)	(0.478)	(0.221)
% true 0s	1.000	0.985	0.997	0.997	–	1.000	0.981	0.989	1.000	0.006	1.000	0.990	0.981	0.999	0.008
	(0.000)	(0.027)	(0.006)	(0.012)	–	(0.000)	(0.025)	(0.016)	(0.004)	(0.002)	(0.000)	(0.012)	(0.030)	(0.003)	(0.002)
% true 1s	1.000	0.913	0.823	0.823	–	1.000	0.920	0.821	0.731	0.999	1.000	0.931	0.862	0.722	0.999
	(0.000)	(0.167)	(0.316)	(0.323)	–	(0.000)	(0.115)	(0.253)	(0.372)	(0.003)	(0.000)	(0.094)	(0.186)	(0.367)	(0.002)
$\hat{\rho}-\rho_0$	0.000	−0.195	−0.252	−0.257	–	0.000	−0.121	−0.258	−0.244	−0.163	0.000	−0.052	−0.247	−0.231	−0.248
	(0.000)	(0.076)	(0.033)	(0.039)	–	(0.000)	(0.060)	(0.039)	(0.054)	(0.067)	(0.000)	(0.035)	(0.073)	(0.081)	(0.054)
$\hat{\beta}-\beta_0$	0.000	0.001	−0.103	−0.093	–	0.000	0.001	−0.099	−0.148	−0.032	0.000	0.000	−0.065	−0.155	−0.021
	(0.000)	(0.012)	(0.175)	(0.162)	–	(0.000)	(0.010)	(0.136)	(0.192)	(0.042)	(0.000)	(0.008)	(0.088)	(0.192)	(0.033)
$\hat{\gamma}-\gamma_0$	0.000	0.129	0.011	0.003	–	0.000	0.123	0.006	−0.041	0.499	0.000	0.040	−0.003	−0.055	0.446
	(0.000)	(0.232)	(0.076)	(0.079)	–	(0.000)	(0.164)	(0.095)	(0.077)	(0.014)	(0.000)	(0.053)	(0.117)	(0.087)	(0.083)

Simulations: Alternative Parameters

p	ρ_0	β_0	γ_0	q	$mse(\hat{\Pi})$	$mse(\hat{W})$	% tr. 0s	% tr. 1s	$\hat{\rho}-\rho_0$	$\hat{\beta}-\beta_0$	$\hat{\gamma}-\gamma_0$
1	0.3	0.4	0.5	0	3.252 (0.918)	0.058 (0.047)	0.991 (0.004)	0.999 (0.005)	−0.011 (0.028)	0.000 (0.020)	0.032 (0.029)
2	0.3	0.4	0.5	0	4.652 (1.040)	0.544 (0.173)	0.960 (0.010)	0.821 (0.082)	−0.110 (0.068)	0.004 (0.021)	0.126 (0.065)
2*	0.3	0.4	0.5	0	4.245 (1.050)	0.329 (0.114)	0.981 (0.006)	0.951 (0.040)	−0.058 (0.044)	0.003 (0.020)	0.014 (0.045)
3	0.3	0.4	0.5	0	4.674 (0.962)	0.924 (0.152)	0.952 (0.011)	0.713 (0.072)	−0.182 (0.088)	0.007 (0.021)	0.107 (0.093)
3*	0.3	0.4	0.5	0	4.386 (0.999)	0.701 (0.140)	0.976 (0.007)	0.828 (0.061)	−0.107 (0.063)	0.003 (0.021)	−0.020 (0.055)
4*	0.3	0.4	0.5	0	4.545 (0.931)	1.077 (0.194)	0.974 (0.007)	0.735 (0.072)	−0.142 (0.079)	0.005 (0.021)	−0.061 (0.063)
5*	0.3	0.4	0.5	0	4.847 (0.943)	1.415 (0.228)	0.971 (0.008)	0.675 (0.083)	−0.164 (0.097)	0.006 (0.021)	−0.098 (0.073)
2	0.1	0.4	0.5	0	3.518 (0.830)	0.889 (0.306)	0.978 (0.007)	0.705 (0.116)	−0.055 (0.053)	0.002 (0.021)	0.005 (0.057)
2	0.5	0.4	0.5	0	4.719 (1.174)	0.414 (0.131)	0.959 (0.012)	0.872 (0.063)	−0.103 (0.052)	0.007 (0.022)	0.146 (0.070)
2	0.7	0.4	0.5	0	4.203 (1.047)	0.327 (0.118)	0.913 (0.052)	0.913 (0.052)	−0.075 (0.034)	0.008 (0.023)	0.174 (0.076)
2	0.9	0.4	0.5	0	2.941 (1.747)	0.877 (0.246)	0.890 (0.018)	0.811 (0.098)	−0.064 (0.026)	0.012 (0.031)	0.456 (0.204)
2	0.3	0	0.5	0	2.542 (0.456)	1.092 (0.383)	0.979 (0.006)	0.655 (0.136)	−0.133 (0.080)	0.001 (0.007)	−0.029 (0.050)
2	0.3	0.8	0.5	0	5.255 (1.209)	0.352 (0.119)	0.965 (0.010)	0.892 (0.056)	−0.073 (0.049)	0.003 (0.021)	0.135 (0.071)
2	0.3	0.4	0.3	0	1.608 (0.402)	1.654 (0.520)	0.982 (0.005)	0.537 (0.167)	−0.177 (0.091)	0.002 (0.024)	0.022 (0.050)
2	0.3	0.4	0.7	0	4.967 (1.042)	0.280 (0.090)	0.967 (0.009)	0.918 (0.046)	−0.063 (0.044)	0.002 (0.021)	0.098 (0.053)
2	0.3	0.4	0.5	0.1	4.021 (1.019)	0.538 (0.182)	0.963 (0.010)	0.823 (0.086)	−0.062 (0.073)	0.002 (0.021)	0.092 (0.061)
2	0.3	0.4	0.5	0.25	3.029 (0.828)	0.553 (0.194)	0.969 (0.009)	0.812 (0.087)	0.010 (0.088)	−0.001 (0.020)	0.030 (0.061)
2	0.3	0.4	0.5	0.5	2.270 (0.622)	0.512 (0.162)	0.955 (0.012)	0.850 (0.077)	0.115 (0.110)	−0.009 (0.018)	0.033 (0.072)
2	0.3	0.4	0.5	0.75	1.818 (0.543)	0.507 (0.181)	0.966 (0.011)	0.850 (0.080)	0.212 (0.133)	−0.014 (0.017)	−0.074 (0.082)
2	0.3	0.4	0.5	1	1.712 (0.795)	0.508 (0.209)	0.981 (0.011)	0.854 (0.081)	0.251 (0.153)	−0.019 (0.017)	−0.161 (0.103)

Simulations: High School Friendships

	∅	EN	AL	SC	OLS	∅	EN	AL	SC	OLS	∅	EN	AL	SC	OLS
	n = 15 , T = 50					n = 15 , T = 100					n = 15 , T = 150				
$mse(\hat{\Pi})$	0.000 (0.000)	1.587 (2.121)	0.935 (0.621)	0.738 (0.555)	2.940 (0.410)	0.000 (0.000)	1.090 (0.996)	0.434 (0.269)	0.365 (0.441)	1.190 (0.134)	0.000 (0.000)	0.537 (0.749)	0.305 (0.162)	0.266 (0.166)	0.748 (0.078)
$mse(\hat{W})$	0.000 (0.000)	0.390 (0.551)	1.586 (1.090)	1.285 (0.966)	3.785 (0.662)	0.000 (0.000)	0.275 (0.261)	0.813 (0.526)	0.620 (0.643)	2.124 (0.546)	0.000 (0.000)	0.114 (0.148)	0.618 (0.382)	0.517 (0.371)	1.577 (0.400)
% true 0s	1.000 (0.000)	0.981 (0.026)	0.942 (0.048)	0.943 (0.043)	0.005 (0.005)	1.000 (0.000)	0.990 (0.010)	0.942 (0.044)	0.973 (0.022)	0.007 (0.006)	1.000 (0.000)	0.992 (0.012)	0.941 (0.043)	0.959 (0.044)	0.007 (0.006)
% true 1s	1.000 (0.000)	0.985 (0.038)	0.932 (0.081)	0.952 (0.076)	1.000 (0.001)	1.000 (0.000)	0.996 (0.018)	0.994 (0.021)	0.979 (0.082)	1.000 (0.000)	1.000 (0.000)	1.000 (0.004)	0.999 (0.009)	0.998 (0.018)	1.000 (0.000)
$\hat{\rho}-\rho_0$	0.000 (0.000)	−0.070 (0.099)	−0.273 (0.027)	−0.265 (0.059)	−0.219 (0.079)	0.000 (0.000)	−0.018 (0.062)	−0.276 (0.026)	−0.264 (0.063)	−0.238 (0.087)	0.000 (0.000)	−0.012 (0.050)	−0.270 (0.042)	−0.250 (0.076)	−0.216 (0.107)
$\hat{\beta}-\beta_0$	0.000 (0.000)	0.011 (0.035)	−0.118 (0.111)	−0.033 (0.111)	0.046 (0.082)	0.000 (0.000)	0.008 (0.029)	−0.013 (0.054)	0.026 (0.104)	0.065 (0.061)	0.000 (0.000)	0.004 (0.019)	0.013 (0.047)	0.033 (0.055)	0.061 (0.054)
$\hat{\gamma}-\gamma_0$	0.000 (0.000)	0.035 (0.082)	0.191 (0.168)	0.232 (0.174)	0.461 (0.081)	0.000 (0.000)	−0.014 (0.052)	0.179 (0.136)	0.157 (0.116)	0.373 (0.134)	0.000 (0.000)	−0.004 (0.034)	0.186 (0.122)	0.168 (0.111)	0.316 (0.151)
	n = 30 , T = 50					n = 30 , T = 100					n = 30 , T = 150				
$mse(\hat{\Pi})$	0.000 (0.000)	3.852 (3.095)	0.679 (0.357)	0.546 (0.323)	5.216 (0.584)	0.000 (0.000)	2.156 (2.367)	0.370 (0.128)	0.514 (0.296)	1.449 (0.099)	0.000 (0.000)	3.702 (1.201)	0.271 (0.191)	0.369 (0.299)	0.839 (0.048)
$mse(\hat{W})$	0.000 (0.000)	0.541 (0.448)	1.071 (0.580)	0.919 (0.546)	5.198 (0.754)	0.000 (0.000)	0.184 (0.203)	0.636 (0.233)	0.893 (0.536)	1.777 (0.234)	0.000 (0.000)	0.245 (0.057)	0.477 (0.213)	0.626 (0.420)	1.222 (0.229)
% true 0s	1.000 (0.000)	0.963 (0.030)	0.977 (0.016)	0.957 (0.029)	0.004 (0.002)	1.000 (0.000)	0.989 (0.012)	0.967 (0.017)	0.980 (0.019)	0.007 (0.003)	1.000 (0.000)	0.978 (0.006)	0.964 (0.015)	0.974 (0.019)	0.008 (0.003)
% true 1s	1.000 (0.000)	0.936 (0.068)	0.772 (0.143)	0.868 (0.149)	0.999 (0.004)	1.000 (0.000)	0.989 (0.023)	0.945 (0.049)	0.840 (0.178)	1.000 (0.001)	1.000 (0.000)	0.996 (0.012)	0.985 (0.023)	0.924 (0.139)	1.000 (0.000)
$\hat{\rho}-\rho_0$	0.000 (0.000)	−0.152 (0.097)	−0.268 (0.022)	−0.272 (0.026)	−0.041 (0.090)	0.000 (0.000)	−0.041 (0.053)	−0.274 (0.015)	−0.274 (0.027)	−0.236 (0.057)	0.000 (0.000)	−0.026 (0.042)	−0.276 (0.021)	−0.275 (0.030)	−0.255 (0.058)
$\hat{\beta}-\beta_0$	0.000 (0.000)	0.020 (0.030)	−0.241 (0.140)	−0.073 (0.132)	−0.029 (0.090)	0.000 (0.000)	0.008 (0.018)	−0.080 (0.054)	−0.159 (0.151)	0.033 (0.045)	0.000 (0.000)	0.013 (0.018)	−0.027 (0.038)	−0.078 (0.136)	0.042 (0.035)
$\hat{\gamma}-\gamma_0$	0.000 (0.000)	0.189 (0.182)	0.190 (0.165)	0.276 (0.188)	0.499 (0.022)	0.000 (0.000)	0.002 (0.037)	0.181 (0.131)	0.100 (0.151)	0.474 (0.060)	0.000 (0.000)	−0.003 (0.044)	0.198 (0.113)	0.141 (0.145)	0.432 (0.095)
	n = 50 , T = 50					n = 50 , T = 100					n = 50 , T = 150				
$mse(\hat{\Pi})$	0.000 (0.000)	0.882 (0.814)	0.438 (0.259)	0.377 (0.236)	– (–)	0.000 (0.000)	1.258 (0.192)	0.310 (0.025)	0.552 (0.059)	1.680 (0.644)	0.000 (0.000)	0.552 (0.836)	0.094 (0.100)	0.426 (0.179)	1.010 (0.044)
$mse(\hat{W})$	0.000 (0.000)	0.352 (0.321)	0.649 (0.389)	0.652 (0.407)	– (–)	0.000 (0.000)	0.355 (0.038)	0.561 (0.120)	1.130 (0.157)	1.798 (0.399)	0.000 (0.000)	0.081 (0.123)	0.183 (0.217)	0.897 (0.397)	1.147 (0.113)
% true 0s	1.000 (0.000)	0.967 (0.030)	0.995 (0.004)	0.977 (0.025)	– (–)	1.000 (0.000)	0.950 (0.006)	0.983 (0.007)	0.999 (0.003)	0.244 (0.424)	1.000 (0.000)	0.993 (0.011)	0.990 (0.012)	0.999 (0.003)	0.008 (0.002)
% true 1s	1.000 (0.000)	0.898 (0.101)	0.546 (0.274)	0.679 (0.269)	– (–)	1.000 (0.000)	0.951 (0.032)	0.774 (0.063)	0.368 (0.109)	0.848 (0.274)	1.000 (0.000)	0.993 (0.016)	0.957 (0.053)	0.534 (0.221)	1.000 (0.001)
$\hat{\rho}-\rho_0$	0.000 (0.000)	−0.194 (0.078)	−0.256 (0.027)	−0.274 (0.022)	– (–)	0.000 (0.000)	−0.116 (0.060)	−0.267 (0.020)	−0.262 (0.027)	−0.146 (0.064)	0.000 (0.000)	−0.022 (0.042)	−0.271 (0.021)	−0.263 (0.028)	−0.247 (0.047)
$\hat{\beta}-\beta_0$	0.000 (0.000)	0.015 (0.023)	−0.296 (0.174)	−0.218 (0.155)	– (–)	0.000 (0.000)	0.020 (0.017)	−0.181 (0.051)	−0.391 (0.039)	−0.101 (0.168)	0.000 (0.000)	0.004 (0.009)	−0.040 (0.050)	−0.312 (0.136)	0.011 (0.032)
$\hat{\gamma}-\gamma_0$	0.000 (0.000)	0.264 (0.252)	0.071 (0.128)	0.109 (0.136)	– (–)	0.000 (0.000)	0.271 (0.076)	0.079 (0.139)	−0.110 (0.081)	0.353 (0.263)	0.000 (0.000)	−0.006 (0.026)	0.033 (0.101)	−0.120 (0.090)	0.479 (0.047)
	n = 73 , T = 50					n = 73 , T = 100					n = 73 , T = 150				
$mse(\hat{\Pi})$	0.000 (0.000)	0.083 (0.188)	0.356 (0.133)	0.331 (0.127)	– (–)	0.000 (0.000)	0.064 (0.163)	0.244 (0.014)	0.256 (0.038)	3.447 (0.242)	0.000 (0.000)	0.105 (0.264)	0.161 (0.010)	0.241 (0.102)	1.266 (0.048)
$mse(\hat{W})$	0.000 (0.000)	0.082 (0.183)	0.480 (0.183)	0.682 (0.309)	– (–)	0.000 (0.000)	0.047 (0.118)	0.507 (0.083)	0.618 (0.129)	3.627 (0.637)	0.000 (0.000)	0.044 (0.111)	0.419 (0.127)	0.574 (0.225)	1.254 (0.069)
% true 0s	1.000 (0.000)	0.989 (0.024)	0.998 (0.001)	0.995 (0.005)	– (–)	1.000 (0.000)	0.994 (0.016)	0.991 (0.003)	0.991 (0.004)	0.005 (0.001)	1.000 (0.000)	0.997 (0.008)	0.987 (0.005)	0.992 (0.007)	0.007 (0.001)
% true 1s	1.000 (0.000)	0.946 (0.122)	0.287 (0.268)	0.354 (0.257)	– (–)	1.000 (0.000)	0.980 (0.052)	0.556 (0.055)	0.546 (0.131)	0.999 (0.004)	1.000 (0.000)	0.987 (0.034)	0.765 (0.047)	0.544 (0.286)	1.000 (0.002)
$\hat{\rho}-\rho_0$	0.000 (0.000)	−0.252 (0.063)	−0.252 (0.029)	−0.270 (0.020)	– (–)	0.000 (0.000)	−0.149 (0.066)	−0.258 (0.025)	−0.265 (0.023)	0.026 (0.068)	0.000 (0.000)	−0.032 (0.049)	−0.262 (0.023)	−0.249 (0.032)	−0.195 (0.055)
$\hat{\beta}-\beta_0$	0.000 (0.000)	0.004 (0.013)	−0.351 (0.131)	−0.337 (0.130)	– (–)	0.000 (0.000)	0.003 (0.009)	−0.257 (0.040)	−0.270 (0.051)	−0.039 (0.077)	0.000 (0.000)	0.002 (0.006)	−0.140 (0.036)	−0.240 (0.128)	−0.008 (0.030)
$\hat{\gamma}-\gamma_0$	0.000 (0.000)	0.101 (0.234)	0.013 (0.093)	−0.057 (0.088)	– (–)	0.000 (0.000)	0.039 (0.104)	−0.053 (0.082)	−0.127 (0.084)	0.499 (0.035)	0.000 (0.000)	−0.008 (0.027)	−0.101 (0.090)	−0.137 (0.088)	0.499 (0.014)

Conclusion

- In this project, we study identification of social connections under standard hypothesis in the literature on social interactions.
- Sparsity inducing methods can be used for estimation (though further research is welcome!).
- Empirical illustration: Besley and Case [1995].

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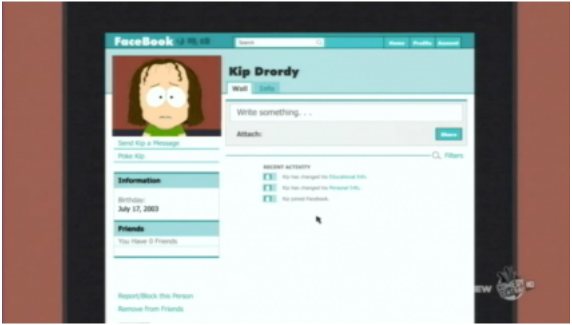
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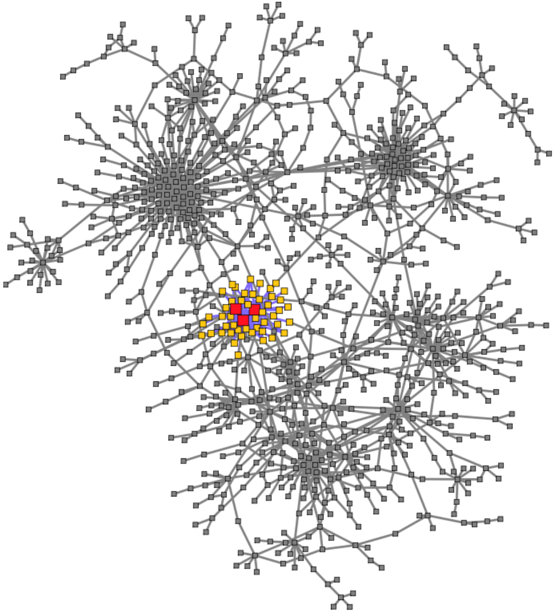
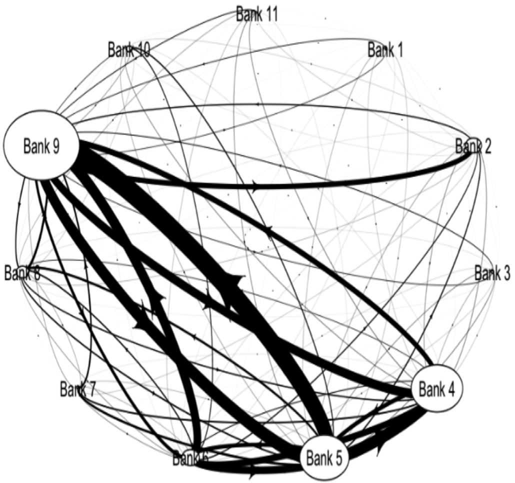
Kip Drordy
Wall
Info
Write something. . .
Attach:
Filters
Send Kip a Message
Poke Kip
Information
Birthday:
July 17, 2003
Friends
You Have 0 Friends
Report/Block this Person
Remove from Friends

Thank You!





Source: Denbee, Julliard, Li and Yuan (2014)

- This system can be obtained from interaction models with maximizing agents with quadratic payoffs.

  - Example: Blume, Brock, Durlauf and Jayaraman [2015]. Bayes-Nash equilibrium with

$$U_i(\mathbf{y}; W) = \left(\alpha + \beta_0 x_i + \gamma_0 \sum_{j\neq i} W_{ij} x_j + z_i\right) y_i + \rho_0 \sum_{j\neq i} W_{ij} y_i y_j - \frac{1}{2} y_i^2.$$

  - Example: Calvo-Armengól, Patacchini and Zenou [2009]. Nash equilibrium with $y_i = e_i + \epsilon_i$ and

$$U_i(e_i, \epsilon; W) = \left(\beta_0 x_i + \gamma_0 \sum_{j\neq i} W_{ij} x_j\right) e_i - \frac{1}{2} e_i^2 + (\alpha W_i \mathbf{1} + \nu_i)\epsilon_i - \frac{1}{2}\epsilon_i^2 + \tilde{\rho}_0 \sum_{j=1}^{N} W_{ij}\epsilon_i\epsilon_j$$

$$\Rightarrow \mathbf{y} = \frac{\alpha}{\tilde{\rho}_0}(\mathbf{I} - \tilde{\rho}_0 W)^{-1}\tilde{\rho}_0 W \mathbf{1} + (\beta_0 \mathbf{I} + \gamma_0 W)\mathbf{x} + (\mathbf{I} - \tilde{\rho}_0 W)^{-1}\nu.$$

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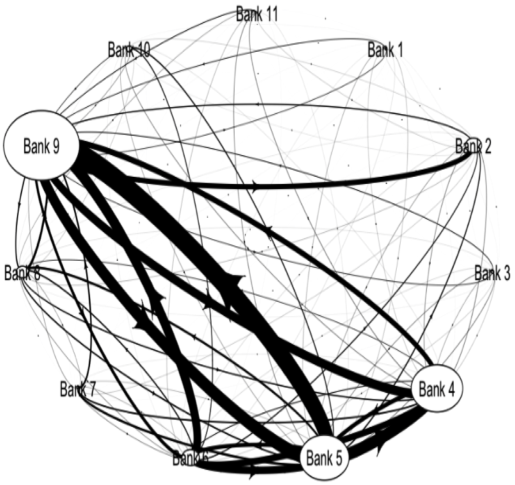

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