(Arbitrage-Free, Practical) Yield-Curve Modeling

November 22, 2016

Some Background

Articles:

Diebold and Li (2006), "Forecasting the Term Structure of Government Bond Yields," *J. Econometrics.*

Diebold, Rudebusch, and Aruoba (2006), "The Macro-Economy and the Yield Curve: A Dynamic Latent Factor Approach," *J. Econometrics*.

Diebold, Li, and Yu (2008), "Global Yield Curve Dynamics and Interactions: A Generalized Nelson-Siegel Approach," *J. Econometrics*.

Christensen, Diebold, and Rudebusch, G.D. (2011), "The Affine Arbitrage-Free Class of Nelson-Siegel Term Structure Models," *Journal of Econometrics*.

Book:

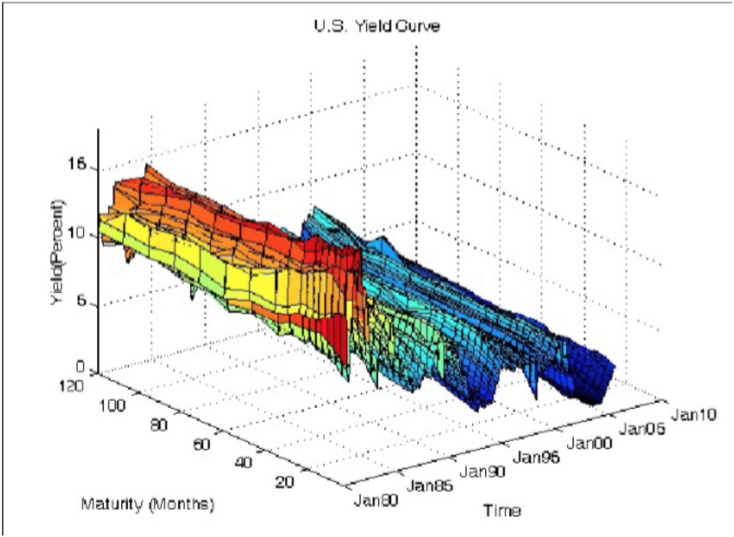
- Diebold and Rudebusch, Yield Curve Modeling and Forecasting: The Dynamic Nelson-Siegel Approach. Princeton University Press (The Tinbergen Lectures), 2013, `http://press.princeton.edu/titles/9895.html`

Definitions and Notation

$$P_t(\tau) = e^{-\tau y_t(\tau)}$$

$$f_t(\tau) = -\frac{P_t'(\tau)}{P_t(\tau)}$$

$$y_t(\tau) = \frac{1}{\tau} \int_0^\tau f_t(u) du$$


U.S. Yield Curve
Yield(Percent)
15
10
5
0
120
100
80
60
40
20
Maturity (Months)
Jan80
Jan85
Jan90
Jan95
Jan00
Jan05
Jan10
Time

A more detailed look:

`http://www.nytimes.com/interactive/2015/03/19/upshot/3d-yield-curve-economic-growth.html?action=click&contentCollection=The%20Upshot&region=Footer&module=WhatsNext&version=WhatsNext&contentID=WhatsNext&moduleDetail=undefined&pgtype=Multimedia`

Incompletely-Satisfying Advances in Arbitrage-Free Modeling

"Cross Sectional Flavor" (e.g. HJM, 1992 *Econometrica*)

"Time Series flavor" (e.g. Vasicek, 1977 JFE)

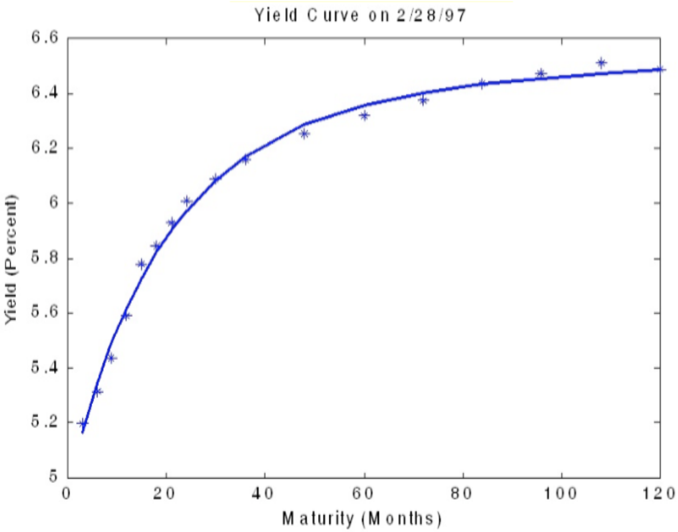
We Take a Classic Yield Curve Model (Nelson-Siegel) and:

- Show that it has a modern interpretation
- Show that it is flexible, fits well, and forecasts well
- Explore a variety of implications and extensions
- Make it arbitrage free (yet still tractable)

Nelson-Siegel (1989)

$$y(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\lambda \tau}}{\lambda \tau} \right) - \beta_3 (e^{-\lambda \tau}) + \varepsilon(\tau)$$

The Model Fits Well



The Federal Reserve Fits it Every Day

`https://www.federalreserve.gov/econresdata/feds/2006/index.htm`

(Scroll down to Gurkaynak, Sack, and Wright)

Actually they fit a slight generalization due to Lars Svensson. We will discuss later.

Dynamic Nelson-Siegel

Before:

$$y(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \beta_3 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) + \varepsilon(\tau)$$

Now:

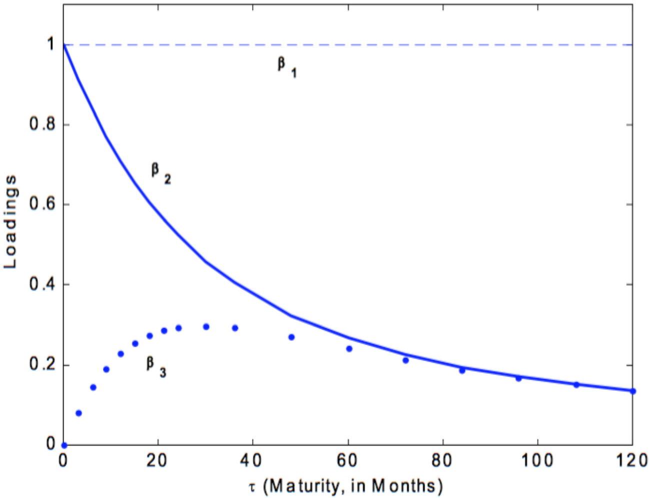
$$y_t(\tau) = L_t + S_t \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_t \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) + \varepsilon_t(\tau)$$

where

$$F_t = AF_{t-1} + \eta_t$$

and $F_t = (L_t, S_t, C_t)'$

Dynamic Nelson-Siegel Factor Loadings



The Model is Flexible

Yield Curve Facts:

(1) Average curve is increasing and concave

(2) Many shapes

(3) Yield dynamics are persistent

(4) Spread dynamics are much less persistent

(5) Short rates are more volatile than long rates

(6) Long rates are more persistent than short rates

DNS is Easily Fit

Two-step method:

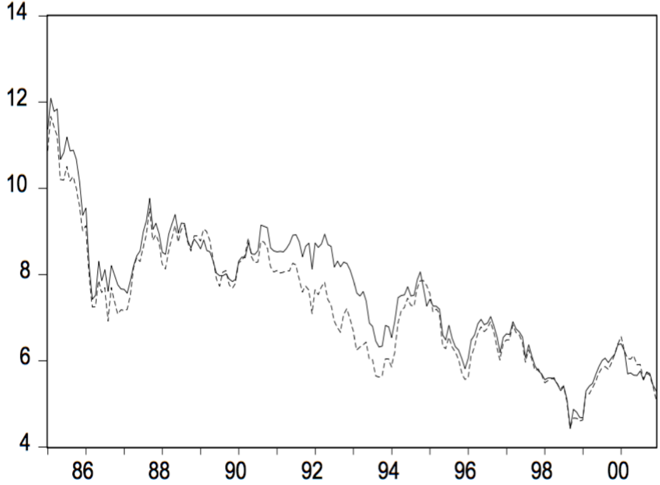
Step 1: Fit separate curves to each time-t cross-section, $t = 1, ..., T$:

$$y_t(\tau) = L_t + S_t \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_t \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) + \varepsilon_t(\tau)$$

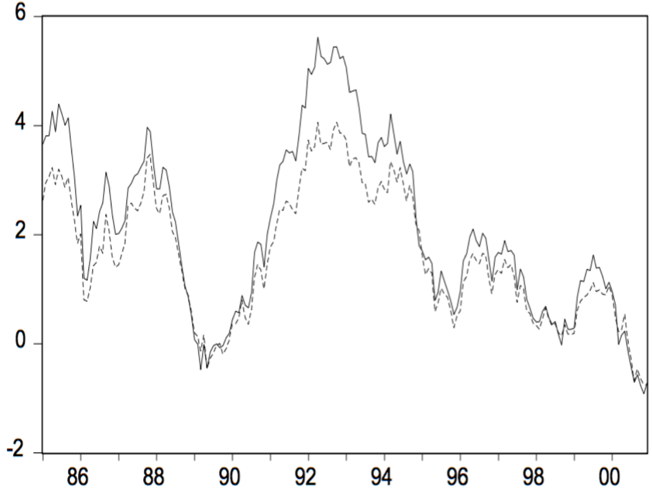
(Just OLS if λ is calibrated.)

Step 2: Fit a dynamic model to the 3-variate series $\{\hat{L}_t, \hat{S}_t, \hat{C}_t\}_{t=1}^{T}$ obtained from Step 1.

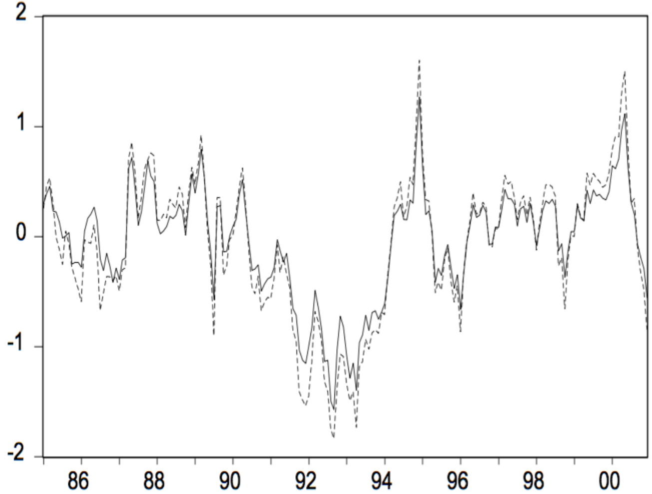
Empirical Level and Estimated Level Factor



Empirical Slope and Estimated Slope Factor



Empirical Curvature and Estimated Curvature Factor



The Model Forecasts Well

Just use the dynamic model for the 3-variate series $\{\hat{L}_t, \hat{S}_t, \hat{C}_t\}_{t=1}^{T}$ obtained from Step 1, to *forecast* $\{L_t, S_t, C_t\}$, which translates into a forecast of the entire curve. We will explore DNS with $AR(1)$ Factors.

$$\hat{y}_{t+h|t}(\tau) = \hat{L}_{t+h|t} + \hat{S}_{t+h|t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \hat{C}_{t+h|t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right)$$

$$\begin{aligned}
\hat{L}_{t+h|t} &= \hat{c}_L + \hat{a}_L \hat{L}_t \\
\hat{S}_{t+h|t} &= \hat{c}_S + \hat{a}_S \hat{S}_t \\
\hat{C}_{t+h|t} &= \hat{c}_C + \hat{a}_C \hat{C}_t
\end{aligned}$$

Random Walk

$$\hat{y}_{t+h|t}(\tau) = y_t(\tau)$$

Slope Regression

$$\hat{y}_{t+h|t}(\tau) - y_t(\tau) = \hat{c}(\tau) + \hat{\gamma}(\tau)(y_t(\tau) - y_t(3))$$

Fama-Bliss Forward Regression

$$\hat{y}_{t+h|t}(\tau) - y_t(\tau) = \hat{c}(\tau) + \hat{\gamma}(\tau)(f_t^h(\tau) - y_t(\tau))$$

Cochrane-Piazzesi Forward Regression

$$\hat{y}_{t+h|t}(\tau) - y_t(\tau) = \hat{c}(\tau) + \hat{\gamma}(\tau) y_t(\tau) + \hat{\gamma}_k(\tau) \sum_{k=1}^{9} f_t^{12k}(12)$$

NOTICE: There is a $\gamma_k(\tau)$ outside the summation (this notation was in the original slide deck). Should $\gamma_k(\tau)$ be inside the summation, or should the k be removed?

AR(1) on Yield Levels

$$\hat{y}_{t+h|t}(\tau) = \hat{c} + \gamma y_t(\tau)$$

VAR(1) on Yield Levels

$$\hat{y}_{t+h|t}(\tau) = \hat{c} + \Gamma y_t(\tau)$$

VAR(1) on Yield Changes

$$\hat{z}_{t+h|t}(\tau) = \hat{c} + \Gamma z_t(\tau)$$

$$z_t = [y_t(3) - y_{t-1}(3), y_t(12) - y_{t-1}(12), y_t(36) - y_{t-1}(36), y_t(60) - y_{t-1}(60), y_t(120) - y_{t-1}(120)]$$

ECM(1) With One Common Trend

$$\hat{z}_{t+h|t}(\tau) = \hat{c} + \Gamma z_t(\tau)$$

$$z_t = [y_t(3) - y_{t-1}(3), y_t(12) - y_{t-1}(3), y_t(36) - y_{t-1}(3), y_t(60) - y_{t-1}(3), y_t(120) - y_{t-1}(3)]$$

ECM(1) With Two Common Trends

$$\hat{z}_{t+h|t}(\tau) = \hat{c} + \Gamma z_t(\tau)$$

$$z_t = [y_t(3) - y_{t-1}(3), y_t(12) - y_{t-1}(12), y_t(36) - y_{t-1}(3), y_t(60) - y_{t-1}(3), y_t(120) - y_{t-1}(3)]$$

ECM(1) With Three Common Trends

$$\hat{z}_{t+h|t}(\tau) = \hat{c} + \Gamma z_t(\tau)$$

$$z_t = [y_t(3) - y_{t-1}(3), y_t(12) - y_{t-1}(12), y_t(36) - y_{t-1}(36), y_t(60) - y_{t-1}(3), y_t(120) - y_{t-1}(3)]$$

1-Month-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(1)$	$\hat{\rho}(12)$
3 months	-0.045	0.170	0.176	0.247	0.017
1 year	0.023	0.235	0.236	0.425	-0.213
3 years	-0.056	0.273	0.279	0.332	-0.117
5 years	-0.091	0.277	0.292	0.333	-0.116
10 years	-0.062	0.252	0.260	0.259	-0.115

Random Walk

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(1)$	$\hat{\rho}(12)$
3 months	0.033	0.176	0.179	0.220	0.053
1 year	0.021	0.240	0.241	0.340	-0.153
3 years	0.007	0.279	0.279	0.341	-0.133
5 years	-0.003	0.276	0.276	0.275	-0.131
10 years	-0.011	0.254	0.254	0.215	-0.145

6-Month-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(6)$	$\hat{\rho}(18)$
3 months	0.083	0.510	0.517	0.301	-0.190
1 year	0.131	0.656	0.669	0.168	-0.174
3 years	-0.052	0.748	0.750	0.049	-0.189
5 years	-0.173	0.758	0.777	0.069	-0.273
10 years	-0.251	0.676	0.721	0.058	-0.288

Random Walk

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(6)$	$\hat{\rho}(18)$
3 months	0.220	0.564	0.605	0.381	-0.214
1 year	0.181	0.758	0.779	0.139	-0.150
3 years	0.099	0.873	0.879	0.018	-0.211
5 years	0.048	0.860	0.861	0.008	-0.249
10 years	-0.020	0.758	0.758	0.019	-0.271

1-Year-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.150	0.724	0.739	-0.288	0.001
1 year	0.173	0.823	0.841	-0.332	-0.004
3 years	-0.123	0.910	0.918	-0.408	0.015
5 years	-0.337	0.918	0.978	-0.412	0.003
10 years	-0.531	0.825	0.981	-0.433	-0.003

Random Walk

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.416	0.930	1.019	-0.118	-0.109
1 year	0.388	1.132	1.197	-0.268	-0.019
3 years	0.236	1.214	1.237	-0.419	0.060
5 years	0.130	1.184	1.191	-0.481	0.072
10 years	-0.033	1.051	1.052	-0.508	0.069

1-Year-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.150	0.724	0.739	-0.288	0.001
1 year	0.173	0.823	0.841	-0.332	-0.004
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10 years	-0.531	0.825	0.981	-0.433	-0.003

Slope Regression

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	NA	NA	NA	NA	NA
1 year	0.896	1.235	1.526	-0.187	-0.024
3 years	0.641	1.316	1.464	-0.212	0.024
5 years	0.515	1.305	1.403	-0.255	0.035
10 years	0.362	1.208	1.261	-0.268	0.042

1-Year-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.150	0.724	0.739	-0.288	0.001
1 year	0.173	0.823	0.841	-0.332	-0.004
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5 years	-0.337	0.918	0.978	-0.412	0.003
10 years	-0.531	0.825	0.981	-0.433	-0.003

Fama-Bliss Forward Regression

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.942	1.010	1.381	-0.046	-0.096
1 year	0.875	1.276	1.547	-0.142	-0.039
3 years	0.746	1.378	1.567	-0.291	0.035
5 years	0.587	1.363	1.484	-0.352	0.040
10 years	0.547	1.198	1.317	-0.403	0.062

1-Year-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.150	0.724	0.739	-0.288	0.001
1 year	0.173	0.823	0.841	-0.332	-0.004
3 years	-0.123	0.910	0.918	-0.408	0.015
5 years	-0.337	0.918	0.978	-0.412	0.003
10 years	-0.531	0.825	0.981	-0.433	-0.003

Cochrane-Piazzesi Forward Regression

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	NA	NA	NA	NA	NA
1 year	-0.162	1.275	1.285	-0.179	-0.079
3 years	-0.377	1.275	1.330	-0.274	-0.028
5 years	-0.529	1.225	1.334	-0.301	-0.021
10 years	-0.760	1.088	1.327	-0.307	-0.020

1-Year-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.150	0.724	0.739	-0.288	0.001
1 year	0.173	0.823	0.841	-0.332	-0.004
3 years	-0.123	0.910	0.918	-0.408	0.015
5 years	-0.337	0.918	0.978	-0.412	0.003
10 years	-0.531	0.825	0.981	-0.433	-0.003

VAR(1) on Yield Levels

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	-0.276	1.006	1.043	-0.219	-0.099
1 year	-0.390	1.204	1.266	-0.322	-0.058
3 years	-0.467	1.240	1.325	-0.345	-0.015
5 years	-0.540	1.201	1.317	-0.348	-0.012
10 years	-0.744	1.060	1.295	-0.328	-0.010

1-Year-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.150	0.724	0.739	-0.288	0.001
1 year	0.173	0.823	0.841	-0.332	-0.004
3 years	-0.123	0.910	0.918	-0.408	0.015
5 years	-0.337	0.918	0.978	-0.412	0.003
10 years	-0.531	0.825	0.981	-0.433	-0.003

VAR(1) on Yield Changes

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.717	1.072	1.290	-0.068	-0.127
1 year	0.704	1.240	1.426	-0.223	-0.041
3 years	0.627	1.341	1.480	-0.399	0.051
5 years	0.559	1.281	1.398	-0.459	0.070
10 years	0.408	1.136	1.207	-0.491	0.072

1-Year-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.150	0.724	0.739	-0.288	0.001
1 year	0.173	0.823	0.841	-0.332	-0.004
3 years	-0.123	0.910	0.918	-0.408	0.015
5 years	-0.337	0.918	0.978	-0.412	0.003
10 years	-0.531	0.825	0.981	-0.433	-0.003

ECM(1) with one Common Trend

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.738	0.982	1.228	-0.163	-0.123
1 year	0.767	1.143	1.376	-0.239	-0.072
3 years	0.546	1.203	1.321	-0.278	-0.013
5 years	0.379	1.191	1.250	-0.278	-0.003
10 years	0.169	1.095	1.108	-0.224	0.009

1-Year-Ahead Forecast Error Analysis, 1990.01 - 2000.12

Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.150	0.724	0.739	-0.288	0.001
1 year	0.173	0.823	0.841	-0.332	-0.004
3 years	-0.123	0.910	0.918	-0.408	0.015
5 years	-0.337	0.918	0.978	-0.412	0.003
10 years	-0.531	0.825	0.981	-0.433	-0.003

ECM(1) with Two Common Trends

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.778	1.037	1.296	-0.175	-0.129
1 year	0.868	1.247	1.519	-0.286	-0.033
3 years	0.586	1.186	1.323	-0.288	-0.034
5 years	0.425	1.155	1.231	-0.304	-0.014
10 years	0.220	1.035	1.058	-0.274	0.015

1-Year-Ahead Forecast Error Analysis, 1990.01 - 2000.12

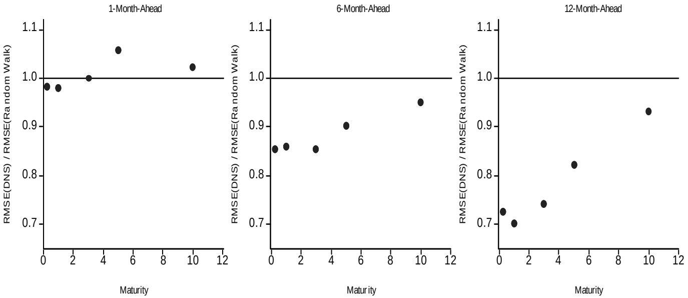
Nelson-Siegel with AR(1) Factor Dynamics

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.150	0.724	0.739	-0.288	0.001
1 year	0.173	0.823	0.841	-0.332	-0.004
3 years	-0.123	0.910	0.918	-0.408	0.015
5 years	-0.337	0.918	0.978	-0.412	0.003
10 years	-0.531	0.825	0.981	-0.433	-0.003

ECM(3) with Three Common Trends

Maturity (τ)	Mean	Std. Dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
3 months	0.810	0.951	1.249	-0.245	-0.082
1 year	0.786	1.261	1.486	-0.248	-0.064
3 years	0.613	1.453	1.577	-0.289	0.028
5 years	0.306	1.236	1.273	-0.246	-0.069
10 years	0.063	1.141	1.143	-0.191	-0.086

Out-of-Sample Forecasting, DNS vs. Random Walk



Incorporating Additional Factors

Svensson (1995):

$$y(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} \right) + \beta_3 \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} - e^{-\lambda_1 \tau} \right) + \beta_4 \left(\frac{1 - e^{-\lambda_2 \tau}}{\lambda_2 \tau} - e^{-\lambda_2 \tau} \right) + \varepsilon(\tau)$$

Dynamic Svensson:

$$y_t(\tau) = L_t + S_t \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} \right) + C_t^1 \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} - e^{-\lambda_1 \tau} \right) + C_t^2 \left(\frac{1 - e^{-\lambda_2 \tau}}{\lambda_2 \tau} - e^{-\lambda_2 \tau} \right) + \varepsilon_t(\tau)$$

Term Structures of Credit Spreads

$$y_t^1(\tau) = L_t^1 + S_t^1 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_t^1 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right)$$

$$y_t^2(\tau) = L_t^2 + S_t^2 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_t^2 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right)$$

$$\left(y_t^1(\tau) - y_t^2(\tau)\right) = \left(L_t^1 - L_t^2\right) + \left(S_t^1 - S_t^2\right) \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \left(C_t^1 - C_t^2\right) \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right)$$

Generalized Duration

Discount Bond:

$$-\frac{dP_t(\tau)}{P_t(\tau)} = \tau dL_t + dS_t\left(\frac{1-e^{-\lambda\tau}}{\lambda}\right) + dC_t\left(\frac{1-e^{-\lambda\tau}}{\lambda} - e^{-\lambda\tau}\right)$$

Coupon Bond:

$$-\frac{dP_{ct}(\tau)}{P_{ct}(\tau)} = \sum_{i=1}^{n}(w_i x_i \tau_i)dL_t + dS_t\sum_{i=1}^{n}\left(w_i x_i \frac{1-e^{-\lambda\tau_i}}{\lambda}\right) + dC_t\sum_{i=1}^{n}\left(w_i x_i \frac{1-e^{-\lambda\tau_i}}{\lambda} - w_i x_i \tau_i e^{-\lambda\tau}\right)$$

Dynamic Nelson-Siegel has a Natural State-Space Structure

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\tau_1\lambda}}{\tau_1\lambda} & \frac{1-e^{-\tau_1\lambda}}{\tau_1\lambda} - e^{-\tau_1\lambda} \\ 1 & \frac{1-e^{-\tau_2\lambda}}{\tau_2\lambda} & \frac{1-e^{-\tau_2\lambda}}{\tau_2\lambda} - e^{-\tau_2\lambda} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\tau_N\lambda}}{\tau_N\lambda} & \frac{1-e^{-\tau_N\lambda}}{\tau_N\lambda} - e^{-\tau_N\lambda} \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} + \begin{pmatrix} \varepsilon_t(\tau_1) \\ \varepsilon_t(\tau_2) \\ \vdots \\ \varepsilon_t(\tau_N) \end{pmatrix}$$

Compactly

$$y_t = c + Z(\tau)F_t + \varepsilon_t$$

$$F_t = AF_{t-1} + \eta_t$$

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right)$$

where $F_t^{'} = (L_t, S_t, C_t)$

State-Space Representation

- Powerful framework
- One-step exact maximum-likelihood estimation
- Optimal extraction of latent factors
- Optimal point and interval forecasts

More

- Heteroskedasticity, confidence tunnels, density forecasts
- Regime switching
- Bayesian estimation and analysis

Inclusion of Macro and Policy Variances

$$y_t = c + Z(\tau)F_t + \varepsilon_t$$

$$F_t = AF_{t-1} + \eta_t$$

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix} \right)$$

where $F_t^{'} = (L_t, S_t, C_t, CU_t, FFR_t, INFL_t)$

One-Step vs Two-step

One-step:

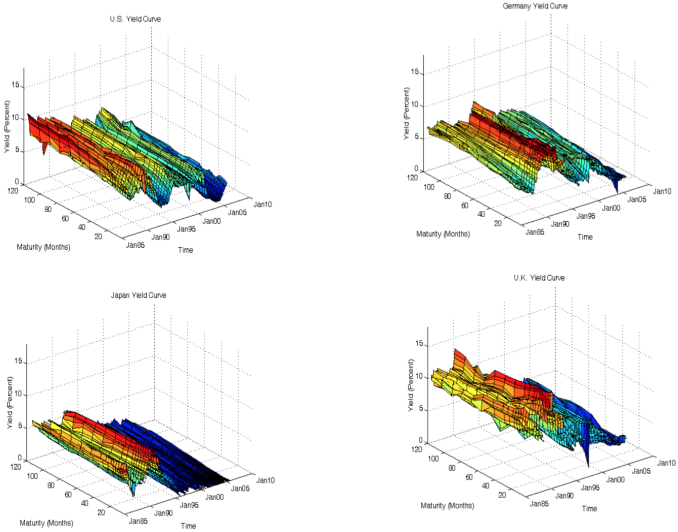
- λ calibrated
- λ fixed but estimated
- Time-varying λ (structured)

Two-step:

- λ calibrated
- Time-varying λ (unstructured)

Two-step proves appealing for tractability
Fixed λ linked to absence of arbitrage

Yied Curves Across Countries and Time



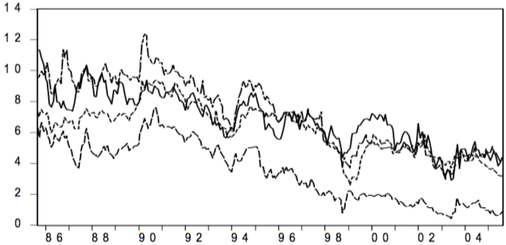
Single-Country Models

$$y_i(\tau) = L_i + S_i \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_i \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) + v_i(\tau)$$

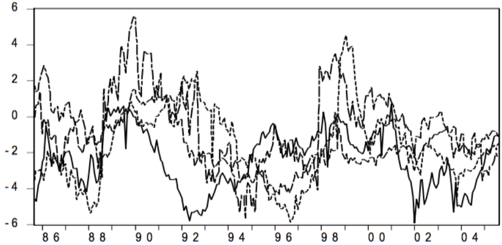
$$y_{it}(\tau) = L_{it} + S_{it} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_{it} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) + v_{it}(\tau)$$

$$y_{it}(\tau) = L_{it} + S_{it} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + v_{it}(\tau)$$

Estimated Country Level Factors



Estimated Country Slope Factors



Multi-Country Model, I

$$Y_t(\tau) = L_t + S_t \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + V_t(\tau)$$

$$\begin{pmatrix} L_t \\ S_t \end{pmatrix} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} L_{t-1} \\ S_{t-1} \end{pmatrix} + \begin{pmatrix} U_t^L \\ U_t^S \end{pmatrix}$$

Multi-Country Model, II

$$l_{it} = \alpha_i^l + \beta_i^l L_t + \varepsilon_{it}^l$$

$$s_{it} = \alpha_i^s + \beta_i^s S_t + \varepsilon_{it}^s$$

$$\begin{pmatrix} \varepsilon_{it}^l \\ \varepsilon_{it}^s \end{pmatrix} = \begin{pmatrix} \phi_{i,11} & \phi_{i,12} \\ \phi_{i,21} & \phi_{i,22} \end{pmatrix} \begin{pmatrix} \varepsilon_{i,t-1}^l \\ \varepsilon_{i,t-1}^s \end{pmatrix} + \begin{pmatrix} u_{it}^L \\ u_{it}^S \end{pmatrix}$$

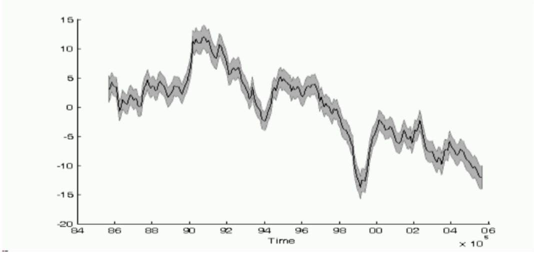
State Space Representation

$$
\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = A \begin{pmatrix} \alpha_1^l \\ \alpha_2^l \\ \vdots \\ \alpha_N^c \end{pmatrix} + B \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} + A \begin{pmatrix} \varepsilon_{1t}^l \\ \varepsilon_{1t}^s \\ \vdots \\ \varepsilon_{Nt}^c \end{pmatrix} + \begin{pmatrix} v_{1t}^l \\ v_{1t}^s \\ \vdots \\ v_{Nt}^c \end{pmatrix}
$$

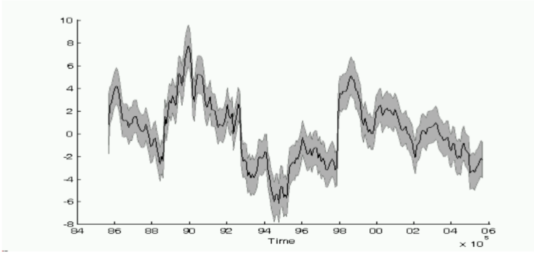
$$A = \begin{pmatrix} 1 & \left(\frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1}\right) & \left(\frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1}\right) & 0 & \dots & 0 \\ 1 & \left(\frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2}\right) & \left(\frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2}\right) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \left(\frac{1-e^{-\lambda\tau_J}}{\lambda\tau_J}\right) & \left(\frac{1-e^{-\lambda\tau_J}}{\lambda\tau_J} - e^{-\lambda\tau_J}\right) \end{pmatrix}$$

$$B = \begin{pmatrix} \beta_1^l & \beta_1^s \left(\frac{1 - e^{-\lambda \tau_1}}{\lambda \tau_1} \right) & \beta_1^c \left(\frac{1 - e^{-\lambda \tau_1}}{\lambda \tau_1} - e^{-\lambda \tau_1} \right) \\ \beta_1^l & \beta_1^s \left(\frac{1 - e^{-\lambda \tau_2}}{\lambda \tau_2} \right) & \beta_1^c \left(\frac{1 - e^{-\lambda \tau_2}}{\lambda \tau_2} - e^{-\lambda \tau_2} \right) \\ \dots & \dots & \dots \\ \beta_N^l & \beta_N^s \left(\frac{1 - e^{-\lambda \tau_J}}{\lambda \tau_J} \right) & \beta_N^c \left(\frac{1 - e^{-\lambda \tau_J}}{\lambda \tau_J} - e^{-\lambda \tau_J} \right) \end{pmatrix}$$

Extracted Global Level Factor



Extracted Global Slope Factor



Two Approaches to Yield Curves

I. Dynamic Nelson-Siegel (Diebold-Li,...)

- Popular in practice
- Level, slope, curvature
- Easy to estimate, with good fits and forecasts
- *But*, doe snot enforce absence of arbitrage

II. Affine Equilibrium (Duffie-Kan,...)

- Popular in theory
- Enforces absence of arbitrage
- *But*, difficult to estimate and evaluate

Affine Equilibrium (Duffie-Kan, 1996,...)

Risk-neutral dynamics: $r_t = \rho_0 + \rho_1^{'} X_t,$

where $dX_t = K(\theta - X_t)d_t + \Sigma dW_t$

Freedom from arbitrage requires:

$$y_t(\tau) = -\frac{1}{\tau} B(\tau)^{'} X_t - \frac{1}{\tau}\Pi(\tau),$$

where $B(\tau)$ and $\Pi(\tau)$ solve Duffie-Kan ODEs

Making DNS Arbitrage-Free

Set:

$$B(\tau)' = \left(-\tau, -\left(\frac{1-e^{-\lambda\tau}}{\lambda}\right), -\left(\frac{1-e^{-\lambda\tau}}{\lambda} - \tau e^{-\lambda\tau}\right)\right)$$

and find ρ_1 and K s.t. Duffie-Kan ODE is satisfied.

Duffie-Kan ODE:

$$\frac{dB(\tau)}{d\tau} = \rho_1 + K^{'} B(\tau)$$

Solution:

$$\rho_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix}$$

Proposition (Arbitrage-Free Nelson Siegel)

Suppose that the isntantaneous risk-free rate is

$$r_t = X_{1t} + X_{2t}$$

with risk-neutral state dynamics:

$$\begin{pmatrix} dX_{1t} \\ dX_{2t} \\ dX_{3t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \left[\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} - \begin{pmatrix} X_{1t} \\ X_{2t} \\ X_{3t} \end{pmatrix} \right] dt + \Sigma dW_t$$

Then yield dynamics are abitrage-free:

$$y_t(\tau) = -\frac{1}{\tau} B(\tau)^{'} X_t - \frac{1}{\tau} \Pi(\tau),$$

where $B(\tau)$ has Nelson-Siegel form:

$$B(\tau)^{'} = \left(-\tau, -\left(\frac{1 - e^{-\lambda\tau}}{\lambda} \right), -\left(\frac{1 - e^{-\lambda\tau}}{\lambda} - \tau e^{-\lambda\tau} \right), \ldots \right)$$

...and where the yield adjustment term is:

$$\frac{d\Pi(\tau)}{dt} = -B(\tau)^{'} K\theta - \frac{1}{2}\sum_{j=1}^{3}(\Sigma^{'} B(s)B(s)^{'}\Sigma)_{j,j}$$

$$\Pi(\tau) = (K\theta)^{'} \int_{0}^{\tau} B(s)ds + \frac{1}{2}\sum_{j=1}^{3}\int_{0}^{\tau}(\Sigma^{'} B(s)B(s)^{'}\Sigma)_{j,j}ds$$

Observations

$$y_t(\tau) = L_t + S_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) - \frac{\Pi(\tau)}{\tau}$$

- Three-factor "Gaussian" model
- X_{1t} has a unit-root
- X_{2t} reverts to a stochastic mean
- X_{3t} *is* that stochastic mean
- Same λ in slope and curvature dynamics

Factor Dynamics Under the Physical Measure

Essentailly affine risk premium (Duffee, 2002):

$$dW_t = dW_t^{'} + \Gamma_t dt$$
$$\Gamma_t = \gamma_0 + \gamma_1 X_t$$

Same dynamic structure under the P measure:

$$dX_t = K^p(\theta^p - X_t)dt + \Sigma dW_t^p$$

Yield Data and Model Estimation

January 1987 - December 2002

Sixteen maturities (in years):
$.25, .5, .75, 1, 1.5, 2, 3, 4, 5, 7, 8, 9, 10, 15, 20, 30$

Linear, Gaussian state space structure
$\Rightarrow$ Estimate using Kalman filter

Models (e.g. Independent Factor)

1. DNS Independent
(Diebold-Li)

$$y_t = Z(\tau)F_t + \varepsilon_t$$

$$F_t = AF_{t-1} + \Sigma\eta_t$$

A, Σ diagonal

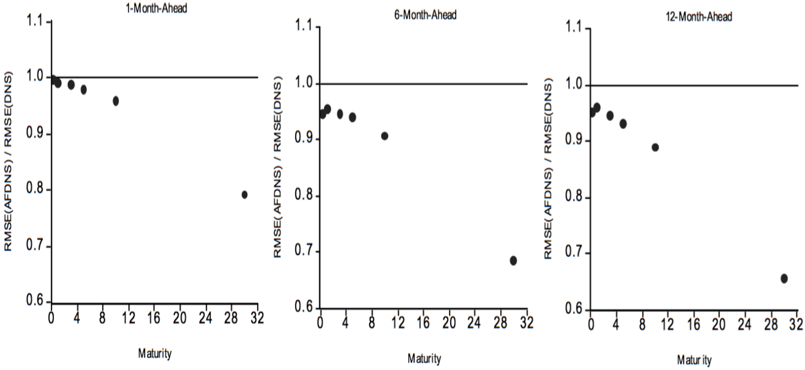
1. AFNS Independent

$$y_t = -\frac{1}{\tau} B(\tau)^{'} X_t - \frac{1}{\tau} \Pi(\tau) + \varepsilon_t$$

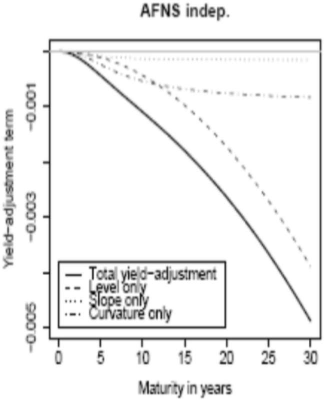
$$dX_t = K^p(\theta^p - X_t) d_t + \Sigma dW_t^p$$

K^p, Σ diagonal

Out-of-Sample Forecasting, Independent Case, AFNS vs DNS



Yield Adjustment Term, $-\frac{\Pi(\tau)}{\tau}$



	Forecast Horizon in Months	
Model	h=6	h=12
<u>3-Month Yield</u>		
DNS_{indep}	96.87	173.39
DNS_{corr}	$\boxed{87.43}$	166.91
AFNS_{indep}	91.63	164.70
AFNS_{corr}	88.49	$\boxed{161.94}$
<u>1-Year Yield</u>		
DNS_{indep}	103.25	170.85
DNS_{corr}	102.71	173.14
AFNS_{indep}	$\boxed{98.49}$	$\boxed{163.46}$
AFNS_{corr}	98.63	165.50
<u>3-Year Yield</u>		
DNS_{indep}	92.22	135.24
DNS_{corr}	99.55	145.82
AFNS_{indep}	$\boxed{86.99}$	$\boxed{126.95}$
AFNS_{corr}	90.64	135.79
<u>5-Year Yield</u>		
DNS_{indep}	87.87	122.09
DNS_{corr}	94.95	132.40
AFNS_{indep}	$\boxed{82.41}$	$\boxed{112.85}$
AFNS_{corr}	88.15	124.87
<u>10-Year Yield</u>		
DNS_{indep}	74.71	105.02
DNS_{corr}	79.48	112.37
AFNS_{indep}	$\boxed{67.48}$	$\boxed{92.39}$
AFNS_{corr}	90.21	123.89
<u>30-Year Yield</u>		
DNS_{indep}	71.35	96.90
DNS_{corr}	72.71	99.68
AFNS_{indep}	$\boxed{48.06}$	$\boxed{61.97}$
AFNS_{corr}	71.38	96.75

	Forecast Horizon in Months	
Maturity/Model	h=6	h=12
<u>6-Month Yield</u>		
Random Walk	40.0	48.4
Preferred $A_0(3)$	36.5	42.1
AFNS$_{indep}$	$\boxed{34.0}$	$\boxed{41.3}$
<u>2-Year Yield</u>		
Random Walk	65.2	76.2
Preferred $A_0(3)$	56.6	60.0
AFNS$_{indep}$	$\boxed{54.3}$	$\boxed{59.0}$
<u>10-Year Yield</u>		
Random Walk	66.9	81.5
Preferred $A_0(3)$	63.6	73.8
AFNS$_{indep}$	$\boxed{60.7}$	$\boxed{71.8}$

Incorporating Additional Factors

Svensson (1995):

$$y(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} \right) + \beta_3 \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} - e^{-\lambda_1 \tau} \right) + \beta_4 \left(\frac{1 - e^{-\lambda_2 \tau}}{\lambda_2 \tau} - e^{-\lambda_2 \tau} \right) + \varepsilon(\tau)$$

Dynamic Svensson:

$$y_t(\tau) = L_t + S_t \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} \right) + C_t^1 \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} - e^{-\lambda_1 \tau} \right) + C_t^2 \left(\frac{1 - e^{-\lambda_2 \tau}}{\lambda_2 \tau} - e^{-\lambda_2 \tau} \right) + \varepsilon_t(\tau)$$

Generalized Dynamic Svensson:

$$y_t(\tau) = L_t + S_t^1 \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} \right) + S_t^2 \left(\frac{1 - e^{-\lambda_2 \tau}}{\lambda_2 \tau} \right) + C_t^1 \left(\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} - e^{-\lambda_1 \tau} \right) + C_t^2 \left(\frac{1 - e^{-\lambda_2 \tau}}{\lambda_2 \tau} - e^{-\lambda_2 \tau} \right) + \varepsilon_t(\tau)$$

Arbitrage-Free Dynamic Nelson-Siegel-Svensson

(Christensen, Diebold, and Rudebusch)

If:

$$r_t = X_{1t} + X_{2t} + X_{3t}$$

$$\begin{pmatrix} dX_{1t} \\ dX_{2t} \\ dX_{3t} \\ dX_{4t} \\ dX_{5t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & -\lambda_1 & 0 \\ 0 & 0 & \lambda_2 & 0 & -\lambda_2 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{pmatrix} \left[\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{pmatrix} - \begin{pmatrix} X_{1t} \\ X_{2t} \\ X_{3t} \\ X_{4t} \\ X_{5t} \end{pmatrix} \right] + \Sigma dW_t$$

Then...

Conclusions

- AFNS delivers *tractable* rigorous modeling
- AFNS delivers *rigorous* tractable modeling
  - AF restrictions may help forecasts

Moving Forward...

- Is DNS/AFNS “special”?
- Zero lower bound (ZLB)
- Spanning

Moving Forward: Is DNS/AFNS "special"?

- Old world: Maximally-flexible $A_0(3)$
  (Not even identified – Hamilton, Wu, et al.) See Singleton's book.
- New world: Joslin-Singleton-Zhu (JSZ) model class
  (Flexible, tractable, and identified, and AFNS is a special case)
- In between (?): AFNS

  Observations:

  - NS motivation (See Diebold-Rudebusch book)
  - Krippner approximation theory (See Krippner JAE paper)
  - JSZ fail to reject AFNS (p-value $\approx .5$) (See JSZ RFS paper)
  - So DNS/AFNS constraints are benefits, not costs

Moving Forward: The Zero Lower Bound

Creal, Koopman, and Lucas (2013): "Generalized Vector Autoregressive Score Models with Applications". *Journal of Applied Econometrics*

Harvey (2013): "Dynamic Models for Volatility and Heavy Tails". *Econometric Society Monographs*

Duffie Kan (1996): "A Yield Factor Model of Interest Rates". *Mathematical Finance*

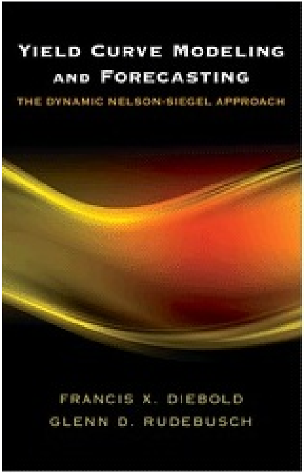
Cox, Ingersoll, and Ross (1985): "A Theory of the Term Structure of Interest Rates". *Econometrica*

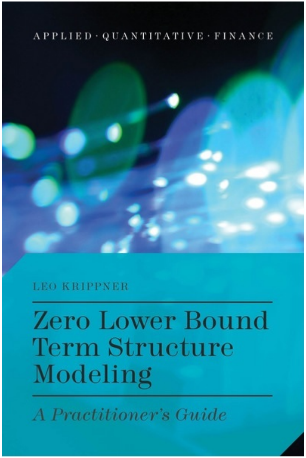
Gourieroux and Jasiak (2006): "Autoregressive Gamma Process". *Journal of Forecasting*

Monfort et al (2015): "Staying at Zero with Affine Processes"

Favorite Books

No ZLB in the first, all ZLB in the second.





Lots of Constrained Series in Finance

"Soft" barriers:

- Exchange rate target zones
- Inflation corridors

"Hard" barriers:

- Volatilities: e.g., asset returns
- Durations: e.g., intertrade
- Event counts: e.g., bankruptcies
- Nominal bond yields (whether ZLB or ELB)

Lots of Associated Constrained Stochastic Processes Studied in Financial Econometrics

GARCH, stochastic volatility, ACD, GAS, MEM, more...
(Creal, Koopman, and Lucas, 2013; Harvey, 2013)

What About Bond Yields?

Duffie-Kan (1996) Gaussian affine term structure model (GATSM):

State x_t is an affine diffusion under the risk-neutral measure:

$$dx_t = K(\theta - x_t)dt + \Sigma\, dW_t$$

Instantaneous risk-free rate r_t is affine in x_t:

$$r_t = \rho_0 + \rho_1' x_t$$

Duffie-Kan arbitrage-free result:

$$y_t(\tau) = -\frac{1}{\tau} B(\tau)' x_t - \frac{1}{\tau} C(\tau)$$

– Arbitrage-free
– Analytic closed-form solution
– But fails to enforce the ZLB
(sometimes irrelevant; sometimes relevant)

Constrained Processes for Bonds

- Square root: $dx_t = k(\theta - x_t)\,dt + \sigma\sqrt{x_t}\,dW_t$
  (Cox, Ingersol and Ross, 1976)
- Others: lognormal, quadratic
- Crude: $x_t = max(\mu(1-\rho) + \rho x_{t-1} + \varepsilon_t,\ 0)$
- Autoregressive gamma (ARG)
  (Gourieroux and Jasiak, 2006)
- Autoregressive gamma with mass point at 0 ($ARG0$)
  (MPRR, 2015)

$ARG(1)$

x_t is an $ARG(1)$ process if
$x_t|x_{t-1}$ is distributed non-central gamma with:

- Non-centrality parameter βx_{t-1}
- Scale parameter $c > 0$
- Degree of freedom parameter $\delta > 0$

– Non-negative (obvious)

– Diffusion limit is CIR (not obvious)

An Alternative $ARG(1)$ Characterization

If $x_t \sim ARG(1)$, then

$$x_t | z_t \sim Gamma(\delta + z_t, c)$$

$$z_t | x_{t-1} \sim Poisson(\beta x_{t-1})$$

This will be useful later.

$ARG(1)$ Conditional Moments

$$E(x_t|x_{t-1}) = \rho x_{t-1} + c\delta$$

$$V(x_t|x_{t-1}) = 2c\rho x_{t-1} + c^2\delta$$

where $\rho = \beta c > 0$

$ARG(1)$ Conditional Over-Dispersion

Recall that conditional over-dispersion is said to exist if and only if

$$V(x_t|x_{t-1}) > (E(x_t|x_{t-1}))^2.$$

The stationary $ARG(1)$ process with $\delta < 1$ has:

- marginal over-dispersion.
- conditional under- or over-dispersion, depending on the value of x_{t-1}.

Remark: The ACD model always has conditional over-dispersion.

$ARG(1)$ Approach

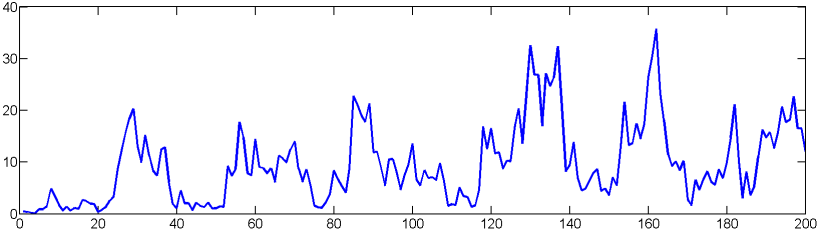
$$x_t | z_t \sim Gamma(\delta + z_t, c)$$

$$z_t | x_{t-1} \sim Poisson(\beta x_{t-1})$$

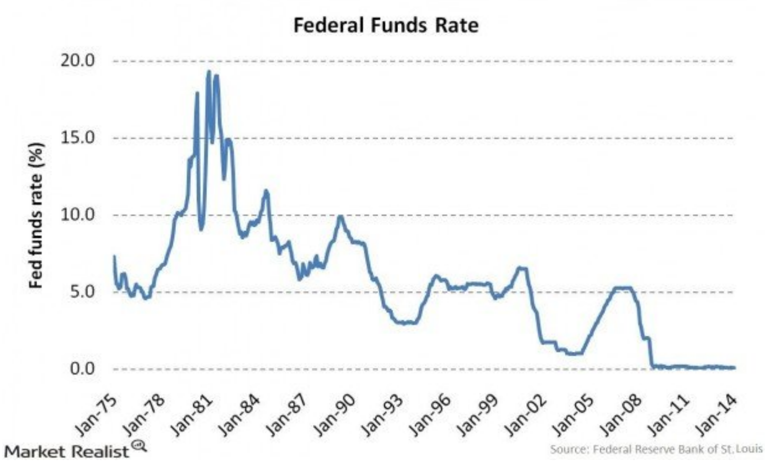
1. Arbitrage-free
2. Analytic closed-form solution
3. Respects the ZLB

End of story? No!

Here's a Simulated $ARG(1)$ Realization...



...But Here's What Really Happens



Extension

Creal (2013) considers the non-linear state space model,

Measurement

$$y_t | h_t, x_t \sim p(y_t | h_t, x_t; \theta)$$

Transition

$$\begin{aligned} h_t | z_t &\sim Gamma(\delta + z_t, c) \\ z_t | h_{t-1} &\sim Poisson(\rho h_{t-1}), \end{aligned}$$

where x_t is an exogenous regressor.

- When $y_t | h_t, x_t = h_t$, the process is ARG.
- Various other models fit this form.

Example 1: Stochastic volatility models

Measurement

$$y_t | h_t, x_t = \mu + x_t \beta + \sqrt{h_t} e_t, \quad e_t \sim N(0,1)$$

Transition

$$\begin{aligned} h_t | z_t &\sim Gamma(\delta + z_t, c) \\ z_t | h_{t-1} &\sim Poisson(\rho h_{t-1}) \end{aligned}$$

Example 2: Stochastic duration and intensity models

Measurement

$$y_t | h_t, x_t \sim \mathit{Gamma}\left(\alpha,\ h_t \exp(x_t \beta)\right)$$

Transition

$$\begin{aligned} h_t | z_t &\sim \mathit{Gamma}(\delta + z_t, c) \\ z_t | h_{t-1} &\sim \mathit{Poisson}(\rho h_{t-1}) \end{aligned}$$

Example 3: Stochastic count models

Measurement

$$y_t | h_t, x_t \sim \mathit{Poisson}\left(h_t \exp(x_t \beta)\right)$$

Transition

$$\begin{aligned} h_t | z_t &\sim \mathit{Gamma}(\delta + z_t, c) \\ z_t | h_{t-1} &\sim \mathit{Poisson}(\rho h_{t-1}) \end{aligned}$$

$ARG0(1)$

Recall that if $x_t \sim ARG(1)$, then

$$
\begin{aligned}
x_t | z_t &\sim Gamma(\delta + z_t, c) \\
z_t | x_{t-1} &\sim Poisson(\beta x_{t-1})
\end{aligned}
$$

$ARG0(1)$ easier to characterize this way
(as opposed to characterizing its conditional density).
If $x_t \sim ARG0(1)$, then

$$
\begin{aligned}
x_t | z_t &\sim Gamma(z_t, c) \\
z_t | x_{t-1} &\sim Poisson(\alpha + \beta x_{t-1})
\end{aligned}
$$

- $ARG0$ takes $\delta \to 0$, which makes $x_t = 0$ a mass point.
  (As $\delta \to 0$, $G(\delta, c) \to$ Dirac's delta.)
- Introduces α, which governs probability of escaping the ZLB.
  (Note that $\alpha = 0 \implies x_t = 0$ is an absorbing state.)

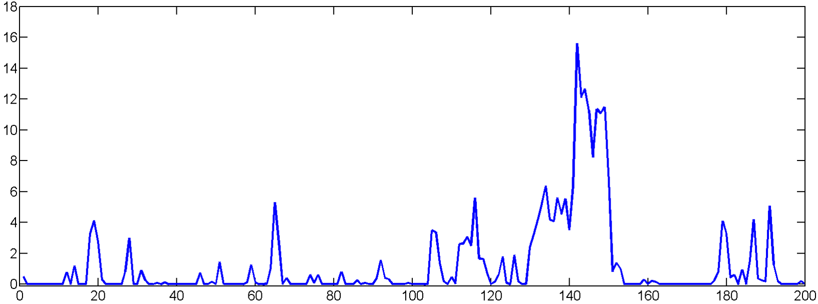
$ARG0(1)$ Conditional Moments

$$E(x_t|x_{t-1}) = \alpha c + \rho x_{t-1}$$

$$V(x_t|x_{t-1}) = 2c^2\alpha + 2c\rho x_{t-1}$$

where $\rho = \beta c > 0$

Simulated $ARG0(1)$ Realization



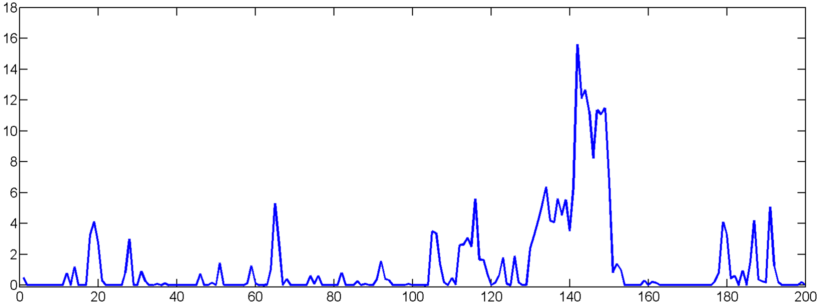
$ARG0$ Approach

$$x_t | z_t \sim Gamma(z_t, c)$$
$$z_t | x_{t-1} \sim Poisson(\alpha + \beta x_{t-1})$$

1. Arbitrage-free
2. Simple (closed-form)
3. Respects the ZLB

End of story?

But Are we Really Happy with Realizations Like This?



What About "Crude" Approaches?

One crude approach:

$$x_t = max(\mu(1-\rho) + \rho x_{t-1} + \varepsilon_t,\ 0)$$

– Can stay at 0, but not for long
– Still not very appealing

A different "crude" approach:

$$x_t = max(x_{s,t}, 0)$$

$$x_{s,t} = \mu(1-\rho) + \rho x_{s,t-1} + \varepsilon_t$$

– Actually not crude at all

Shadow-Rate Approach (Shadow/ZLB GATSM)

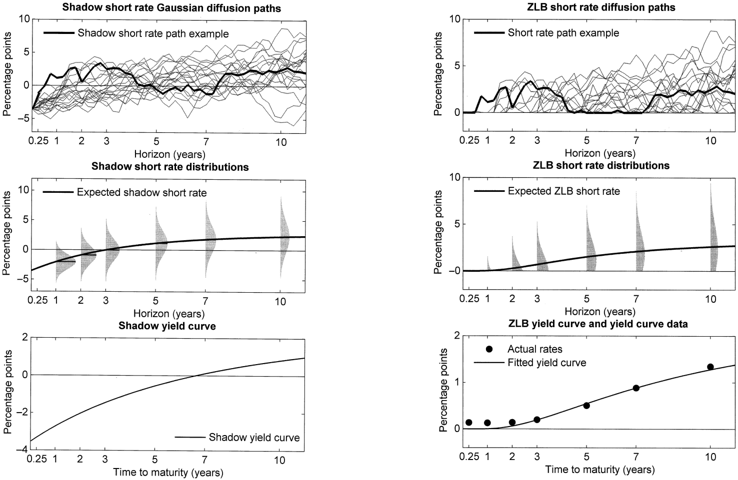
Where would the rate be if it could go negative
(i.e., if money could pay negative interest)?

Shaddow rate: $x_{s,t} = \mu(1-\rho) + \rho x_{s,t-1} + \varepsilon_t$

Observed rate: $x_t = max(x_{s,t}, 0)$

1. Arbitrage-free
2. Simple (simulation)
3. Respects the ZLB

Shadow Rates and ZLB Rates



Shadow-Rate Approach (Shadow/ZLB GATSM)

$$x_{s,t} = \mu(1-\rho) + \rho x_{s,t-1} + \varepsilon_t$$

$$x_t = max(x_{s,t}, 0)$$

1. Arbitrage-free
2. Simple (simulation)
3. Respects the ZLB
4. Sample path feature probabilities (e.g., lift-off from ZLB)
5. Sample path integral densities (e.g., effective stimulus)

   But Monfort et al. could also do points 4 and 5...

6. Shadow rate path and shadow yield curve

Final Thoughts on Relative Performance

Much boils down to:

– Value of the shadow rate path and shadow yield curve

– Views about "simplicity"

I tip slightly toward shadow/ZLB GATSM

Interesting question:
With appropriate constraints on the Gamma and Poisson processes, can Monfort et al. "replicate" a shadow/ZLB GATSM, but without the mechanism of shadow short rates and the shadow yield curve?

ZLB Web Sites

FRB Atlanta shadow rate,
`https://www.frbatlanta.org/cqer/research/shadow_rate.aspx?panel=1`

Krippner book, `http://www.palgrave.com/us/book/9781137408327`

Moving Forward: Spanning

- Spanning: Time-t yield curve has embedded in it all time-t macro information of relevance for predicting future yields.
- Simple argument: Efficient markets
- Does spanning hold in the data?
  - Several rejections (e.g., Ludvigson-Ng, Joslin-Priebsch-Singleton)
  - Several reconciliations (e.g., Bauer-Hamilton, Bauer-Rudebusch)