Suppose that seasonally-adjusted U.S. quarterly real GDP growth, y_t, follows a covariance stationary $AR(2)$ process with weak white noise innovations.

1. Provide a detailed characterization of y_t via its Wold decomposition. Is it a complete characterization? Are the innovations associated with its Wold representation uncorrelated? Independent? Gaussian?

2. What is the unconditional innovation variance of y_t? Must it be finite? What is the conditional innovation variance of y_t? Is it necessarily smaller than the unconditional variance?

   Now suppose that y_t follows a covariance stationary $AR(2)$ process with conditionally-Gaussian $GARCH(1,1)$ innovations.

3. Write down the full conditionally-Gaussian $AR(2)-GARCH(1,1)$ process for y_t. What must be true of the AR and $GARCH$ parameters to ensure covariance stationarity? How would you modify the process to allow the response of volatility to depend on the signs of innovations? Write down the modified process. Why/when might such a modification be useful?

4. How would you estimate the model by Gaussian MLE, and what are the properties of the resulting estimates?

   Now suppose instead that you don't *know* that y_t follows a covariance stationary $AR(2)$ process with conditionally-Gaussian $GARCH(1,1)$ innovations, but you *think* that it does, so you fit the $AR(2)-GARCH(1,1)$ model.

5. How would you diagnose the specification adequacy (as regards conditional mean dynamics, conditional variance dynamics, and conditional density) of your fitted $AR(2) - GARCH(1,1)$ model?

6. Get the data, do the fitting, and do the diagnosis. Discuss.