Lecture Notes for Econ 898

KAKUTANI'S FIXED POINT THEOREM AND ITS APPLICATION TO NASH'S THEOREM

Julio Dávila

Department of Economics
University of Pennsylvania

Brouwer's Fixed Point theorem guarantees the existence of a fixed point for every continuous function mapping a nonempty, compact, convex set to itself. When considered as a singleton-valued correspondence a continuous function is clearly convex-valued and has a closed graph. These two properties play the role of continuity in order to generalize Brouwer's Fixed Point Theorem to correspondences Γ that are not singleton-valued. Such generalization is **Kakutani's Fixed Point Theorem**.

Kakutani's theorem is not only a generalization of Brouwer's, but it is also proved by means of it. In the simplest case, when there exists a continuous selection f within the graph of a convex-valued and closed-graph correspondence Γ that takes nonempty values in a nonempty, compact, convex set, contained in the same set, then Brouwer's Fixed Point Theorem immediately applies to this continuous selection, and the existence of a fixed point x of f is guaranteed. Since f is a selection of Γ, then $f(x) \in \Gamma(x)$, but since $x = f(x)$, then $x \in \Gamma(x)$ indeed.

The problem is that there needs not be a continuous selection of a convex-valued and closed-graph correspondence (think of $\Gamma(x)$ equal to $\{-1\}$ for all $x \in [-1, 0)$, $\{1\}$ for all $x \in (0, 1]$, and $\Gamma(0) = [-1, 1]$, for instance). Therefore, in general the fixed point of such a correspondence Γ is obtained as the limit of a convergent sequence of fixed points x_n of a sequence of functions f_n to which Brouwer's theorem applies. More specifically, Γ is first approximated by a sequence of closed-valued, convex-valued, lower hemicontinuous correspondences Γ_n whose graphs have all their points within a distance $\frac{1}{n}$ of some point of the graph of Γ. Then Michael's selection theorem allows to extract a continuous selection f_n from the graph of each of these correspondences Γ_n.[1] For each f_n Brouwer's theorem applies and thus a sequence of fixed points x_n of each Γ_n is obtained, in such a way that (x_n, x_n) is contained in the graph of Γ_n for all $n \in \mathbb{N}$. Since the sequence x_n is contained in the compact domain of Γ, a convergent subsequence $x_{h(n)}$ can be extracted from it whose limit x is such that (x, x) is within an arbitrarily small distance of some $(x_{h(n)}, x_{h(n)})$ in the graph of Γ_n, itself within an arbitrarily small distance of some

---

[1] Note that Michael's Selection Theorem does not apply directly to the correspondences Γ addressed by Kakutani's theorem since Γ needs not be lower hemicontinuous, hence the need for an approximation by lower hemicontinuous correspondences.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX

point in the graph of Γ. Thus (x, x) is necessarily a closure point of the closed graph of Γ and is hence in it, i.e. x is a fixed point of Γ.

Nash's theorem for the existence of an equilibrium in a game of finitely many players with nonempty, compact, convex strategy sets in finite-dimensional real vector spaces and quasi-concave in their own strategy and continuous payoffs is a direct consequence of Kakutani's theorem, the equilibrium being a fixed point of the correspondence of best replies. The details of both theorems follow.

THEOREMS

S1. Kakutani's Fixed Point Theorem.

If

(1) *X is a finite-dimensional real inner product space,*
(2) *$\Gamma \in \mathcal{P}(X)^X$ is convex-valued, has a closed graph, and is such that*
    (*i*) *$X' = \Gamma^{-1}(\mathcal{P}(X) \setminus \{\phi\})$ is nonempty, compact, and convex, and,*
    (*ii*) *for all $x \in X'$, $\Gamma(x) \subset X'$,*

then there exists $x \in X'$ such that

$$x \in \Gamma(x).$$

Proof. Let X be a finite dimensional real inner product space, $\Gamma \in \mathcal{P}(X)^X$ be convex-valued, have a closed graph, and be such that $X' = \Gamma^{-1}(\mathcal{P}(X) \setminus \{\phi\})$ is nonempty, compact, and convex, and, for all $x \in X'$, $\Gamma(x) \subset X'$.

Let $n \in \mathbb{N}$.

(1) Let $\Gamma_n \in \mathcal{P}(X)^X$ such that $\Gamma_n^{-1}(\mathcal{P}(X) \setminus \{\phi\}) = X'$ and, for all $x \in X'$,

$$\Gamma_n(x) = \text{ClCo} \bigcup_{x' \in B_{\frac{1}{n}}(x)} \Gamma(x').$$

Since X is a finite-dimensional normed real vector space, $\Gamma \in \mathcal{P}(Y)^X$ is convex-valued, has closed graph, and is such that $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is compact, nonempty, and convex, and $\Gamma_*(X) \subset \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, then[2] Γ_n is closed-valued, convex-valued, lower hemicontinuous, and

$$\text{Gr}_{\Gamma_n} \subset \bigcup_{(x,x') \in \text{Gr}_\Gamma} B_{\frac{1}{n}}(x, x').$$

(2) Since Γ_n is closed-valued, convex-valued, lower hemicontinuous, and Gr_{Γ_n} is included in $\bigcup_{(x,x') \in \text{Gr}_\Gamma} B_{\frac{1}{n}}(x, x')$, then[3] there exists $f_n \in X^X$ continuous and such that, $f_n^{-1}(X) = \Gamma_n^{-1}(\mathcal{P}(X) \setminus \{\phi\}) = X'$ and, for all $x \in X'$,

$$f_n(x) \in \Gamma_n(x).$$

---

[2] See Lemma ? in Handout ?.

[3] By Michael's Selection Theorem.

(3) Since $B_{\frac{1}{n}}(x) \subset X$ and $\bigcup_{x' \in X} \Gamma(x') \subset X'$, then $\bigcup_{x' \in B_{\frac{1}{n}}(x)} \Gamma(x') \subset X'$. Since $\bigcup_{x' \in B_{\frac{1}{n}}(x)} \Gamma(x') \subset X'$ and X' is convex, then

$$\begin{aligned} \mathrm{Co} \bigcup_{x' \in B_{\frac{1}{n}}(x)} \Gamma(x') &\subset \mathrm{Co} X' \\ &= X'. \end{aligned}$$

Since, for all $x \in X'$, $f_n(x) \in \Gamma_n(x)$, $\Gamma_n(x) = \mathrm{ClCo} \bigcup_{x' \in B_{\frac{1}{n}}(x)} \Gamma(x')$, $\mathrm{Co} \bigcup_{x' \in B_{\frac{1}{n}}(x)} \Gamma(x') \subset X'$, and X' is compact and hence closed, then, for all $x \in X'$,

$$\begin{aligned} f_n(x) &\in \\ \Gamma_n(x) &= \\ \mathrm{ClCo} \bigcup_{x' \in B_{\frac{1}{n}}(x)} \Gamma(x') &\subset \mathrm{Cl} X' \\ &= X' \\ &= f_n^{-1}(X), \end{aligned}$$

i.e.

$$f_n(X) \subset f_n^{-1}(X).$$

(4) Since X is a finite-dimensional real inner product space, f_n is continuous, $f_n^{-1}(X)$ is nonempty, compact, and closed, and $f_n(X) \subset f_n^{-1}(X)$, then there exists $x_n \in f^{-1}(X) = X'$ such that

$$\begin{aligned} x_n &= f_n(x_n) \\ &\in \Gamma_n(x_n). \end{aligned}$$

(5) Since $x_n \in \Gamma_n(x_n)$ and Gr_{Γ_n} is included in $\bigcup_{(x,x') \in \mathrm{Gr}_\Gamma} B_{\frac{1}{n}}(x, x')$, then

$$\begin{aligned} (x_n, x_n) &\in \mathrm{Gr}_{\Gamma_n} \\ &\subset \bigcup_{(x,x') \in \mathrm{Gr}_\Gamma} B_{\frac{1}{n}}(x, x'). \end{aligned}$$

Since $(x_n, x_n) \in \bigcup_{(x,x') \in \mathrm{Gr}_\Gamma} B_{\frac{1}{n}}(x, x')$, then there exists $(\tilde{x}_n, \tilde{x}'_n) \in \mathrm{Gr}_\Gamma$ such that

$$(x_n, x_n) \in B_{\frac{1}{n}}(\tilde{x}_n, \tilde{x}'_n).$$

Since $(x_n, x_n) \in B_{\frac{1}{n}}(\tilde{x}_n, \tilde{x}'_n)$, then

$$\|(x_n, x_n) - (\tilde{x}_n, \tilde{x}'_n)\|_{X \times X} < \frac{1}{n}.$$

(6) Since, for all $n \in \mathbb{N}$, $x_n \in X'$and X' is compact, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and $x \in X'$ such that $x = \lim_{n \to \infty} x_{h(n)}$. Since, for all $n \in \mathbb{N}$, $\|(x_n, x_n) - (\tilde{x}_n, \tilde{x}'_n)\|_{X \times X} < \frac{1}{n}$, and h is increasing then, for all $n \in \mathbb{N}$,

$$\begin{aligned} \|(x_{h(n)}, x_{h(n)}) - (\tilde{x}_{h(n)}, \tilde{x}'_{h(n)})\|_{X \times X} &< \frac{1}{h(n)} \\ &< \frac{1}{n}, \end{aligned}$$

(7) Let $\varepsilon > 0$. Since

$$\begin{aligned} x &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} x_{h(n)} \end{aligned}$$

then there exists $n^* \in \mathbb{N}$ such that, for all $n > n^*$,

$$\|x_{h(n)} - x\|_X < \frac{\varepsilon}{4}$$

and

$$\frac{1}{n} < \frac{\varepsilon}{2}$$

Since, for all $n > n^*$, $\|x_{h(n)} - x\|_X < \frac{\varepsilon}{4}$ and $\frac{1}{n} < \frac{\varepsilon}{2}$, then, for all $n > n_1$,

$$\begin{aligned} \|(x_{h(n)}, x_{h(n)}) - (x, x)\|_{X\times X} &= \\ \|x_{h(n)} - x\|_X + \|x_{h(n)} - x\|_X &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2} \end{aligned}$$

and

$$\begin{aligned} \|(x_{h(n)}, x_{h(n)}) - (\tilde{x}_{h(n)}, \tilde{x}'_{h(n)})\|_{X\times X} &< \frac{1}{n} \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Since, for all $n > n^*$, $\|(x_{h(n)}, x_{h(n)}) - (x, x)\|_{X\times X} < \frac{\varepsilon}{2}$ and $\|(x_{h(n)}, x_{h(n)}) - (\tilde{x}_{h(n)}, \tilde{x}'_{h(n)})\|_{X\times X} < \frac{\varepsilon}{2}$ then, for all $n > n^*$,

$$\begin{aligned} \|(x, x) - (\tilde{x}_{h(n)}, \tilde{x}'_{h(n)})\|_{X\times X} &\leq \|(x_{h(n)}, x_{h(n)}) - (x, x)\|_{X\times X} \\ &\qquad + \|(x_{h(n)}, x_{h(n)}) - (\tilde{x}_{h(n)}, \tilde{x}'_{h(n)})\|_{X\times X} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since, for all $\varepsilon > 0$, there exists $n^* \in \mathbb{N}$ such that, for all $n > n^*$, $\|(x, x) - (\tilde{x}_{h(n)}, \tilde{x}'_{h(n)})\|_{X\times X} < \varepsilon$, then

$$(x, x) = \lim_{n\to\infty} (\tilde{x}_{h(n)}, \tilde{x}'_{h(n)}).$$

(8) Since $(x, x) = \lim_{n\to\infty} (\tilde{x}_{h(n)}, \tilde{x}'_{h(n)})$, for all $n \in \mathbb{N}$, $(\tilde{x}_{h(n)}, \tilde{x}'_{h(n)}) \in \mathrm{Gr}_\Gamma$, and Gr_Γ is closed, then

$$\begin{aligned} (x, x) &\in \mathrm{Cl}\mathrm{Gr}_\Gamma \\ &= \mathrm{Gr}_\Gamma. \end{aligned}$$

i.e.

$$x \in \Gamma(x).$$

Q.E.D.

S2. Nash's Equilibrium Existence Theorem.

If, for all $i = 1, \ldots, m$*,* X_i *is a finite-dimensional real inner product space,* $S_i \subset X_i$ *is nonempty, compact, and convex, and* $f_i \in \mathbb{R}^{\times_{i=1}^m X_i}$ *is continuous, quasi-concave on* X_i*,*[4] *and* $f_i^{-1}(\mathbb{R}) = \times_{i=1}^m S_i$*, then, for all* $i = 1, \ldots, m$*, there exists* $x_i \in S_i$ *such that,*[5] *for all* $i = 1, \ldots, m$*,*

$$x_i \in \arg\max_{y_i \in S_i} f_i(y_i, x_{-i}).$$

Proof. Let, for all $i = 1, \ldots, m$, X_i be a finite-dimensional real inner product space, $S_i \subset X_i$ be nonempty, compact, and convex, $f_i \in \mathbb{R}^{\times_{i=1}^m S_i}$ be continuous, quasi-concave on X_i, $f_i^{-1}(\mathbb{R}) = \times_{i=1}^m S_i$.

Let $\Gamma \in \mathcal{P}(\times_{i=1}^m X_i)^{\times_{i=1}^m X_i}$ be such that $\Gamma^{-1}(\mathcal{P}(\times_{i=1}^m X_i) \setminus \{0\}) = \times_{i=1}^m S_i$ and, for all $x \in \times_{i=1}^m S_i$,

$$\Gamma(x) = \times_{i=1}^m \arg\max_{y_i \in S_i} f_i(y_i, x_{-i}).$$

Therefore

(1) $\times_{i=1}^m X_i$ is a finite dimensional real inner product space.

(2) Γ is convex-valued:

Let $x \in \times_{i=1}^m S_i$ and $x', x'' \in \Gamma(x)$. Since $x', x'' \in \Gamma(x)$, then, for all $i = 1, \ldots, m$,

$$x_i', x_i'' \in \arg\max_{y_i \in S_i} f_i(y_i, x_{-i}).$$

Let $i = 1, \ldots, m$. Since $x_i', x_i'' \in \arg\max_{x_i \in S_i} f_i(x_i, x_{-i})$, then $x_i', x_i'' \in S_i$ and

$$f_i(x_i', x_{-i}) = f_i(x_i'', x_{-i}).$$

Let $\alpha \in [0,1]$. Since S_i is convex and $x_i', x_i'' \in S_i$, then $\alpha x_i' + (1-\alpha)x_i'' \in S_i$. Since $\alpha x_i' + (1-\alpha)x_i'' \in S_i$ and $x_i' \in \arg\max_{x_i \in S_i} f_i(x_i, x_{-i})$, then

$$f_i(\alpha x_i' + (1-\alpha)x_i'', x_{-i}) \leq f_i(x_i', x_{-i}).$$

Since f_i is quasi-concave on X_i, then, for all $\alpha \in [0,1]$,

$$\begin{aligned} f_i(\alpha x_i' + (1-\alpha)x_i'', x_{-i}) &\geq \min\{f_i(x_i', x_{-i}), f_i(x_i'', x_{-i})\} \\ &= f_i(x_i'', x_{-i}) \\ &= f_i(x_i', x_{-i}). \end{aligned}$$

Since, $f_i(\alpha x_i' + (1-\alpha)x_i'', x_{-i}) \leq f_i(x_i', x_{-i})$ and $f_i(\alpha x_i' + (1-\alpha)x_i'', x_{-i}) \geq f_i(x_i', x_{-i})$, then

$$f_i(\alpha x_i' + (1-\alpha)x_i'', x_{-i}) = f_i(x_i', x_{-i}).$$

---

[4] That is to say, such that, for all $j \neq i$, all $x_j \in S_j$, all $x_i', x_i'' \in S_i$, and all $\alpha \in [0,1]$, $f_i(\alpha x_i' + (1-\alpha)x_i'', x_{-i}) \geq \min\{f_i(x_i', x_{-i}), f_i(x_i'', x_{-i})\}$.

[5] In what follows, for all $(x_1, \ldots, x_i, \ldots, x_m) \in \times_{i=1}^m S_i$ and all $i = 1, \ldots, m$, x_{-i} stands for $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ and $f_i(x_i, x_{-i})$ is a customary notation for $f_i(x_1, \ldots, x_i, \ldots, x_m)$.

Since $f_i(\alpha x_i' + (1-\alpha)x_i'', x_{-i}) = f_i(x_i', x_{-i})$, then

$$\alpha x_i' + (1-\alpha)x_i'' \in \arg\max_{y_i \in S_i} f_i(y_i, x_{-i}).$$

Since, for all $i = 1, \ldots, m$, $\alpha x_i' + (1-\alpha)x_i'' \in \arg\max_{y_i \in S_i} f_i(y_i, x_{-i})$, then

$$\alpha x' + (1-\alpha)x'' \in \Gamma(x).$$

(3) Γ has a closed graph:

Let $(x, \tilde{x}) \in \mathrm{ClGr}_\Gamma$. Since $(x, \tilde{x}) \in \mathrm{ClGr}_\Gamma$, then there exists $\{(x^n, \tilde{x}^n)\}_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $(x^n, \tilde{x}^n) \in \mathrm{Gr}_\Gamma$ and $(x, \tilde{x}) = \lim_{n\to\infty}(x^n, \tilde{x}^n)$. Since, for all $n \in \mathbb{N}$, $(x^n, \tilde{x}^n) \in \mathrm{Gr}_\Gamma$, then $\tilde{x}^n \in \Gamma(x^n)$, i.e., for all $i = 1, \ldots, m$,

$$\begin{aligned} \tilde{x}_i^n &\in \arg\max_{y_i \in S_i} f_i(y_i, x_{-i}^n) \\ &= \Gamma_i(x_{-i}^n) \end{aligned}$$

i.e.

$$(x_{-i}^n, \tilde{x}_i^n) \in \mathrm{Gr}_{\Gamma_i}$$

where Γ_i has a closed graph by the Theorem of the Maximum. Since $(x, \tilde{x}) = \lim_{n\to\infty}(x^n, \tilde{x}^n)$, then, for all $\varepsilon > 0$, there exists $n^* \in \mathbb{N}$ such that, for all $n > n^*$,

$$\begin{aligned} &\sum_{i=1}^m \|x_i^n - x_i\|_{X_i} + \sum_{i=1}^m \|\tilde{x}_i^n - \tilde{x}_i\|_{X_i} = \\ &\|x^n - x\|_{\times_{i=1}^m X_i} + \|\tilde{x}^n - \tilde{x}\|_{\times_{i=1}^m X_i} = \|(x^n, \tilde{x}^n) - (x, \tilde{x})\|_{(\times_{i=1}^m X_i)\times(\times_{i=1}^m X_i)} \\ &\qquad < \frac{\varepsilon}{m} \end{aligned}$$

and hence, for all $i = 1, \ldots, m$,

$$\begin{aligned} \|x_i^n - x_i\|_{X_i} &< \frac{\varepsilon}{m} \\ \|\tilde{x}_i^n - \tilde{x}_i\|_{X_i} &< \frac{\varepsilon}{m}. \end{aligned}$$

Therefore, for all $i = 1, \ldots, m$ and all $n > n^*$,

$$\begin{aligned} &\|(\tilde{x}_i^n, x_{-i}^n) - (\tilde{x}_i, x_{-i})\|_{(\times_{i=1}^m X_i)} = \\ &\|\tilde{x}_i^n - \tilde{x}_i\|_{X_i} + \sum_{j\neq i} \|x_j^n - x_j\|_{X_j} < m\frac{\varepsilon}{m} \\ &\qquad = \varepsilon, \end{aligned}$$

and hence, for all $i = 1, \ldots, m$,

$$\begin{aligned} (x_{-i}, \tilde{x}_i) &\in \mathrm{ClGr}_{\Gamma_i} \\ &= \mathrm{Gr}_{\Gamma_i}, \end{aligned}$$

i.e.

$$\begin{aligned}\tilde{x}_i &\in \Gamma_i(x_{-i}) \\ &= \arg\max_{y_i \in S_i} f_i(y_i, x_{-i}),\end{aligned}$$

that is to say,

$$(x, \tilde{x}) \in \mathrm{Gr}_\Gamma.$$

Since Gr_Γ contains all its closure points, then Gr_Γ is closed.

(4) $\times_{i=1}^m S_i$ is nonempty, compact, and convex:

In effect,

$\times_{i=1}^m S_i$ is nonempty:

Since, for all $i = 1, \ldots, m$, S_i is nonempty, then $\times_{i=1}^m S_i$ is nonempty.

$\times_{i=1}^m S_i$ is compact:

If $\{x^n\}_{n\in\mathbb{N}} \subset \times_{i=1}^m S_i$ and hence, for all $i = 1, \ldots, n$, $x_i^n \in S_i$, then

i) since, for all $n \in \mathbb{N}$, $x_1^n \in S_1$ and S_1 is compact, there exists $x_1 \in S_1$ and $h_1 \in \mathbb{N}^\mathbb{N}$ increasing such that $x_1 = \lim_{n\to\infty} x_1^{h_1(n)}$;

ii) since, for all $n \in \mathbb{N}$, $x_2^n \in S_2$ and S_2 is compact, there exists $x_2 \in S_2$ and $h_2 \in \mathbb{N}^\mathbb{N}$ increasing such that $x_2 = \lim_{n\to\infty} x_2^{h_2(n)}$; moreover, since $x_1 = \lim_{n\to\infty} x_1^{h_1(n)}$, then $x_1 = \lim_{n\to\infty} x_1^{h_2(h_1(n))}$; and so on.

Let $h = h_m \circ h_{m-1} \circ \ldots h_1$. Since, for all $i = 1, \ldots, m$, $x_i = \lim_{n\to\infty} x_i^{h(n)}$ and $x_i \in S_i$, then $x = \lim_{n\to\infty} x^{h(n)}$ and $x \in \times_{i=1}^m S_i$. Since, for all $\{x^n\}_{n\in\mathbb{N}} \subset \times_{i=1}^m S_i$, there exist $x \in \times_{i=1}^m S_i$ and $h \in \mathbb{N}^\mathbb{N}$ increasing such that $x = \lim_{n\to\infty} x^{h(n)}$, then $\times_{i=1}^m S_i$ is compact.

$\times_{i=1}^m S_i$ is convex:

If $x', x'' \in \times_{i=1}^m S_i$, then, for all $i = 1, \ldots, m$, $x_i', x_i'' \in S_i$. Since, for all $i = 1, \ldots, m$, $x_i', x_i'' \in S_i$ and S_i is convex, then, for all $\alpha \in [0,1]$ and all $i = 1, \ldots, m$,

$$\alpha x_i' + (1-\alpha)x_i'' \in S_i.$$

Since, for all $\alpha \in [0,1]$ and all $i = 1, \ldots, m$, $\alpha x_i' + (1-\alpha)x_i'' \in S_i$, then, for all $\alpha \in [0,1]$,

$$\alpha x' + (1-\alpha)x'' \in \times_{i=1}^m S_i.$$

Since for all $x', x'' \in \times_{i=1}^m S_i$ and all $\alpha \in [0,1]$, $\alpha x' + (1-\alpha)x'' \in \times_{i=1}^m S_i$, then $\times_{i=1}^m S_i$ is convex.

(5) for all $x \in \times_{i=1}^m S_i$, $\Gamma(x) \subset \times_{i=1}^m S_i$:

Let $x \in \times_{i=1}^m S_i$. Since

$$\Gamma(x) = \times_{i=1}^m \arg\max_{y_i \in S_i} f_i(y_i, x_{-i})$$

and, for all $i = 1, \ldots, m$, $\arg\max_{y_i \in S_i} f_i(y_i, x_{-i}) \subset S_i$, then, for all $x \in \times_{i=1}^m S_i$,

$$\Gamma(x) \subset \times_{i=1}^m S_i.$$

Since $\times_{i=1}^{m} X_i$ is a finite dimensional real inner product space, Γ is convex-valued, has a closed graph, $\times_{i=1}^{m} S_i$ is nonempty, compact, and convex, and for all $x \in \times_{i=1}^{m} S_i$, $\Gamma(x) \subset \times_{i=1}^{m} S_i$, then[6] there exists $x \in \times_{i=1}^{m} S_i$ such that

$$x \in \Gamma(x),$$

i.e.

$$x \in \times_{i=1}^{m} \arg\max_{y_i \in S_i} f_i(y_i, x_{-i}),$$

or, equivalently, for all $i = 1, \ldots, m$,

$$x_i \in \arg\max_{y_i \in S_i} f_i(y_i, x_{-i}).$$

Q.E.D.

---

[6] By Kakutani's Fixed Point Theorem.