Lecture Notes for Econ 898

APPROXIMATIONS AND SELECTIONS OF CORRESPONDENCES

Julio Dávila

Department of Economics
University of Pennsylvania

Kakutani's Fixed Point Theorem establishes the existence of a fixed point for a correspondence by means of repeatedly applying Brouwer's Fixed Point Theorem to a sequence of continuous functions contained in the graphs of a sequence of correspondences that approximate arbitrarily closely the given correspondence. Therefore, Kakutani's claim depends on the conditions under which such continuous selections can be shown to exist. The following lemmas aim at making clear these conditions.

The first one, known as **Von Neumann's Approximation Lemma**, establishes that, for every convex-valued, lower hemicontinuous correspondence Γ with compact domain in a normed vector space X and range in another normed vector space Y there exists a continuous function f from the domain of the correspondence to Y with points of $\Gamma(x)$ within any given positive distance ε of $f(x)$, for all x in the domain. In effect, it turns out that the family of upper inverses[1] $\Gamma_+^{-1}(B_\varepsilon(y))$ of the balls $B_\varepsilon(y)$ centered at any $y \in \Gamma(x)$, for all x in the domain of the correspondence, is an open cover of this domain. As a consequence, since the domain is compact, one can extract a finite subcover from this one, i.e. a finite set of $\tilde{x}$'s and $\tilde{y}$'s within each $\Gamma(\tilde{x})$, whose $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ cover the domain entirely. One can then use the distances $g_{\tilde{x}\tilde{y}}(x)$ from any x to the complements to the domain of the sets $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ as weights in a linear combination of the points $\tilde{y}$. This linear combination can be made to be convex dividing every weight by the nonzero sum $\sum_{\tilde{x}\in\tilde{X},\tilde{y}\in\tilde{Y}_{\tilde{x}}} g_{\tilde{x}\tilde{y}}(x)$ of all of them.[2] This convex linear combination of the $\tilde{y}$'s obviously depends on x and the function f that associates it to each x is such that $f(x)$

---

[1] Recall that for any $\Gamma \in \mathcal{P}(Y)^X$ and any $B \subset Y$, the upper inverse $\Gamma_+^{-1}(B)$ of B by Γ is the set of all the points x in the domain of Γ whose $\Gamma(x)$ has a nonempty intersection with B; and the lower inverse $\Gamma_-^{-1}(B)$ of B by Γ is the set of all the points x in the domain of Γ whose $\Gamma(x)$ is completely contained in B.

[2] Should this sum be zero, then x would be at a distance zero to the complement of every $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, i.e. a closure point of all of them. Since these complements are closed (recall that the domain is compact and hence closed, and that $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ is open because it is the upper inverse of an open set by a lower hemicontinuous correspondence), then x would have to be in all of them or, equivalently, in none of the $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, which would contradict the these sets cover the domain!

is always, by construction, within a distance ε of some point of $\Gamma(x)$. In effect, any point x of the domain is in some upper inverse $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, for some $\tilde{x}$ and some $\tilde{y}$ in $\Gamma(\tilde{x})$, which means that there is some common point $y_{\tilde{x}\tilde{y}}$ in $\Gamma(x)$ and $B_\varepsilon(\tilde{y})$, i.e. within a distance ε of $\tilde{y}$. For any such $\tilde{x}$ and $\tilde{y}$, the distance $g_{\tilde{x}\tilde{y}}(x)$ from x to the complement of $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ is strictly positive.[3] For any other $\tilde{x}$ and $\tilde{y}$ in $\Gamma(\tilde{x})$ such that $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ does not contain x this distance $g_{\tilde{x}\tilde{y}}(x)$ is zero. Let for these points $y_{\tilde{x}\tilde{y}}$ denote any point in $\Gamma(x)$. Thus the convex linear combination of the points $y_{\tilde{x}\tilde{y}}$ in $\Gamma(x)$ with the weights $g_{\tilde{x}\tilde{y}}(x)$ (relative to their nonzero sum) is, on the one hand, in $\Gamma(x)$ since Γ is convex-valued and, on the other hand, within a distance ε of the same convex linear combination of the $\tilde{y}$'s, which is $f(x)$.

Note, nonetheless, that Von Neumann's approximation Lemma does not guarantee that the graph of the continuous function f is entirely contained in that of Γ, but only within an arbitrarily small distance from it, in a pointwise manner. Thus even if we managed to find a fixed point for the function f it may not be a fixed point for the correspondence Γ yet. In order to get a continuous selection from a correspondence, **Michael's Selection Theorem** actually uses Von Neumann's approximation lemma to construct a sequence of Cauchy continuous functions that are increasingly close to the correspondence Γ and whose limit is contained in the graph of the correspondence. For this strategy to work the correspondence must be convex-valued, closed-valued, lower hemicontinuous, have a compact domain and take values in a finite-dimensional normed real vector space.

But for the purposes of establishing Kakutani's fixed point theorem, Michael's selection theorem cannot be applied directly, since in Kakutani's theorem the correspondence needs not be lower hemicontinuous. Hence the need to establish in a final lemma that any convex-valued correspondence with closed graph, nonempty, compact, convex domain, and range within a compact, can be approximated arbitrarily closely by a convex-valued, closed-valued, lower hemicontinuous correspondence, to which Michael's theorem then can be applied. The detailed statements and proofs follow.

S1. Von Neumann's Approximation Lemma.

If

(1) *X is a normed vector space over K,*
(2) *Y is a normed vector space over K',*
(3) *$\Gamma \in \mathcal{P}(Y)^X$ is convex-valued, lower hemicontinuous and such that $\Gamma^{-1}(\mathcal{P}(Y)\setminus \{\phi\})$ is compact,*

then, for all $\varepsilon \in K'_{++}$, there exists $f \in Y^X$ continuous and such that

(1) *$f^{-1}(Y) = \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ and,*
(2) *for all $x \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$,*

$$\Gamma(x) \cap B_\varepsilon(f(x)) \neq \phi.$$

Proof. Let X be a normed vector space over K, Y be a normed vector space over K', $\Gamma \in \mathcal{P}(Y)^X$ be convex-valued, lower hemicontinuous and such that $\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ is compact.

[3] Should it be zero, x would be a closure point of the complement to the domain of $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, which is a compact and hence closed set. Therefore x would not be in $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, which means that $\Gamma(x)$ and $B_\varepsilon(\tilde{y})$ have an empty intersection.

Let $\varepsilon \in K'_{++}$.

(1) There exist $\tilde{X} \subset \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ finite and, for all $\tilde{x} \in \tilde{X}$, $\tilde{Y}_{\tilde{x}} \subset \Gamma(\tilde{x})$ finite, such that[4]

$$\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \subset \bigcup_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \Gamma_+^{-1}(B_\varepsilon(\tilde{y})) :$$

(i) $\{\Gamma_+^{-1}(B_\varepsilon(y))\}_{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\}),y\in\Gamma(x)}$ is an open cover of $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$.

(i.a) for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and all $y \in \Gamma(x)$, $\Gamma_+^{-1}(B_\varepsilon(y))$ is open.

Let $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and $y \in \Gamma(x)$. Since, $B_\varepsilon(y)$ is open and Γ is lower hemicontinuous, then $\Gamma_+^{-1}(B_\varepsilon(y))$ is open. Therefore, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and all $y \in \Gamma(x)$, $\Gamma_+^{-1}(B_\varepsilon(y))$ is open.

(i.b) $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \subset \bigcup_{\substack{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\}) \\ y\in\Gamma(x)}} \Gamma_+^{-1}(B_\varepsilon(y))$:

Let $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and $y \in \Gamma(x)$. Since $y \in \Gamma(x)$ and $y \in B_\varepsilon(y)$, then $y \in \Gamma(x) \cap B_\varepsilon(y)$. Since $y \in \Gamma(x) \cap B_\varepsilon(y)$, then $\Gamma(x) \cap B_\varepsilon(y) \neq \phi$. Since $\Gamma(x) \cap B_\varepsilon(y) \neq \phi$, then $x \in \Gamma_+^{-1}(B_\varepsilon(y))$. Since, $x \in \Gamma_+^{-1}(B_\varepsilon(y))$, then,

$$\begin{aligned} x &\in \bigcup_{y\in\Gamma(x)} \Gamma_+^{-1}(B_\varepsilon(y)) \\ &\subset \bigcup_{\substack{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\}) \\ y\in\Gamma(x)}} \Gamma_+^{-1}(B_\varepsilon(y)). \end{aligned}$$

Since, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, $x \in \bigcup_{\substack{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\}) \\ y\in\Gamma(x)}} \Gamma_+^{-1}(B_\varepsilon(y))$, then

$$\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \subset \bigcup_{\substack{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\}) \\ y\in\Gamma(x)}} \Gamma_+^{-1}(B_\varepsilon(y)).$$

(ii) Since, $\{\Gamma_+^{-1}(B_\varepsilon(y))\}_{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\}),y\in\Gamma(x)}$ is an open cover of $\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ and $\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ is compact, then there exist $\tilde{X} \subset \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ finite and, for all $\tilde{x} \in \tilde{X}$, $\tilde{Y}_{\tilde{x}} \subset \Gamma(\tilde{x})$ finite, such that

$$\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \subset \bigcup_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \Gamma_+^{-1}(B_\varepsilon(\tilde{y})).$$

[4]Recall that for any $\Gamma \in \mathcal{P}(Y)^X$ and any $B \subset Y$, the upper inverse $\Gamma_+^{-1}(B)$ of B by Γ is the set of all the points x in the domain of Γ whose $\Gamma(x)$ has a nonempty intersection with B; and the lower inverse $\Gamma_-^{-1}(B)$ of B by Γ is the set of all the points x in the domain of Γ whose $\Gamma(x)$ is completely contained in B.

(2) For all $\tilde{x} \in \tilde{X}$ and all $\tilde{y} \in \tilde{Y}_{\tilde{x}}$, there exists $g_{\tilde{x}\tilde{y}} \in \mathbb{R}^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}$ continuous and such that, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$g_{\tilde{x}\tilde{y}}(x) = \min_{x' \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})\setminus\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))} \|x - x'\|$$

and

$$G(x) \equiv \sum_{\substack{\tilde{x}\in\tilde{X} \\ y\in\tilde{Y}_{\tilde{x}}}} g_{\tilde{x}\tilde{y}}(x) \neq 0 :$$

(i) Let $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, $\tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}_{\tilde{x}}$. Since $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is compact and $\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ is open, then $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ is compact. Since $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ is compact and $\|x - \cdot\|$ is continuous, then there exists $\min_{x' \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})\setminus\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))} \|x - x'\|$.

(ii) Since, for all $\tilde{x} \in \tilde{X}$ and all $\tilde{y} \in \tilde{Y}_{\tilde{x}}$, $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ is trivially a compact-valued, constant and, hence, continuous correspondence of x and $\|x - \cdot\|$ is continuous, then, for all $\tilde{x} \in \tilde{X}$ and all $\tilde{y} \in \tilde{Y}_{\tilde{x}}$, $g_{\tilde{x}\tilde{y}}$ is continuous.

(iii) Assume that $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is such that $\sum_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} g_{\tilde{x}\tilde{y}}(x) = 0$. Since $\sum_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} g_{\tilde{x}\tilde{y}}(x) = 0$, then, for all $\tilde{x} \in \tilde{X}$ and all $\tilde{y} \in \tilde{Y}_{\tilde{x}}$,

$$\min_{x' \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})\setminus\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))} \|x - x'\| = g_{\tilde{x}\tilde{y}}(x) = 0.$$

Let $\tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}_{\tilde{x}}$. Since $\min_{x' \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})\setminus\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))} \|x - x'\| = 0$, then $x \in \mathrm{Cl}(\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y})))$. Since $x \in \mathrm{Cl}(\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y})))$ and $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ is compact and hence closed, then, $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$. Since $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, then $x \notin \Gamma_+^{-1}(B_\varepsilon(y))$. Since, for all $\tilde{x} \in \tilde{X}$ and all $\tilde{y} \in \tilde{Y}_{\tilde{x}}$, $x \notin \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, then

$$x \notin \bigcup_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \Gamma_+^{-1}(B_\varepsilon(\tilde{y})).$$

Since $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and $x \notin \bigcup_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, then

$$\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \not\subset \bigcup_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))!!$$

(3) There exists $f \in Y^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}$ continuous such that, for all $x \in \Gamma^{-1}(\mathcal{P}(Y)\setminus \{\phi\})$,

$$f(x) = \sum_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} \cdot \tilde{y}$$

and

$$\Gamma(x) \cap B_\varepsilon(f(x)) \neq \phi.$$

(i) Since, for all $\tilde{x} \in \tilde{X}$ and all $\tilde{y} \in \tilde{Y}_{\tilde{x}}$, $g_{\tilde{x}\tilde{y}}$ is continuous, then f is continuous.

(ii) For all $\tilde{x} \in \tilde{X}$, all $\tilde{y} \in \tilde{Y}_{\tilde{x}}$, and all $x \in \Gamma^{-1}(\mathcal{P}(Y)\backslash\{\phi\})$, $\Gamma(x)\cap B_\varepsilon(\tilde{y}) = \phi$ if, and only if, $g_{\tilde{x}\tilde{y}}(x) = 0$.

Let $\tilde{x} \in \tilde{X}$, $\tilde{y} \in \tilde{Y}_{\tilde{x}}$, and $x \in \Gamma^{-1}(\mathcal{P}(Y)\backslash\{\phi\})$. Assume that $\Gamma(x)\cap B_\varepsilon(\tilde{y}) = \phi$. Since $\Gamma(x) \cap B_\varepsilon(\tilde{y}) = \phi$, then $x \notin \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$. Since $x \notin \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ and $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, then $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$. Since $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, then

$$g_{\tilde{x}\tilde{y}}(x) = \min_{x' \in \Gamma^{-1}(\mathcal{P}(Y)\backslash\{\phi\})\backslash\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))} \|x - x'\| = 0.$$

Assume $g_{\tilde{x}\tilde{y}}(x) = 0$. Since $g_{\tilde{x}\tilde{y}}(x) = 0$, then

$$\min_{x' \in \Gamma^{-1}(\mathcal{P}(Y)\backslash\{\phi\})\backslash\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))} \|x - x'\| = 0.$$

Since $\min_{x' \in \Gamma^{-1}(\mathcal{P}(Y)\backslash\{\phi\})\backslash\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))} \|x - x'\| = 0$, then $x \in \mathrm{Cl}(\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y})))$. Since $x \in \mathrm{Cl}(\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y})))$ and $\Gamma^{-1}(\mathcal{P}(Y)\backslash\{\phi\})\backslash\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$ is closed, then $x \in \Gamma^{-1}(\mathcal{P}(Y)\backslash\{\phi\})\backslash\Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$. Since $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \setminus \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, then $x \notin \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$. Since $x \notin \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, then $\Gamma(x) \cap B_\varepsilon(\tilde{y}) = \phi$.

(iii) Let $x \in \Gamma^{-1}(\mathcal{P}(Y)\backslash\{\phi\})$. Since $x \in \Gamma^{-1}(\mathcal{P}(Y)\backslash\{\phi\})$ and $\Gamma^{-1}(\mathcal{P}(Y)\setminus \{\phi\}) \subset \bigcup_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, then

$$x \in \bigcup_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \Gamma_+^{-1}(B_\varepsilon(\tilde{y})).$$

Since $x \in \bigcup_{\substack{\tilde{x}\in\tilde{X} \\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$, then there exist $\tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}_{\tilde{x}}$ such that

$$x \in \Gamma_+^{-1}(B_\varepsilon(\tilde{y})).$$

(iii.a) For all $\tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}_{\tilde{x}}$ such that $x \in \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$,

$$\Gamma(x) \cap B_\varepsilon(\tilde{y}) \neq \phi.$$

Since $\Gamma(x) \cap B_\varepsilon(\tilde{y}) \neq \phi$, then

$$g_{\tilde{x}\tilde{y}}(x) > 0$$

and there exists $y_{\tilde{x}\tilde{y}} \in \Gamma(x)$ such that

$$\|\tilde{y} - y_{\tilde{x}\tilde{y}}\|_Y < \varepsilon.$$

Since $g_{\tilde{x}_x\tilde{y}}(x) > 0$ and $\|\tilde{y} - y_{\tilde{x}\tilde{y}}\|_Y < \varepsilon$, then

$$g_{\tilde{x}\tilde{y}}(x)\|\tilde{y} - y_{\tilde{x}\tilde{y}}\|_Y < g_{\tilde{x}\tilde{y}}(x)\varepsilon.$$

(iii.b) For all $\tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}_{\tilde{x}}$ such that $x \notin \Gamma_+^{-1}(B_\varepsilon(\tilde{y}))$,

$$\Gamma(x) \cap B_\varepsilon(\tilde{y}) = \phi$$

hence let $y_{\tilde{x}\tilde{y}} \in Y$. Since $\Gamma(x) \cap B_\varepsilon(\tilde{y}) = \phi$, then

$$g_{\tilde{x}\tilde{y}}(x) = 0.$$

Since $g_{\tilde{x}\tilde{y}}(x) = 0$, then

$$g_{\tilde{x}\tilde{y}}(x)\|\tilde{y} - y_{\tilde{x}\tilde{y}}\|_Y \leq g_{\tilde{x}\tilde{y}}(x)\varepsilon.$$

Since, for all $\tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}_{\tilde{x}}$, $g_{\tilde{x}\tilde{y}}(x)\|\tilde{y} - y_{\tilde{x}\tilde{y}}\|_Y \leq g_{\tilde{x}\tilde{y}}(x)\varepsilon$, and there exist $\tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}_{\tilde{x}}$ such that $g_{\tilde{x}\tilde{y}}(x)\|\tilde{y} - y_{\tilde{x}\tilde{y}}\|_Y < g_{\tilde{x}\tilde{y}}(x)\varepsilon$, then

$$\begin{aligned}
&\|f(x) - \sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} y_{\tilde{x}\tilde{y}}\|_Y = \\
&\|\sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} \cdot \tilde{y} - \sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} \cdot y_{\tilde{x}\tilde{y}}\|_Y = \\
&\|\sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} (\tilde{y} - y_{\tilde{x}\tilde{y}})\|_Y \leq \sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} \|\tilde{y} - y_{\tilde{x}\tilde{y}}\|_Y \\
&< \sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} \cdot \varepsilon \\
&= \varepsilon.
\end{aligned}$$

Since, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, all $\tilde{x} \in \tilde{X}$, and all $\tilde{y} \in \tilde{Y}_{\tilde{x}}$, $y_{\tilde{x}\tilde{y}} \in \Gamma(x)$, $g_{\tilde{x}\tilde{y}}(x) \geq 0$, $\sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} = 1$, and $\Gamma(x)$ is convex, then

$$\sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} y_{\tilde{x}\tilde{y}} \in \Gamma(x).$$

Since $\sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} y_{\tilde{x}\tilde{y}} \in \Gamma(x)$ and $\|f(x) - \sum_{\substack{\tilde{x}\in\tilde{X}\\ \tilde{y}\in\tilde{Y}_{\tilde{x}}}} \frac{g_{\tilde{x}\tilde{y}}(x)}{G(x)} y_{\tilde{x}\tilde{y}}\|_Y < \varepsilon$, then

$$\Gamma(x) \cap B_\varepsilon(f(x)) \neq \phi.$$

Q.E.D.

S2. Michael's Selection Theorem.

If

(1) X *is a normed real vector space,*
(2) Y *is a finite-dimensional normed real vector space,*
(3) $\Gamma \in \mathcal{P}(Y)^X$ *is convex-valued, closed-valued, lower hemicontinuous, and such that* $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ *is compact,*

then there exists $f \in Y^X$ *continuous and such that* $f^{-1}(X) = \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ *and, for all* $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$f(x) \in \Gamma(x).$$

Proof. Let X and Y be normed real vector spaces, Y be finite-dimensional, $\Gamma \in \mathcal{P}(Y)^X$ be convex-valued, closed-valued, lower hemicontinuous, and such that $\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ is compact.

(1) Since Γ is convex-valued, lower hemicontinuous and $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is compact, then there exists $f_1 \in Y^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}$ continuous and such that, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$\Gamma(x) \cap B_{\frac{1}{2}}(f_1(x)) \neq \phi.$$

Let $\Gamma_2 \in \mathcal{P}(Y)^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}$ be such that, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$\Gamma_2(x) = \Gamma(x) \cap B_{\frac{1}{2}}(f_1(x)).$$

Since f_1 is continuous, then $B_{\frac{1}{2}}(f_1(\cdot))$ is convex-valued and lower hemicontinuous. Since $B_{\frac{1}{2}}(f_1(\cdot))$ is convex-valued and lower hemicontinuous, and Γ is convex-valued and lower hemicontinuous, then Γ_2 is convex-valued and lower hemicontinuous.

(2) Since Γ_2 is convex-valued, lower hemicontinuous and $\Gamma_2^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is compact, then there exists $f_2 \in Y^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}$ continuous and such that, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$\Gamma_2(x) \cap B_{\frac{1}{2^2}}(f_2(x)) \neq \phi.$$

Let $\Gamma_3 \in \mathcal{P}(Y)^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}$ be such that, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$\Gamma_3(x) = \Gamma_2(x) \cap B_{\frac{1}{2^2}}(f_2(x)).$$

Since f_2 is continuous, then $B_{\frac{1}{2^2}}(f_2(\cdot))$ is convex-valued and lower hemicontinuous. Since $B_{\frac{1}{2^2}}(f_2(\cdot))$ is convex-valued and lower hemicontinuous, and Γ_2 is convex-valued and lower hemicontinuous, then Γ_3 is convex-valued and lower hemicontinuous.

(3) In general, for all $n \in \mathbb{N}$, since $\Gamma_n \in \mathcal{P}(Y)^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}$ such that, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$\Gamma_n(x) = \Gamma_{n-1}(x) \cap B_{\frac{1}{2^{n-1}}}(f_{n-1}(x)),$$

is convex-valued and lower hemicontinuous, and $\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ is compact, then there exists $f_n \in Y^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}$ continuous and such that, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$\Gamma_n(x) \cap B_{\frac{1}{2^n}}(f_n(x)) \neq \phi.$$

For all $n \in \mathbb{N}$, since $\Gamma_{n+2}(x) = \Gamma_{n+1}(x) \cap B_{\frac{1}{2^{n+1}}}(f_{n+1}(x))$, then

$$\min_{y \in \Gamma_{n+2}(x)} \|f_{n+1}(x) - y\|_Y < \frac{1}{2^{n+1}}.$$

Since $\min_{y \in \Gamma_{n+2}(x)} \|f_{n+1}(x) - y\|_Y < \frac{1}{2^{n+1}}$, then there exists $y \in \Gamma_{n+2}(x)$ such that

$$\|f_{n+1}(x) - y\|_Y < \frac{1}{2^{n+1}}.$$

Since, $y \in \Gamma_{n+2}(x)$ and $\Gamma_{n+2}(x) = \Gamma_{n+1}(x) \cap B_{\frac{1}{2^{n+1}}}(f_{n+1}(x))$, then $y \in \Gamma_{n+1}(x)$. Since, $y \in \Gamma_{n+1}(x)$ and $\Gamma_{n+1}(x) = \Gamma_n(x) \cap B_{\frac{1}{2^n}}(f_n(x))$, then $y \in B_{\frac{1}{2^n}}(f_n(x))$. Since $y \in B_{\frac{1}{2^n}}(f_n(x))$, then

$$\|f_n(x) - y\|_Y < \frac{1}{2^n}.$$

Since $\|f_{n+1}(x) - y\|_Y < \frac{1}{2^{n+1}}$ and $\|f_n(x) - y\|_Y < \frac{1}{2^n}$, then

$$\begin{aligned}
\|f_n(x) - f_{n+1}(x)\|_Y &\leq \|f_n(x) - y\|_Y + \|f_{n+1}(x) - y\|_Y \\
&< \frac{1}{2^n} + \frac{1}{2^{n+1}} \\
&= \frac{3}{2^{n+1}}.
\end{aligned}$$

Since, for all $n \in \mathbb{N}$ and all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$\|f_n(x) - f_{n+1}(x)\|_Y < \frac{3}{2^{n+1}},$$

then, for all $n, n' \in \mathbb{N}$ such that $n < n'$ and all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$\begin{aligned}
\|f_n(x) - f_{n'}(x)\|_Y &\leq \|f_n(x) - f_{n+1}(x)\|_Y + \cdots + \|f_{n'-1}(x) - f_{n'}(x)\|_Y \\
&< \frac{3}{2^{n+1}} + \cdots + \frac{3}{2^{n'}} \\
&= \frac{3}{2^{n+1}}(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n'-n-1}}) \\
&= \frac{3}{2^n}(1 - \frac{1}{2^{n'-n}}) \\
&< \frac{3}{2^n}.
\end{aligned}$$

Since for all $n, n' \in \mathbb{N}$ such that $n < n'$ and all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$,

$$\|f_n(x) - f_{n'}(x)\|_Y < \frac{3}{2^n},$$

then

$$\sup_{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}\|f_n(x)-f_{n'}(x)\|_Y \leq \frac{3}{2^n}.$$

Since for all $n, n' \in \mathbb{N}$ such that $n < n'$, $\sup_{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}\|f_n(x)-f_{n'}(x)\|_Y \leq \frac{3}{2^n}$, and, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $\frac{3}{2^n} < \varepsilon$, then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n, n' > N$,

$$\sup_{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}\|f_n(x)-f_{n'}(x)\|_Y < \varepsilon,$$

i.e. $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{CB}(Y^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})})$ with the sup norm metric. Since Y is a finite dimensional real vector space, then $\mathrm{CB}(Y^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})})$ with the sup norm metric is complete. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{CB}(Y^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})})$ with the sup norm metric and $\mathrm{CB}(Y^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})})$ with the sup norm metric is complete, then there exists $f \in \mathrm{CB}(Y^{\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})})$ such that, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$,

$$\sup_{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}\|f_n(x)-f(x)\|_Y < \varepsilon.$$

Therefore, for all $x \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ and all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$,

$$\begin{aligned}\|f_n(x)-f(x)\|_Y &\leq \sup_{x\in\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})}\|f_n(x)-f(x)\|_Y\\ &< \varepsilon,\end{aligned}$$

Since, for all $x \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ and all $n \in \mathbb{N}$,

$$\begin{aligned}&\phi \neq \Gamma_n(x) \cap B_{\frac{1}{2^n}}(f_n(x))\\ &\quad = \Gamma_{n-1}(x) \cap B_{\frac{1}{2^{n-1}}}(f_{n-1}(x)) \cap B_{\frac{1}{2^n}}(f_n(x))\\ &\quad = \Gamma_{n-2}(x) \cap B_{\frac{1}{2^{n-2}}}(f_{n-2}(x)) \cap B_{\frac{1}{2^{n-1}}}(f_{n-1}(x)) \cap B_{\frac{1}{2^n}}(f_n(x))\\ &\quad = \ldots,\end{aligned}$$

then, for all $x \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ and all $n \in \mathbb{N}$,

$$\Gamma(x) \cap \Big(\bigcap_{i=1}^{n} B_{\frac{1}{2^i}}(f_i(x))\Big) \neq \phi.$$

Since, for all $x \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ and all $n \in \mathbb{N}$, $\Gamma(x)\cap\Big(\bigcap_{i=1}^{n} B_{\frac{1}{2^i}}(f_i(x))\Big) \neq \phi$ and $\bigcap_{i=1}^{n} B_{\frac{1}{2^i}}(f_i(x)) \subset B_{\frac{1}{2^n}}(f_n(x))$, then, for all $x \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ and all $n \in \mathbb{N}$,

$$\Gamma(x) \cap B_{\frac{1}{2^n}}(f_n(x)) \neq \phi.$$

i.e. there exists $y \in \Gamma(x)$ such that

$$\|y - f_n(x)\|_Y < \frac{1}{2^n}.$$

Since,

(1) for all $\varepsilon' > 0$, there exists $\varepsilon \in (0, \varepsilon')$ and $N' = [\log_2 \frac{1}{\varepsilon' - \varepsilon}] \in \mathbb{N}$ such that, for all $n > N'$,

$$\frac{1}{2^n} + \varepsilon < \varepsilon',$$

(2) for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$,

$$\|f_n(x) - f(x)\|_Y < \varepsilon,$$

and,

(3) for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and all $n \in \mathbb{N}$, there exists $y \in \Gamma(x)$ such that

$$\|y - f_n(x)\|_Y < \frac{1}{2^n},$$

then, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and all $\varepsilon' > 0$, there exist $N^* = \max\{N, N'\} \in \mathbb{N}$ and $y \in \Gamma(x)$ such that, for all $n > N^*$,

$$\begin{aligned} \|y - f(x)\|_Y &\leq \|y - f_n(x)\|_Y + \|f_n(x) - f(x)\|_Y \\ &< \frac{1}{2^n} + \varepsilon \\ &< \varepsilon', \end{aligned}$$

i.e. for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and all $\varepsilon' > 0$, there exists $y \in \Gamma(x)$ such that,

$$\|y - f(x)\|_Y < \varepsilon',$$

or, equivalently, for all $\varepsilon' > 0$, $\Gamma(x) \cap B_{\varepsilon'}(f(x)) \neq \phi$. Since, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ and all $\varepsilon' > 0$, $\Gamma(x) \cap B_{\varepsilon'}(x) \neq \phi$, then, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, $f(x)$ is a closure point of $\Gamma(x)$. Since, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, $f(x)$ is a closure point of $\Gamma(x)$ and $\Gamma(x)$ is closed, then, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, $f(x) \in \Gamma(x)$. Q.E.D.

S3. Approximation Lemma.

If

(1) *X is a normed real vector space,*
(2) *Y is a finite-dimensional normed real vector space,*
(3) *$\Gamma \in \mathcal{P}(Y)^X$ is convex-valued, has closed graph, and is such that $\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is compact, nonempty, and convex, and there exists $Y' \subset Y$ compact such that $\Gamma_*(X) \subset Y'$,*

then, for all $\varepsilon > 0$, there exists $\Gamma' \in \mathcal{P}(Y)^X$ convex-valued, closed-valued, lower hemicontinuous and such that ${\Gamma'}^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is compact, and

$$\mathrm{Gr}_{\Gamma'} \subset \bigcup_{(x,y) \in \mathrm{Gr}_\Gamma} B_\varepsilon(x, y).$$

Proof. Let X and Y be normed real vector spaces, Y be finite-dimensional and compact, $\Gamma \in \mathcal{P}(Y)^X$ be convex-valued, have closed graph, and be such that

$\Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is compact, nonempty, and convex, and there exists $Y' \subset Y$ compact such that $\Gamma_*(X) \subset Y'$.

Let, for all $n \in \mathbb{N}$, $\Gamma_n \in \mathcal{P}(Y)^X$ be such that

$$\Gamma_n(x) = \begin{cases} \mathrm{ClCo}\Gamma_*(B_{\frac{1}{n}}(x)) & \forall x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \\ \phi & \forall x \notin \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}). \end{cases}$$

(1) For all $n \in \mathbb{N}$, $\Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is compact.

In effect, let $n \in \mathbb{N}$. If $x \in \Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, then $\Gamma_n(x) \neq \phi$. Since $\Gamma_n(x) \neq \phi$, then $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$. Since, for all $x \in \Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, then

$$\Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \subset \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}).$$

Conversely, if $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, then $\Gamma(x) \neq \phi$, and $\Gamma_n(x) = \mathrm{ClCo}\Gamma_*(B_{\frac{1}{n}}(x))$. Since $\Gamma(x) \neq \phi$, $\Gamma_n(x) = \mathrm{ClCo}\Gamma_*(B_{\frac{1}{n}}(x))$, and $x \in B_{\frac{1}{n}}(x)$, then

$$\begin{aligned} \Gamma_n(x) &= \\ \mathrm{ClCo}\Gamma_*(B_{\frac{1}{n}}(x)) &\supset \mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x)) \\ &\supset \Gamma_*(B_{\frac{1}{n}}(x)) \\ &\supset \Gamma(x) \\ &\neq \phi. \end{aligned}$$

Since $\Gamma_n(x) \neq \phi$, then $x \in \Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$. Since, for all $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, $x \in \Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, then

$$\Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \supset \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}).$$

Since

$$\begin{aligned} \Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) &\subset \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) \\ \text{and } \Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) &\supset \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}), \end{aligned}$$

then

$$\Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\}) = \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\}).$$

Since $\Gamma_n^{-1}(\mathcal{P}(Y)\setminus\{\phi\}) = \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ and $\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ is compact, then $\Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$ is compact.

(2) For all $n \in \mathbb{N}$, Γ_n is closed-valued, by construction.

(3) For all $n \in \mathbb{N}$, Γ_n is convex-valued, by construction.

(4) for all $n \in \mathbb{N}$, Γ_n is lower hemicontinuous.

In effect, let $x \in \Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, and $y \in Y$ and $\varepsilon > 0$ be such that $\Gamma_n(x) \cap B_\varepsilon(y) \neq \phi$. Since

$$\begin{aligned} \Gamma_n(x) \cap B_\varepsilon(y) &= \\ \mathrm{ClCo}\Gamma_*(B_{\frac{1}{n}}(x)) \cap B_\varepsilon(y) &\neq \phi \end{aligned}$$

then $\mathrm{ClCo}\Gamma_*(B_{\frac{1}{n}}(x)) \not\subset B_\varepsilon(y)^C$. Since $B_\varepsilon(y)^C$ is closed, and

$$\begin{aligned} \mathrm{ClCo}\Gamma_*(B_{\frac{1}{n}}(x)) &\not\subset B_\varepsilon(y)^C \\ &= \mathrm{Cl}(B_\varepsilon(y)^C), \end{aligned}$$

then $\mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x)) \not\subset B_\varepsilon(y)^C$. Since $\mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x)) \not\subset B_\varepsilon(y)^C$, then

$$\mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x)) \cap B_\varepsilon(y) \neq \phi.$$

Since $\mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x)) \cap B_\varepsilon(y) \neq \phi$, then there exists $y' \in \mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x)) \cap B_\varepsilon(y)$. Since $y' \in \mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x))$, and Y is finite-dimensional, say N-dimensional, then there exists $y'_i \in \Gamma_*(B_{\frac{1}{n}}(x))$, for all $i = 1, \ldots, N+1$ and $\alpha \in [0,1]^{N+1}$ such that $y' = \sum_{i=1}^{N+1} \alpha_i y'_i$ and $\sum_{i=1}^{N+1} \alpha_i = 1$.

Let $i = 1, \ldots, N+1$. Since $y'_i \in \Gamma_*(B_{\frac{1}{n}}(x))$, then there exists $x'_i \in B_{\frac{1}{n}}(x)$ such that $y'_i \in \Gamma(x'_i)$. Since $x'_i \in B_{\frac{1}{n}}(x)$, then $\|x'_i - x\|_X < \frac{1}{n}$. Since $\|x'_i - x\|_X < \frac{1}{n}$, then $0 < \frac{1}{n} - \|x'_i - x\|_X$. Since $0 < \frac{1}{n} - \|x'_i - x\|_X$, then there exists $\delta_i > 0$ such that $\delta_i < \frac{1}{n} - \|x'_i - x\|_X$. Since $\delta_i < \frac{1}{n} - \|x'_i - x\|_X$, then, for all $x'' \in X$ such that $\|x'' - x\|_X < \delta_i$,

$$\begin{aligned}\|x'_i - x''\|_X &\leq \|x'_i - x\|_X + \|x'' - x\|_X \\ &< \|x'_i - x\|_X + \delta_i \\ &< \frac{1}{n}\end{aligned}$$

i.e. for all $x'' \in B_{\delta_i}(x)$, $x'_i \in B_{\frac{1}{n}}(x'')$.

If $\delta = \min_{i=1,\ldots,N+1} \delta_i$, then, for all $x'' \in B_\delta(x)$ and all $i = 1, \ldots, N+1$, $x'_i \in B_{\frac{1}{n}}(x'')$. Since, for all $i = 1, \ldots, N+1$, $y'_i \in \Gamma(x'_i)$ and, for all $x'' \in B_\delta(x)$, $x'_i \in B_{\frac{1}{n}}(x'')$, then, for all $x'' \in B_\delta(x)$ and all $i = 1, \ldots, N+1$,

$$\begin{aligned}y'_i &\in \Gamma_*(B_{\frac{1}{n}}(x'')) \\ &\subset \mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x'')).\end{aligned}$$

Since, for all $i = 1, \ldots, N+1$, $y'_i \in \mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x''))$, then

$$\begin{aligned}y' &= \\ \sum_{i=1}^{N+1} \alpha_i y'_i &\in \mathrm{Co}\Gamma_*(B_{\frac{1}{n}}(x'')) \\ &\subset \mathrm{ClCo}\Gamma_*(B_{\frac{1}{n}}(x'')) \\ &= \Gamma_n(x'').\end{aligned}$$

Since, for all $x'' \in B_\delta(x)$, $y' \in \Gamma_n(x'')$ and $y' \in B_\varepsilon(y)$, then, for all $x'' \in B_\delta(x)$, $y' \in \Gamma_n(x'') \cap B_\varepsilon(y)$, i.e. for all $x'' \in B_\delta(x)$, $\Gamma_n(x'') \cap B_\varepsilon(y) \neq \phi$. Therefore, since, for all $n \in \mathbb{N}$, and all $x \in \Gamma_n^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, and for all $y \in Y$ and all $\varepsilon > 0$ such that $\Gamma_n(x) \cap B_\varepsilon(y) \neq \phi$ there exists $\delta > 0$ such that, for all $x'' \in B_\delta(x)$, $\Gamma_n(x'') \cap B_\varepsilon(y) \neq \phi$, then, for all $n \in \mathbb{N}$, Γ_n is lower hemicontinuous.

(5) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that,

$$\mathrm{Gr}_{\Gamma_n} \subset \bigcup_{(x,y) \in \mathrm{Gr}_\Gamma} B_\varepsilon(x, y).$$

In effect, assume that there exists $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$,

$$\mathrm{Gr}_{\Gamma_n} \not\subset \bigcup_{(x,y)\in \mathrm{Gr}_\Gamma} B_\varepsilon(x,y).$$

Since for all $n \in \mathbb{N}$, $\mathrm{Gr}_{\Gamma_n} \not\subset \bigcup_{(x,y)\in \mathrm{Gr}_\Gamma} B_\varepsilon(x,y)$, then, for all $n \in \mathbb{N}$, there exists $(x_n, y_n) \in \mathrm{Gr}_{\Gamma_n}$ such that, for all $(x,y) \in \mathrm{Gr}_\Gamma$, $(x_n, y_n) \notin B_\varepsilon(x,y)$, i.e. for all $n \in \mathbb{N}$, there exists $(x_n, y_n) \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\}) \times Y$ such that

$$\begin{aligned} y_n &\in \Gamma_n(x_n) \\ &= \mathrm{ClCo}\Gamma_*(B_{\frac{1}{n}}(x_n)) \end{aligned}$$

and, for all $(x,y) \in \mathrm{Gr}_\Gamma$,

$$\|(x,y) - (x_n, y_n)\|_{X\times Y} \geq \varepsilon.$$

Since $\{x_n\}_{n\in\mathbb{N}} \subset \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ and $\Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ is compact, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and $x \in \Gamma^{-1}(\mathcal{P}(Y)\setminus\{\phi\})$ such that

$$x = \lim_{n\to\infty} x_{h(n)}.$$

Since, for all $n \in \mathbb{N}$, $y_{h(n)} \in \mathrm{ClCo}\Gamma_*(B_{\frac{1}{h(n)}}(x_{h(n)}))$, then, for all $n \in \mathbb{N}$,

$$\mathrm{Co}\Gamma_*(B_{\frac{1}{h(n)}}(x_{h(n)})) \cap B_{\frac{1}{h(n)}}(y_{h(n)}) \neq \phi.$$

Since, for all $n \in \mathbb{N}$, $\mathrm{Co}\Gamma_*(B_{\frac{1}{h(n)}}(x_{h(n)})) \cap B_{\frac{1}{h(n)}}(y_{h(n)}) \neq \phi$, then, for all $n \in \mathbb{N}$, there exists $\tilde{y}^i_{h(n)} \in \Gamma_*(B_{\frac{1}{h(n)}}(x_{h(n)}))$, for all $i = 1, \ldots, N+1$, and $\alpha_{h(n)} \in [0,1]^{N+1}$ such that, for all $n \in \mathbb{N}$, $\sum_{i=1}^{N+1} \alpha^i_{h(n)} \tilde{y}^i_{h(n)} \in B_{\frac{1}{h(n)}}(y_{h(n)})$, i.e.

$$\Big\| \sum_{i=1}^{N+1} \alpha^i_{h(n)} \tilde{y}^i_{h(n)} - y_{h(n)} \Big\|_Y < \frac{1}{h(n)},$$

and

$$\sum_{i=1}^{N+1} \alpha^i_{h(n)} = 1.$$

(1) Since $\{\tilde{y}^1_{h(n)}\}_{n\in\mathbb{N}} \subset Y'$ and Y' is compact, then there exists $h'_1 \in \mathbb{N}^{\mathbb{N}}$ increasing and $\tilde{y}^1 \in Y'$ such that

$$\tilde{y}^1 = \lim_{n\to\infty} \tilde{y}^1_{h'_1(h(n))}.$$

Since $\{\alpha^1_{h'_1(h(n))}\}_{n\in\mathbb{N}} \subset [0,1]$ and $[0,1]$ is compact, then there exists $h''_1 \in \mathbb{N}^{\mathbb{N}}$ increasing and $\alpha^1 \in [0,1]$ such that

$$\alpha^1 = \lim_{n\to\infty} \alpha^1_{h''_1(h'_1(h(n)))}$$

and since $\tilde{y}^1 = \lim_{n\to\infty} \tilde{y}^1_{h'_1(h(n))}$, then

$$\tilde{y}^1 = \lim_{n\to\infty} \tilde{y}^1_{h''_1(h'_1(h(n)))}.$$

Let $\tilde{h}_1 = h''_1 \circ h'_1$.

(2) Since $\{\tilde{y}^2_{\tilde{h}_1(h(n))}\}_{n\in\mathbb{N}} \subset Y'$ and Y is compact, then there exists $h'_2 \in \mathbb{N}^{\mathbb{N}}$ increasing and $\tilde{y}^2 \in Y'$ such that

$$\tilde{y}^2 = \lim_{n\to\infty} \tilde{y}^2_{h'_2(\tilde{h}_1(h(n)))}.$$

Since $\{\alpha^2_{h'_2(\tilde{h}_1(n))}\}_{n\in\mathbb{N}} \subset [0,1]$ and $[0,1]$ is compact, then there exists $h''_2 \in \mathbb{N}^{\mathbb{N}}$ increasing and $\alpha^2 \in [0,1]$ such that

$$\alpha^2 = \lim_{n\to\infty} \alpha^2_{h''_2(h'_2(\tilde{h}_1(h(n))))}$$

and since $\tilde{y}^2 = \lim_{n\to\infty} \tilde{y}^2_{h'_2(\tilde{h}_1(h(n)))}$, then

$$\tilde{y}^2 = \lim_{n\to\infty} \tilde{y}^2_{h''_2(h'_2(\tilde{h}_1(h(n))))}.$$

Let $\tilde{h}_2 = h''_2 \circ h'_2$. Since $\alpha^1 = \lim_{n\to\infty} \alpha^1_{\tilde{h}_1(h(n))}$ and $\tilde{y}^1 = \lim_{n\to\infty} \tilde{y}^1_{\tilde{h}_1(h(n))}$, then

$$\alpha^1 = \lim_{n\to\infty} \alpha^1_{\tilde{h}_2(\tilde{h}_1(h(n)))}$$

and

$$\tilde{y}^1 = \lim_{n\to\infty} \tilde{y}^1_{\tilde{h}_2(\tilde{h}_1(h(n)))},$$

and so on.

If $\tilde{h} = \tilde{h}_{N+1} \circ \tilde{h}_N \circ \cdots \circ \tilde{h}_1 \circ h$, then, for all $i = 1, \ldots, N+1$,

$$\begin{aligned}\tilde{y}^i &= \lim_{n\to\infty} \tilde{y}^i_{\tilde{h}(n)}\\ \alpha^i &= \lim_{n\to\infty} \alpha^i_{\tilde{h}(n)}\end{aligned}$$

and

$$\sum_{i=1}^{N+1} \alpha^i =$$

$$\begin{aligned}\sum_{i=1}^{N+1} \lim_{n\to\infty} \alpha^i_{\tilde{h}(n)} &= \lim_{n\to\infty} \sum_{i=1}^{N+1} \alpha^i_{\tilde{h}(n)}\\ &= \lim_{n\to\infty} 1\\ &= 1.\end{aligned}$$

Since, for all $i = 1, \ldots, N+1$ and all $n \in \mathbb{N}$,

$$\tilde{y}^i_{\tilde{h}(n)} \in \bigcup_{x' \in B_{\frac{1}{\tilde{h}(n)}}(x_{\tilde{h}(n)})} \Gamma(x'),$$

then, for all $i = 1, \ldots, N+1$ and all $n \in \mathbb{N}$, there exists $\tilde{x}^i_{\tilde{h}(n)} \in B_{\frac{1}{\tilde{h}(n)}}(x_{\tilde{h}(n)})$ such that

$$\tilde{y}^i_{\tilde{h}(n)} \in \Gamma(\tilde{x}^i_{\tilde{h}(n)}).$$

Since $x = \lim_{n\to\infty} x_{h(n)}$ and, for all $i = 1, \ldots, N+1$, and all $n \in \mathbb{N}$, $\tilde{x}^i_{\tilde{h}(n)} \in B_{\frac{1}{\tilde{h}(n)}}(x_{\tilde{h}(n)})$, then $x = \lim_{n\to\infty} \tilde{x}^i_{\tilde{h}(n)}$. Since Γ has a closed graph, and $x \in \Gamma^{-1}(\mathcal{P}(Y) \setminus \{\phi\})$, Γ is closed at x. Since, Γ is closed at x and, for all $i = 1, \ldots, N+1$, and all $n \in \mathbb{N}$, $x = \lim_{n\to\infty} \tilde{x}^i_{\tilde{h}(n)}$, $\tilde{y}^i_{\tilde{h}(n)} \in \Gamma(\tilde{x}^i_{\tilde{h}(n)})$, $\tilde{y}^i = \lim_{n\to\infty} \tilde{y}^i_{\tilde{h}(n)}$, then, for all $i = 1, \ldots, N+1$,

$$\tilde{y}^i \in \Gamma(x).$$

Since, for all $i = 1, \ldots, N+1$, $\tilde{y}^i \in \Gamma(x)$, and $\Gamma(x)$ is convex, then

$$\sum_{i=1}^{N+1} \alpha^i \tilde{y}^i \in \Gamma(x),$$

i.e. $\left(x, \sum_{i=1}^{N+1} \alpha^i \tilde{y}^i\right) \in \Gamma$. Since, for all $i = 1, \ldots, N+1$, $\tilde{y}^i = \lim_{n\to\infty} \tilde{y}^i_{\tilde{h}(n)}$ and $\alpha^i = \lim_{n\to\infty} \alpha^i_{\tilde{h}(n)}$, then, for all $i = 1, \ldots, N+1$,

$$\alpha^i \tilde{y}^i = \lim_{n\to\infty} \alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)}.$$

Since, for all $i = 1, \ldots, N+1$, $\alpha^i \tilde{y}^i = \lim_{n\to\infty} \alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)}$, then, for all $i = 1, \ldots, N+1$ and all $\varepsilon > 0$, there exists $N^i_y \in \mathbb{N}$ such that, for all $n > N^i_y$,

$$\|\alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - \alpha^i \tilde{y}^i\|_Y < \frac{1}{N+1} \cdot \frac{\varepsilon}{4}.$$

Since, for all $i = 1, \ldots, N+1$ and all $n > N^i_y$, $\|\alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - \alpha^i_{\tilde{h}(n)} \tilde{y}^i\|_Y < \frac{1}{N+1} \cdot \frac{\varepsilon}{4}$, then, if $N'_y = \max_{i=1,\ldots,N+1}\{N^i_y\}$, for all $n > N'_y$,

$$\begin{aligned}
&\Big\|\sum_{i=1}^{N+1} \alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - \sum_{i=1}^{N+1} \alpha^i \tilde{y}^i\Big\|_Y = \\
&\quad\Big\|\sum_{i=1}^{N+1} (\alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - \alpha^i \tilde{y}^i)\Big\|_Y \le \sum_{i=1}^{N+1} \|\alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - \alpha^i \tilde{y}^i\|_Y \\
&\qquad < \sum_{i=1}^{N+1} \frac{1}{N+1} \cdot \frac{\varepsilon}{4} \\
&\qquad = \frac{\varepsilon}{4}.
\end{aligned}$$

Since there exists $N''_y \in \mathbb{N}$ such that, for all $n > N''_y$, $\frac{1}{\tilde{h}(n)} < \frac{\varepsilon}{4}$, and, for all $n \in \mathbb{N}$, $\left\|\sum_{i=1}^{N+1} \alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - y_{\tilde{h}(n)}\right\|_Y < \frac{1}{\tilde{h}(n)}$, then, for all $n > N''_y$,

$$\Big\|\sum_{i=1}^{N+1} \alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - y_{\tilde{h}(n)}\Big\|_Y < \frac{\varepsilon}{4}.$$

Since, if $N_y = \max\{N'_y, N''_y\}$, for all $n > N_y$,

$$\|\sum_{i=1}^{N+1} \alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - \sum_{i=1}^{N+1} \alpha^i_{\tilde{h}(n)} \tilde{y}^i\|_Y < \frac{\varepsilon}{4}$$

and $\left\|\sum_{i=1}^{N+1} \alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - y_{\tilde{h}(n)}\right\|_Y < \frac{\varepsilon}{4}$, then, for all $n > N_y$,

$$\begin{aligned}\left\|\sum_{i=1}^{N+1} \alpha^i \tilde{y}^i - y_{\tilde{h}(n)}\right\|_Y &\leq \left\|\sum_{i=1}^{N+1} \alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - \sum_{i=1}^{N+1} \alpha^i \tilde{y}^i\right\|_Y \\ &\qquad + \left\|\sum_{i=1}^{N+1} \alpha^i_{\tilde{h}(n)} \tilde{y}^i_{\tilde{h}(n)} - y_{\tilde{h}(n)}\right\|_Y \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2}.\end{aligned}$$

Since, $x = \lim_{n\to\infty} x_{h(n)}$, then $x = \lim_{n\to\infty} x_{\tilde{h}(n)}$. Since $x = \lim_{n\to\infty} x_{\tilde{h}(n)}$, then there exists $N_x \in \mathbb{N}$ such that, for all $n > N_x$,

$$\|x - x_{\tilde{h}(n)}\| < \frac{\varepsilon}{2}.$$

Since, if $N^* = \max\{N_x, N_y\}$, for all $n > N^*$, $\|x - x_{\tilde{h}(n)}\| < \frac{\varepsilon}{2}$ and $\left\|\sum_{i=1}^{N+1} \alpha^i \tilde{y}^i - y_{\tilde{h}(n)}\right\|_Y < \frac{\varepsilon}{2}$, then, for all $n > N^*$,

$$\begin{aligned}&\left\|\left(x, \sum_{i=1}^{N+1} \alpha^i \tilde{y}^i\right) - (x_{\tilde{h}(n)}, y_{\tilde{h}(n)})\right\|_{X\times Y} = \\ &\|x - x_{\tilde{h}(n)}\|_X + \left\|\sum_{i=1}^{N+1} \alpha^i \tilde{y}^i - y_{\tilde{h}(n)}\right\|_Y < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\qquad = \varepsilon.\end{aligned}$$

Since $\left(x, \sum_{i=1}^{N+1} \alpha^i \tilde{y}^i\right) \in \Gamma$, then there exists $n > N^*$ such that it is not true that, for all $(x, y) \in \mathrm{Gr}_\Gamma$,

$$\|(x, y) - (x_n, y_n)\|_{X\times Y} \geq \varepsilon.$$

Therefore, it is not true that there exists $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$, $\mathrm{Gr}_{\Gamma_n} \not\subset \bigcup_{(x,y)\in \mathrm{Gr}_\Gamma} B_\varepsilon(x, y)$, i.e. for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that,

$$\mathrm{Gr}_{\Gamma_n} \subset \bigcup_{(x,y)\in \mathrm{Gr}_\Gamma} B_\varepsilon(x, y).$$

Q.E.D.