Lecture Notes for Econ 898

VECTOR SPACES

Julio Dávila

Department of Economics
University of Pennsylvania

Theorems like Blackwell's sufficient characterization of a contraction, Banach's Fixed Point Theorem, and the Theorem of the Maximum (along with Bellman's Principle of Optimality), which allowed us to establish the existence of a solution to the standard dynamic programming problem as well as a procedure to find that solution, have been obtained in the framework of metric spaces. Note that a metric space consists only of a set and a metric on it. There is no relation between the elements of the set other than the distance at which they are from each other. The introduction of operations, i.e. functions of elements of the set (and maybe elements of other sets) that take value in the same set, unfolds, by relating the elements of the set among themselves, into new concepts and properties that are important for our purposes. The new structure arising from these operations is that of a vector space.

The bridge between the metric spaces and vector spaces that allows to have the properties of both worlds simultaneously is the concept of norm, a generalization to vector spaces of the notion of length that implicitly defines a metric as well. A vector space that is endowed with a norm is known as a normed vector space. An even more structured kind of vector space is any such space that is endowed with an inner product, an operation among vectors that makes sense of the idea of orthogonality. Inner products generate norms and hence implicitly metrics as well. It is in the framework of inner product spaces, i.e. vector spaces endowed with an inner product, that we will be able to obtain properties like Brower's and Kakutani's Fixed Point Theorems. Kakutani's theorem at the heart of Nash's Theorem on the existence of an equilibrium in a game of finitely many players with compact, convex sets of strategies in finite-dimensional normed real vector spaces, and continuous payoffs that are quasiconcave in their own strategies. The development of the new concepts needed to arrive there follows.

A **vector space** consists of a set with an addition, a field of scalars, and an operation that associates a vector to each vector and scalar and satisfying (i) distributivity properties with respect to both additions, i.e. the addition of scalars and the addition of vectors; (ii) a mixed associativity property with the multiplication of scalars; and (iii) leaving invariant any vector multiplied by the one of the field. The elements of a vector space are called **vectors**. The zero of the vector space is known as the **null vector**.

Associated to any set A of vectors there is another set $\langle A \rangle$, known as the **span** of A, that contains every **linear combination** $\sum_{x' \in A'} \alpha_{x'} x'$ of vectors of A, i.e. every vector that can be obtained through finitely many operations with elements of A (i.e. A' is finite) and scalars. Trivially every set is contained in its own span. If, moreover, the span of a set is contained in the set, in such a way that both sets are indeed the same set, then the set is said to be a **vector subspace**. Vector subspaces are also characterized by containing the null vector and their images by the operations of the vector space. Therefore all the properties of the vector space are inherited by any of its vector subspaces and, hence, they are vector spaces themselves over the same field.

If the field of the vector space is moreover ordered by a linear order consistent with the operations of the field, then to every set A of vectors there is associated another set, CoA, known as the **convex hull** of A, that contains every **convex linear combination** $\sum_{x' \in A} \alpha_{x'} x'$ of vectors of A, i.e. every vector that can be obtained through finitely many operations with elements of A and positive scalars not bigger than 1, that add up to 1. Trivially every set is contained in its convex hull. If, moreover, the convex hull of a set is itself contained in the set, in such a way that both sets are the same set, then the set is said to be a **convex set**.

The null vector is always in the span of any set A of vectors, since it can be obtained as a linear combination of vectors of A whose scalars are all 0. Nonetheless, if that is the only way in which the null vector can be obtained as a linear combination of vectors of A, then A is said to be **linearly independent**. Note that the null vector cannot be in any linearly independent set. A linearly independent set that moreover spans the entire space is known as a **basis**.[1] Since a basis of a vector space spans the entire space, every vector x is a linear combination $\sum_{x' \in A} \alpha_{x'} x'$ of a set A of vectors the basis. Moreover, there is just one such linear combination.[2] The trivial extension of the unique scalar-valued function α with domain A such that assigns the scalar 0 to every vector of the basis not in A is known as the **coordinate function** of the vector x with respect to the basis.

The bases are the maximals of the linearly independent sets of the vector space with respect to $\subset$.[3] As a consequence, every nontrivial vector space, i.e. every

[1] More precisely, a Hamel basis.

[2] In effect, if $\sum_{x' \in A'} \alpha'_{x'} x'$ happens to be x as well, then $\sum_{x' \in A \cup A'} \tilde{\alpha}_{x'} x'$, with $\tilde{\alpha}_{x'}$ being $\alpha_{x'}$ if $x' \in A \setminus A'$, $\alpha'_{x'}$ if $x' \in A' \setminus A$, and $\alpha_{x'} + (-\alpha_{x'})$ if $x' \in A \cap A'$, is necessarily the null vector. Since $A \cup A'$ is a set of vectors of the basis, which is a linearly independent set, then $\sum_{x' \in A \cup A'} \tilde{\alpha}_{x'} x'$ must be the trivial linear combination that gives the null vector, i.e. $\tilde{\alpha}_{x'}$ is 0 always. Therefore there cannot be any element x' in $A \setminus A'$, otherwise $\tilde{\alpha}_{x'}$, i.e. $\alpha_{x'}$ should be 0, which is not true. Hence every element of A is an element of A' as well, i.e. $A \subset A'$. Similarly $A \supset A'$, and hence A and A' are the same set. As a consequence, $\alpha_{x'} + (-\alpha'_{x'})$ must be 0, i.e. $\alpha_{x'} = \alpha'_{x'}$, for all x' as well.

[3] In effect, on the one hand, should there be a linearly independent set B' that contains a basis B but is distinct from it, then, since any $x \in B' \setminus B$ has to be a linear combination $\sum_{x' \in A'} \alpha_{x'} x'$ of vectors A' of the basis B contained in B', there would be a nontrivial linear combination of vectors of B', namely $(-x) + \sum_{x' \in A'} \alpha_{x'} x'$, that would be the null vector 0, contradicting that B' is linearly independent. On the other hand, adding any vector x that is not already in a maximal linearly independent set B to it, gives a set $B \cup \{x\}$ that contains the maximal B but is distinct from it, and hence cannot be linearly independent. As a consequence, there must exist a nontrivial linear combination of vectors of $B \cup \{x\}$, among which necessarily is x, that is the null vector 0. Therefore, x must be a linear combination of vectors of B and hence be in its span. Thus the linearly independent set B must span the entire space and hence be a basis.

vector space that is not a singleton, has a basis.[4]

All the bases of a vector space have the same number of vectors, in the sense that, given any two bases B_1 and B_2 of the space, there exist an injective function f^* that maps all the vectors of any of them to the other, in such a way that none of them can have less vectors than the other.[5] Any two sets with that property are said to have the same cardinality. The common cardinality of all its bases is therefore a characteristic of any vector space known as its **dimension**. When a vector space has a linearly independent set that is not finite, and hence necessarily all its bases are not finite, then it is said to be an **infinite-dimensional vector space**, otherwise, i.e. if there is a finite basis and hence all the basis are finite, then it is said to be a **finite dimensional vector space**.

A **norm** is a function that assigns to every non-null vector of a vector space a positive scalar in such a way that (i) the norm of the multiplication of any vector by any scalar coincides with the multiplication of the norm of the vector and the absolute value of the scalar, and (ii) it satisfies a triangular inequality by which the norm of any sum of vector cannot exceed the sum of their norms. A vector space endowed with a norm is a **normed vector space**.

Finally, an **inner product** for a real vector space[6] is a function that assigns to any couple of vectors a real in such a way that (i) the inner product of any vector with itself is nonnegative, and it is zero only if it is the null vector, (ii) it is commutative, and (iii) linear with respect to the first vector.[7] A real vector space endowed with an inner product is a **inner product space**.

Detailed definitions and statements of these and other properties follow.

Vector Space

D1. Vector space. *If $(K,+,\cdot)$ is a field, X is a set, $+ \in X^{X\times X}$ is an addition, and $\cdot \in X^{K\times X}$,[8] then $\{(X,+),(K,+,\cdot),\cdot\}$ is a vector space[9] if, and only if,*

(1) *for all $\alpha \in K$ and all $x, x', \in X$, $\alpha(x+x') = \alpha x + \alpha x'$,*
(2) *for all $\alpha, \beta \in K$ and all $x \in X$, $(\alpha+\beta)x = \alpha x + \beta x$,*
(3) *for all $\alpha, \beta \in K$ and all $x \in X$, $\alpha(\beta x) = (\alpha\beta)x$,*
(4) *for all $x \in X$, $1x = x$.*

---

[4] Any linearly independent set (e.g. any singleton containing a non null vector) can be extended to become a basis. In effect, any chain $\mathcal{B}^*$ extracted from the set $\mathcal{B}$ of all the linearly independent sets containing a given linearly independent set A has an upper bound in $\mathcal{B}$, namely $\cup_{B\in\mathcal{B}^*} B$. (In a partially ordered set, like the power set of the vector space X with respect to $\subset$, a chain is a subset for which the partial order is actually complete.) Therefore, Zorn's lemma guarantee the existence of a maximal of $\mathcal{B}$, i.e. a maximal linearly independent set of vectors that moreover contains A, that is to say a basis that contains A.

[5] In effect, f^* is the maximal of the set $\mathcal{F}$ that contains every injective function f from, say, B_1 to B_2, and such that the union of its range $f(B_1)$ and those vectors of B_1 that are not in its domain, $B_1 \setminus f^{-1}(B_2)$, is linearly independent.

[6] Inner products can be defined also for vector spaces over non ordered fields as the complex numbers, but we will only consider inner products that are real vector spaces.

[7] This is enough, the commutativity guarantees that it is linear with respect to the second vector as well.

[8] Notice that the same symbol denotes the two additions: one in X, the other in K. The same symbol (or, rather, the absence of symbol) denotes also the distributive multiplication of the field and the multiplication of a scalar and a vector. The context will make unambiguous which of them they are standing for in each occurrence. Notice also that the same symbol 0 is used to denote the zeros of both additions. The same remark applies.

[9] From now of we shall say X is a vector space over a field K, or also X is a K-vector space.

The elements of the vector space X are called **vectors** *of the space.*

Span and Convex hull

D2. Span $\langle A \rangle$ of a set of vectors A. *If X is a vector space over a field K, and $A \subset X$, then $\langle A \rangle \subset X$ is the span of A if, and only if,*

(1) *for all $x \in \langle A \rangle$ there exist $A' \subset A$ finite and $\alpha \in K^{A'}$ such that*

$$x = \sum_{x' \in A'} \alpha_{x'} x',$$

and

(2) *for all $B \subset X$ such that, for all $x \in B$ there exist $B' \subset B$ finite and $\alpha \in K^{B'}$ such that $x = \sum_{x' \in B'} \alpha_{x'} x'$,*

$$B \subset \langle A \rangle.$$

An element of $\langle A \rangle$ is called a **linear combination of** *A.*

D3. Convex hull $\mathrm{Co}A$ of a set of vectors A. *If X is a vector space over a linearly ordered field K, and $A \subset X$, then $\mathrm{Co}A \subset X$ is the convex hull of A if, and only if,*

(1) *for all $x \in \mathrm{Co}A$, there exist $A' \subset A$ finite and $\alpha \in K^{A'}$ such that*

$$x = \sum_{x' \in A'} \alpha_{x'} x'$$

and, for all $x' \in A'$, $0 \leq \alpha_{x'} \leq 1$, and

$$\sum_{x' \in A'} \alpha_{x'} = 1,$$

and

(2) *for all $B \subset X$ such that, for all $x \in B$, there exist $B' \subset B$ finite and $\alpha \in K^{B'}$ such that $x = \sum_{x' \in B'} \alpha_{x'} x'$ and, for all $x' \in B'$, $0 \leq \alpha_{x'} \leq 1$, and $\sum_{x' \in B'} \alpha_{x'} = 1$,*

$$B \subset \mathrm{Co}A.$$

An element of CoA is called a **convex linear combination of** *A.*

Vector subspaces and convex sets

D4. Vector subspace. *If X is a vector space over a field K, and $A \subset X$, then A is a vector subspace if, and only if, $\langle A \rangle \subset A$.*

D5. Convex set. *If X is a vector space over a field K, and $A \subset X$, then A is convex if, and only if, $\mathrm{Co}A \subset A$.*

Linearly independence and Bases

D6. Linearly independent set. *If X is a vector space over a field K, and $A \subset X$, then A is linearly independent if, and only if, for all $A' \subset A$ finite and all $\alpha \in K^{A'}$ such that $0 = \sum_{x' \in A'} \alpha_{x'} x'$,*

$$\alpha = 0.$$

D7. Basis. *If X is a vector space over a field K, and $B \subset X$, then B is a basis*[10] *if, and only if, B is linearly independent and $X \subset \langle B \rangle$.*

D8. Coordinate function. *If X is a vector space over a field K, $B \subset X$ a basis, and $x \in X$, then $\alpha \in K^B$ is the coordinate function of x with respect to B if, and only if, $x = \sum_{x' \in \alpha^{-1}(K \setminus \{0\})} \alpha_{x'} x'$.*

Dimension

D9. Infinite-dimensional vector space. *If X is vector space over a field K, then X is a infinite-dimensional vector space if, and only if, there exists $A \subset X$ linearly independent such that A is not finite.*

D10. Finite-dimensional vector space. *If X is vector space over a field K, then X is a finite-dimensional vector space if, and only if, for all $A \subset X$ linearly independent, A is finite.*

Norms and Inner Products

D11. Norm. *If X is vector space over an oredered field K, and $f \in K^X$ then f is a norm if, and only if,*

1. *$f^{-1}(K) = X$,*
2. *for all $x \in X \setminus \{0\}$, $0 < f(x)$,*
3. *for all $\alpha \in K$ and all $x \in X$, $f(\alpha x) = |\alpha| f(x)$, and*
4. *for all $x, x' \in X$, $f(x + x') \leq f(x) + f(x')$.*

The norm of a vector x is usually denoted $\|x\|$.

D12. Inner product of a real vector space. *If X is vector space over the field $\mathbb{R}$, and $f \in R^{X \times X}$ then f is an inner product if, and only if,*

1. *$f^{-1}(\mathbb{R}) = X \times X$,*
2. *for all $x \in X$, $0 \leq f(x, x)$,*
3. *for all $x, x' \in X$, $f(x, x') = f(x', x)$,*
4. *for all $x, x', x'' \in X$, $f(x + x', x'') = f(x, x'') + f(x', x'')$, and*
5. *for all $x, x' \in X$ and all $\alpha \in \mathbb{R}$, $f(\alpha x, x') = \alpha f(x, x')$.*

The inner product of two vectors x, x' is usually denoted $\langle x, x' \rangle$, $x \cdot x'$ or simply xx'.

[10] More precisely, a Hamel basis.

S1. Every vector subspace contains its images by the operations of the vector space. *If X is a vector space over a field K, and $A \subset X$, then A is a vector subspace of X if, and only if,*

(1) $0 \in A$
(2) *for all* $x, x' \in A$, $x + x' \in A$
(3) *for all* $\alpha \in K$ *and all* $x \in A$, $\alpha x \in A$.

Proof. Let X be a vector space over a field K, and $A \subset X$.

Assume that A is a vector subspace of X. Since A is a vector subspace of X, then $\langle A \rangle \subset A$.

(1) Since $0 \in \langle A \rangle$ and $\langle A \rangle \subset A$, then $0 \in A$.
(2) Let $x, x' \in A$. Since $x + x' \in \langle A \rangle$ and $\langle A \rangle \subset A$, then $x + x' \in A$.
(3) Let $\alpha \in K$ and $x \in A$. Since $\alpha x \in \langle A \rangle$ and $\langle A \rangle \subset A$, then $\alpha x \in A$.

Conversely. assume that

(1) $0 \in A$
(2) for all $x, x' \in A$, $x + x' \in A$
(3) for all $\alpha \in K$ and all $x \in A$, $\alpha x \in A$.

Let $x \in \langle A \rangle$. Since $x \in \langle A \rangle$, then there exist $A' \subset A$ finite and $\alpha \in K^{A'}$ such that $x = \sum_{x' \in A'} \alpha_{x'} x'$. Since, for all $\alpha \in K$ and all $x \in A$, $\alpha x \in A$, and $A' \subset A$, then, for all $x' \in A'$, $\alpha_{x'} x' \in A$. Since, for all $x, x' \in A$, $x + x' \in A$, and, for all $x' \in A'$, $\alpha_{x'} x' \in A$, then $\sum_{x' \in A'} \alpha_{x'} x' \in A$. Since $x = \sum_{x' \in A'} \alpha_{x'} x'$ and $\sum_{x' \in A'} \alpha_{x'} x' \in A$, then $x \in A$. Since, for all $x \in \langle A \rangle$, $x \in A$, then $\langle A \rangle \subset A$. Q.E.D.

S2. The intersection of two vector subspaces is a vector subspace. *If X is a vector space over a field K and $A, A' \subset X$ are vector subspaces, then $A \cap A'$ is a vector subspace.*

Proof. Let X be a vector space over a field K and $A, A' \subset X$ be vector subspaces.

(1) Since A is a vector subspace, then $0 \in A$. Since A' is a vector subspace, then $0 \in A'$. Since $0 \in A$ and $0 \in A'$, then $0 \in A \cap A'$.
(2) Let $x, x' \in A \cap A'$. Since $x, x' \in A \cap A'$, then $x, x' \in A$. Since $x, x' \in A$ and A is a vector subspace, then $x + x' \in A$. Since $x, x' \in A \cap A'$, then $x, x' \in A'$. Since $x, x' \in A'$ and A' is a vector subspace, then $x + x' \in A'$. Since $x + x' \in A$ and $x + x' \in A'$, then $x + x' \in A \cap A'$.
(3) let $x \in A \cap A'$ and $\alpha \in K$. Since $x \in A \cap A'$, then $x \in A$. Since $x \in A$, $\alpha \in K$, and A is a vector subspace, then $\alpha x \in A$. Since $x \in A \cap A'$, then $x \in A'$. Since $x \in A'$, $\alpha \in K$, and A' is a vector subspace, then $\alpha x \in A'$. Since $\alpha x \in A$ and $\alpha x \in A'$, then $\alpha x \in A \cap A'$.

Since $0 \in A \cap A'$, for all $x, x' \in A \cap A'$, $x + x' \in A \cap A'$, and, for all $x \in A \cap A'$ and all $\alpha \in K$, $\alpha x \in A \cap A'$, then $A \cap A'$ is a vector subspace. Q.E.D.

S3. The intersection of two convex sets is a convex set. *If X is a vector space over a linearly ordered field K and $A, A' \subset X$ are convex sets, then $A \cap A'$ is a convex set.*

Proof. Exercise.

S3. Every vector subspace of a vector space over a linearly ordered field is convex. *If X is a vector space over a linearly ordered field K and $A \subset X$ is a vector subspace, then A is a convex set.*

Proof. Exercise.

S4. Any vector has a unique decomposition with respect to a basis. *If X is a vector space over a field K, $B \subset X$ is a basis, $x \in X \setminus \{0\}$, and*

(1) *$A \subset B$ is finite and $\alpha \in (K \setminus \{0\})^A$ is such that $x = \sum_{x' \in A} \alpha_{x'} x'$, and*
(2) *$A' \subset B$ is finite and $\alpha' \in (K \setminus \{0\})^{A'}$ is such that $x = \sum_{x' \in A'} \alpha'_{x'} x'$,*

then $A = A'$ and $\alpha = \alpha'$.

Proof. Let X be a vector space over a field K, $B \subset X$ be a basis, $x \in X \setminus \{0\}$, and $A \subset B$ be finite and $\alpha \in (K \setminus \{0\})^A$ be such that $x = \sum_{x' \in A} \alpha_{x'} x'$, and $A' \subset B$ be finite and $\alpha' \in (K \setminus \{0\})^{A'}$ be such that $x = \sum_{x' \in A'} \alpha'_{x'} x'$.

Since A, A' are finite, then $A \cup A'$ is finite. Let $\tilde{\alpha} \in K^{A \cup A'}$ be such that, for all $x' \in A \cup A'$,

$$\tilde{\alpha}_{x'} = \begin{cases} \alpha_{x'} & \forall x' \in A \setminus A' \\ \alpha_{x'} - \alpha'_{x'} & \forall x' \in A \cap A' \\ -\alpha'_{x'} & \forall x' \in A' \setminus A. \end{cases}$$

Since $x = \sum_{x' \in A} \alpha_{x'} x'$, $x = \sum_{x' \in A'} \alpha'_{x'} x'$, and $\tilde{\alpha} \in K^{A \cup A'}$ is such that, for all $x' \in A \setminus A'$, $\tilde{\alpha}_{x'} = \alpha_{x'}$, for all $x' \in A \cap A'$, $\tilde{\alpha}_{x'} = \alpha_{x'} - \alpha'_{x'}$, and, for all $x' \in A' \setminus A$, $\tilde{\alpha}_{x'} = -\alpha'_{x'}$, then

$$\begin{aligned} 0 &= \sum_{x' \in A} \alpha_{x'} x' - \sum_{x' \in A'} \alpha'_{x'} x' \\ &= \sum_{x' \in A \setminus A'} \alpha_{x'} x' + \sum_{x' \in A \cap A'} \alpha_{x'} x' - \sum_{x' \in A' \cap A} \alpha'_{x'} x' - \sum_{x' \in A' \setminus A} \alpha'_{x'} x' \\ &= \sum_{x' \in A \setminus A'} \alpha_{x'} x' + \sum_{x' \in A \cap A'} \big(\alpha_{x'} - \alpha'_{x'}\big) x' + \sum_{x' \in A' \setminus A} (-\alpha'_{x'})) x' \\ &= \sum_{x' \in A \cup A'} \tilde{\alpha}_{x'} x'. \end{aligned}$$

Since $A \subset B$ and $A' \subset B$, then $A \cup A' \subset B$. Since A is finite and A' is finite, then $A \cup A'$ is finite. Since B is a basis, then B is linearly independent. Since B is linearly independent, $A \cup A' \subset B$ is finite, and $\tilde{\alpha} \in K^{A \cup A'}$ is such that $0 = \sum_{x' \in A \cup A'} \tilde{\alpha}_{x'} x'$, then $\tilde{\alpha} = 0$.

(1) Assume that $A \setminus A' \neq \phi$. Let $x' \in A \setminus A'$. Since $x' \in A \setminus A'$, then $\tilde{\alpha}_{x'} = \alpha_{x'}$. Since $\tilde{\alpha} = 0$, then $\tilde{\alpha}_{x'} = 0$. Since $\tilde{\alpha}_{x'} = \alpha_{x'}$ and $\tilde{\alpha}_{x'} = 0$, then $\alpha_{x'} = 0$!! Therefore, $A \setminus A' = \phi$. Since $A \setminus A' = \phi$, then $A \subset A'$.
(2) Assume that $A' \setminus A \neq \phi$. Let $x' \in A' \setminus A$. Since $x' \in A' \setminus A$, then $\tilde{\alpha}_{x'} = \alpha'_{x'}$. Since $\tilde{\alpha} = 0$, then $\tilde{\alpha}'_{x'} = 0$. Since $\tilde{\alpha}_{x'} = \alpha'_{x'}$ and $\tilde{\alpha}_{x'} = 0$, then $\alpha'_{x'} = 0$!! Therefore, $A' \setminus A = \phi$. Since $A' \setminus A = \phi$, then $A' \subset A$.

Since $A \subset A'$ and $A' \subset A$, then $A = A'$.

Let $x' \in A$. Since $A = A'$, then $A = A \cap A'$. Since $x' \in A$ and $A = A \cap A'$, then $x' \in A \cap A'$. Since $x' \in A \cap A'$, then $\tilde{\alpha}_{x'} = \alpha_{x'} - \alpha'_{x'}$. Since $\tilde{\alpha} = 0$, then $\tilde{\alpha}'_{x'} = 0$. Since $\tilde{\alpha}_{x'} = \alpha_{x'} - \alpha'_{x'}$ and $\tilde{\alpha}_{x'} = 0$, then $\alpha_{x'} - \alpha'_{x'} = 0$. Since $\alpha_{x'} - \alpha'_{x'} = 0$, then $\alpha_{x'} = \alpha'_{x'}$. Since, for all $x' \in A$, $\alpha_{x'} = \alpha'_{x'}$, then $\alpha = \alpha'$. Q.E.D.

S5. The bases are the maximals of the linearly independent sets of vectors with respect to $\subset$. *If X is a vector space over a field K and $B \subset X$, then B is a basis*[11] *if, and only if,*

(1) *B is linearly independent, and*
(2) *for all $B' \subset X$ linearly independent such that $B \subset B'$, $B = B'$.*

Proof. Let X be a vector space over K and $B \subset X$.

Assume that B is a basis.

(1) B is linearly independent:

Since B is a basis, then B is linearly independent.

(2) for all $B' \subset X$ linearly independent such that $B \subset B'$, $B = B'$:

Let $B' \subset X$ be linearly independent and such that $B \subset B'$. Assume that $B \neq B'$. Since $B \neq B'$, then $B \not\supset B'$. Since $B \not\supset B'$, then $B' \setminus B \neq \phi$. Let $x \in B' \setminus B$. Since B is a basis, then $X \subset \langle B \rangle$. Since $x \in B'$ and $B' \subset X$, then $x \in X$. Since $x \in X$ and $x \subset \langle B \rangle$, then $x \in \langle B \rangle$. Since $x \in \langle B \rangle$, then there exist $A \subset B$ finite and $\alpha \in K^A$ such that

$$x = \sum_{x' \in A} \alpha_{x'} x'.$$

Since $x \in B' \setminus B$ and $A \subset B$ then $x \notin A$. Let $\alpha' \in K^{A \cup \{x\}}$ be such that

$$\alpha'_{x'} = \begin{cases} \alpha_{x'} & \text{if } x' \in A \\ -1 & \text{if } x' = x \end{cases}$$

Since $x = \sum_{x' \in A} \alpha_{x'} x'$ and, for all $x' \in A$, $\alpha'_{x'} = \alpha_{x'}$, and $\alpha'_x = -1$, then

$$0 = \sum_{x' \in A \cup \{x\}} \alpha'_{x'} x'.$$

Since $A \subset B$ and $B \subset B'$, then $A \subset B'$. Since $x \in B' \setminus B$, then $\{x\} \subset B'$. Since $A \subset B'$ and $\{x\} \subset B'$, then $A \cup \{x\} \subset B'$. Since A is finite, then $A \cup \{x\}$ is finite. Since $A \cup \{x\} \subset B'$ is finite, $\alpha' \in K^{A \cup \{x\}}$, $\alpha' \neq 0$, and $0 = \sum_{x' \in A \cup \{x\}} \alpha'_{x'} x'$, then B' is not linearly independent!! Therefore $B \supset B'$. Since $B \subset B'$ and $B \supset B'$, then $B = B'$.

Conversely, assume that

(1) B is linearly independent, and
(2) for all $B' \subset X$ linearly independent, either $B = B'$ or $B \not\subset B'$.

Then

(1) B is linearly independent, trivially.
(2) $X \subset \langle B \rangle$:

Let $x \in X$. Either $x \in B$ or $x \notin B$. If $x \in B$, then $x \in \langle B \rangle$. If $x \notin B$, then $B \neq B \cup \{x\}$. Since, (i) for all $B' \subset X$ linearly independent such that $B \subset B'$, $B = B'$, (ii) $B \subset B \cup \{x\}$, and (iii) $B \neq B \cup \{x\}$, then $B \cup \{x\}$ is

[11] More precisely, a Hamel basis.

not linearly independent. Since $B \cup \{x\}$ is not linearly independent, then there exist $A \subset B \cup \{x\}$ finite and $\alpha \in K^A$ such that $\alpha \neq 0$ and

$$0 = \sum_{x' \in A} \alpha_{x'} x'.$$

Since B is linearly independent, then $x \in A$,[12] and $\alpha_x \neq 0$.[13] Since $0 = \sum_{x' \in A} \alpha_{x'} x'$, $x \in A$, and $\alpha_x \neq 0$, then

$$x = \sum_{x' \in A \setminus \{x\}} \alpha_x^{-1}(-\alpha_{x'}) x'.$$

Since $A \subset B \cup \{x\}$, then $A \setminus \{x\} \subset B$. Since A is finite, then $A \setminus \{x\}$ is finite. Since $A \setminus \{x\} \subset B$ is finite and $x = \sum_{x' \in A \setminus \{x\}} \alpha_x^{-1}(-\alpha_{x'}) x'$, then $x \in \langle B \rangle$. Since, for all $x \in X$, $x \in \langle B \rangle$, then $X \subset \langle B \rangle$.

Since B is linearly independent and $X \subset \langle B \rangle$, then B is a basis. Q.E.D.

S6. Any linearly independent set is contained in a basis. *If X is a vector space over a field K and $A \subset X$ is linearly independent, then there exists a basis[14] $B \subset X$ such that $A \subset B$.*

Proof.

Let X be a vector space over a field K and $A \subset X$ be linearly independent.

A. Consider the set $\mathcal{B}$ of all the linearly independent sets containing A:

Let $\mathcal{B} \subset \mathcal{P}(X)$ be such that,

(1) for all $B \in \mathcal{B}$, B is linearly independent and $A \subset B$, and
(2) for all $\mathcal{B}' \subset \mathcal{P}(X)$ such that, for all $B \in \mathcal{B}'$, B is linearly independent and $A \subset B$,

$$\mathcal{B}' \subset \mathcal{B}.$$

B. All the chains in $\mathcal{B}$ w.r.t. $\subset$ have an upper bound in $\mathcal{B}$:

Let $\mathcal{B}^* \subset \mathcal{B}$ be a chain with respect to $\subset$, i.e. such that, for all $B, B' \in \mathcal{B}^*$, either $B \subset B'$ or $B \supset B'$.

(1) $\cup_{B \in \mathcal{B}^*} B$ is an upper bound of $\mathcal{B}^*$ with respect to $\subset$:

Let $B' \in \mathcal{B}^*$. Since $B' \in \mathcal{B}^*$, then $B' \subset \cup_{B \in \mathcal{B}^*} B$. Since, for all $B' \in \mathcal{B}^*$, $B' \subset \cup_{B \in \mathcal{B}^*} B$, then $\cup_{B \in \mathcal{B}^*} B$ is an upper bound of $\mathcal{B}^*$ with respect to $\subset$.

(2) $\cup_{B \in \mathcal{B}^*} B \in \mathcal{B}$:

(2.a) $\cup_{B \in \mathcal{B}^*} B$ is linearly independent:

Assume that $\cup_{B \in \mathcal{B}^*} B$ is not linearly independent.

---

[12] Otherwise, $A \subset B$ is finite, $\alpha \in K^A$, $\alpha \neq 0$ and $0 = \sum_{x' \in A} \alpha_{x'} x'$, and hence B would not be linearly independent!!

[13] Otherwise $A \setminus \{x\} \subset B$ is finite, $\alpha \in K^{A \setminus \{x\}}$, $\alpha \neq 0$ and $0 = \sum_{x' \in A \setminus \{x\}} \alpha_{x'} x'$, and hence B would not be linearly independent!!

[14] More precisely, a Hamel basis.

(2.a.i) Since $\cup_{B\in\mathcal{B}^*}B$ is not linearly independent, then there exists $B' \subset \cup_{B\in\mathcal{B}^*}B$ finite and $\alpha \in K^{B'}$ such that $0 = \sum_{x'\in B'} \alpha_{x'}x'$ and $\alpha \neq 0$. Since B' is finite and $B' \subset \cup_{B\in\mathcal{B}^*}B$, then there exist $\mathcal{B'}^* \subset \mathcal{B}^*$ finite and such that $B' \subset \cup_{B\in\mathcal{B'}^*}B$. Since $\mathcal{B'}^* \subset \mathcal{B}^*$ and $\mathcal{B}^*$ is a chain with respect to $\subset$, then $\mathcal{B'}^*$ is a chain with respect to $\subset$. Since $\mathcal{B'}^*$ is a chain with respect to $\subset$ and $\mathcal{B'}^*$ is finite, then there exists $B'' \in \mathcal{B'}^*$ maximum with respect to $\subset$. Since B'' is a maximum of $\mathcal{B'}^*$ with respect to $\subset$ and $B' \in \mathcal{B'}^*$, then $B' \subset B''$. Since $B'' \in \mathcal{B'}^*$, $\mathcal{B'}^* \subset \mathcal{B}^*$, and $\mathcal{B}^* \subset \mathcal{B}$, then $B'' \in \mathcal{B}$. Since $B'' \in \mathcal{B}$, then B'' is linearly independent.

(2.a.ii) Since there exists $B' \subset B''$ finite and $\alpha \in K^{B'}$ such that $0 = \sum_{x'\in B'} \alpha_{x'}x'$ and $\alpha \neq 0$, then B'' is not linearly independent!!

Therefore $\cup_{B\in\mathcal{B}^*}B$ is linearly independent.

(2.b) $A \subset \cup_{B\in\mathcal{B}^*}B$:

Since, for all $B \in \mathcal{B}$, $A \subset B$, and $\mathcal{B}^* \subset \mathcal{B}$, then, for all $B \in \mathcal{B}^*$, $A \subset B$. Since for all $B \in \mathcal{B}^*$, $A \subset B$, then $A \subset \cup_{B\in\mathcal{B}^*}B$.

Since (2.a) $\cup_{B\in\mathcal{B}^*}B$ is linearly independent and (2.b) $A \subset \cup_{B\in\mathcal{B}^*}B$, then

$$\cup_{B\in\mathcal{B}^*}B \in \mathcal{B}.$$

C. There exists a maximal B of $\mathcal{B}$ w.r.t. $\subset$:

Since, for all chain $\mathcal{B}^* \subset \mathcal{B}$ there exists an upper bound of $\mathcal{B}^*$ with respect to $\subset$ that is in $\mathcal{B}$, then[15] there exists $B \in \mathcal{B}$ such that, for all $B' \in \mathcal{B}$ such that $B \subset B'$,

$$B = B'.$$

D. B is a basis:

(1) B is linearly independent:

Since $B \in \mathcal{B}$, then B is linearly independent.

(2) for all $B' \subset X$ linearly independent such that $B \subset B'$, $B = B'$:

Let $B' \subset X$ be linearly independent and such that $B \subset B'$. Since $B \in \mathcal{B}$, then $A \subset B$. Since $A \subset B$ and $B \subset B'$, then $A \subset B'$. Since B' is linearly independent and $A \subset B'$, then $B' \in \mathcal{B}$. Since (i) $B' \in \mathcal{B}$, (ii) B is a maximal of $\mathcal{B}$ with respect to $\subset$, and (iii) $B \subset B'$, then

$$B = B'.$$

Since (1) B is linearly independent and (2) for all $B' \subset X$ linearly independent such that $B \subset B'$, $B = B'$, then B is a basis.

E. $A \subset B$:

Since $B \in \mathcal{B}$, then $A \subset B$. Q.E.D.

[15] By Zorn's Lemma.

S7. Every nontrivial vector space has a basis. *If X is a vector space over a field K and X is not a singleton, then there exists $B \subset X$ such that B is a basis.*[16]

Proof. Let X be a vector space over K and X not be a singleton. Since X is a vector space over a field K and X is not a singleton, then $X \setminus \{0\} \neq \phi$. Let $x \in X \setminus \{0\}$. Since $x \in X \setminus \{0\}$, then $x \neq 0$. Since $x \neq 0$, then $\{x\}$ is linearly independent. Since $\{x\}$ is linearly independent, then there exists a basis $B \subset X$ such that $\{x\} \subset B$. Q.E.D.

S8. All the bases of a vector space have the same number of vectors. *If X is a vector space over a field K and $B_1, B_2 \subset X$ are bases,*[17] *then there exists $f^* \in B_2^{B_1}$ injective such that ${f^*}^{-1}(B_2) = B_1$.*

Proof.

Let X be a vector space over K and $B_1, B_2 \subset X$ be bases.

A. Consider the set $\mathcal{F}$ of all the injective functions from B_1 to B_2 such that the union of range and the complement of the domain in B_1 is linearly independent:

Let $\mathcal{F} \subset B_2^{B_1}$ be such that, for all $f \in \mathcal{F}$, f is injective and $f(B_1) \cup (B_1 \setminus f^{-1}(B_2))$ is linearly independent.

B. $\mathcal{F} \neq \phi$:

Let $x \in B_2$. Since B_1 is a basis, then there exist a unique $A \subset B_1$ finite and a unique $\alpha \in (K \setminus \{0\})^A$ such that $x = \sum_{x' \in A} \alpha_{x'} x'$.

Let $x' \in A$.

Let $f \in B_2^{\{x'\}}$ be such that $f(x') = x$.

(1) $f \in \mathcal{F}$:

(1.a) f is injective:

Since, for all $(x_1, x), (x_2, x) \in f$, $x_1 = x'$ and $x_2 = x'$, then f is injective.

(1.b) $f(B_1) \cup (B_1 \setminus f^{-1}(B_2))$ is linearly independent:

Assume that $f(B_1) \cup (B_1 \setminus f^{-1}(B_2))$ is not linearly independent.

Since $f(B_1) \cup (B_1 \setminus f^{-1}(B_2))$ is not linearly independent and $f(B_1) \cup (B_1 \setminus f^{-1}(B_2)) = \{x\} \cup (B_1 \setminus \{x'\})$, then there exist $A' \subset \{x\} \cup (B_1 \setminus \{x'\})$ finite and $\alpha' \in K^{A'}$ such that

$$0 = \sum_{x' \in A'} \alpha'_{x'} x'$$

and $\alpha \neq 0$. Since B_1 is linearly independent, then $x \in A'$,[18] and $\alpha'_x \neq 0$.[19] Since $0 = \sum_{x' \in A'} \alpha'_{x'} x'$, $x \in A'$, and $\alpha'_x \neq 0$, then

$$x = \sum_{x' \in A' \setminus \{x\}} {\alpha'}_x^{-1} (-\alpha'_{x'}) x'.$$

---

[16] More precisely, a Hamel basis.

[17] More precisely, Hamel bases.

[18] Otherwise, $A' \subset B_1 \setminus \{x'\} \subset B_1$ is finite, $\alpha' \in K^{A'}$, $\alpha' \neq 0$ and $0 = \sum_{x' \in A'} \alpha'_{x'} x'$, and hence B_1 would not be linearly independent!!

[19] Otherwise $A' \setminus \{x\} \subset B_1 \setminus \{x'\} \subset B_1$ is finite, $\alpha' \in K^{A' \setminus \{x\}}$, $\alpha' \neq 0$ and $0 = \sum_{x' \in A' \setminus \{x\}} \alpha'_{x'} x'$, and hence B_1 would not be linearly independent!!

Since B_2 is a basis, then B_2 is linearly independent. Since B_2 is linearly independent and $x \in B_2$, then $x \neq 0$. Since $x \neq 0$ and $x = \sum_{x' \in A' \setminus \{x\}} {\alpha'}_x^{-1}(-\alpha'_{x'})x'$, then there exists $x' \in A' \setminus \{x\}$ such that

$${\alpha'}_x^{-1}(-\alpha'_{x'}) \neq 0.$$

Since $A' \subset \{x\} \cup (B_1 \setminus \{x'\})$, then $A' \setminus \{x\} \subset B_1 \setminus \{x'\}$. Since $A' \setminus \{x\} \subset B_1 \setminus \{x'\}$ and $x' \in A$, then $A' \setminus \{x\} \neq A$. Since (i) $x = \sum_{x' \in A' \setminus \{x\}} {\alpha'}_x^{-1}(-\alpha'_{x'})x'$, (ii) ${\alpha'}_x^{-1}(-\alpha'_{x'}) \neq 0$, and (iii) $A' \setminus \{x\} \neq A$, then it is not true that there exists a unique $A \subset B_1$ finite and $\alpha \in (K \setminus \{0\})^A$ such that $x = \sum_{x' \in A} \alpha_{x'} x'$!!

Therefore, $f(B_1) \cup (B_1 \setminus f^{-1}(B_2))$ is linearly independent.

Since (1.a) f is injective and (1.b) $f(B_1) \cup (B_1 \setminus f^{-1}(B_2))$ is linearly independent, then $f \in \mathcal{F}$.

C. All the chains in $\mathcal{F}$ w.r.t. $\subset$ have an upper bound in $\mathcal{F}$:

Let $\mathcal{F}^* \subset \mathcal{F}$ be a chain with respect to $\subset$.
Let $f' = \cup_{f \in \mathcal{F}^*} f$.

(1) f' is an upper bound of $\mathcal{F}^*$ with respect to $\subset$:

Since $f' = \cup_{f \in \mathcal{F}^*} f$, then, for all $f \in \mathcal{F}^*$, $f \subset f'$. Since, for all $f \in \mathcal{F}^*$, $f \subset f'$, then f' is an upper bound of $\mathcal{F}^*$ with respect to $\subset$.

(2) $f' \in \mathcal{F}$:

(2.a) f' is injective:

Let $(x_1, x), (x_2, x) \in f'$.
Since $(x_1, x) \in f'$, then there exists $f_1 \in \mathcal{F}^*$ such that $(x_1, x) \in f_1$.
Since $(x_2, x) \in f'$, then there exists $f_2 \in \mathcal{F}^*$ such that $(x_2, x) \in f_2$.
Since $f_1, f_2 \in \mathcal{F}^*$ and $\mathcal{F}^* \subset \mathcal{F}$, then $f_1, f_2 \in \mathcal{F}$.
Since $f_1, f_2 \in \mathcal{F}$, then f_1, f_2 are injective.
Since $f_1, f_2 \in \mathcal{F}^*$ and $\mathcal{F}^*$ is a chain with respect to $\subset$, then either $f_1 \subset f_2$ or $f_2 \subset f_1$.

(i) If $f_1 \subset f_2$, then $(x_1, x), (x_2, x) \in f_2$. Since $(x_1, x), (x_2, x) \in f_2$ and f_2 is injective, then $x_1 = x_2$.

(ii) If $f_2 \subset f_1$, then $(x_1, x), (x_2, x) \in f_1$. Since $(x_1, x), (x_2, x) \in f_1$ and f_1 is injective, then $x_1 = x_2$.

Since, for all $(x_1, x), (x_2, x) \in f'$, $x_1 = x_2$, then f' is injective.

(2.b) $f'(B_1) \cup (B_1 \setminus {f'}^{-1}(B_2))$ is linearly independent:

Assume that $f'(B_1) \cup (B_1 \setminus {f'}^{-1}(B_2))$ is not linearly independent.

Since $f'(B_1) \cup (B_1 \setminus {f'}^{-1}(B_2))$ is not linearly independent, then there exist $A \subset f'(B_1) \cup (B_1 \setminus {f'}^{-1}(B_2))$ finite and $\alpha \in K^A$ such that $\alpha \neq 0$ and

$$0 = \sum_{x' \in A} \alpha_{x'} x'.$$

Since B_1 is linearly independent, then[20]

$$A \cap f'(B_1) \neq \phi.$$

Since, for all $x \in A \cap f'(B_1)$, there exists $x'_x \in B_1$ such that $(x'_x, x) \in f'$, and $f' = \cup_{f \in \mathcal{F}^*} f$, then, for all $x \in A \cap f'(B_1)$, there exist $x'_x \in B_1$ and $f_x \in \mathcal{F}^*$ such that

$$(x'_x, x) \in f_x.$$

(2.b.i) $\exists x^* \in A \cap f'(B_1) | \forall x \in A \cap f'(B_1), f_x \subset f_{x^*}$:

Since A is finite, then $A \cap f'(B_1)$ is finite.
Since (i) $A \cap f'(B_1)$ is finite,
(ii) for all $x \in A \cap f'(B_1)$, $f_x \in \mathcal{F}^*$, and
(iii) $\mathcal{F}^*$ is a chain with respect to $\subset$,
then there exists $x^* \in A \cap f'(B_1)$ such that, for all $x \in A \cap f'(B_1)$,

$$f_x \subset f_{x^*}.$$

(2.b.ii) $A \cap f'(B_1) \subset A \cap f_{x^*}(B_1)$:

Let $x \in A \cap f'(B_1)$.
Since $x \in A \cap f'(B_1)$, then there exists $x'_x \in B_1$ such that $(x'_x, x) \in f_x$.
Since $(x'_x, x) \in f_x$ and $f_x \subset f_{x^*}$, then $(x'_x, x) \in f_{x^*}$.
Since, $x'_x \in B_1$ and $(x'_x, x) \in f_{x^*}$, then $x \in f_{x^*}(B_1)$.
Since $x \in A$ and $x \in f_{x^*}(B_1)$, then $x \in A \cap f_{x^*}(B_1)$.
Since, for all $x \in A \cap f'(B_1)$, $x \in A \cap f_{x^*}(B_1)$, then

$$A \cap f'(B_1) \subset A \cap f_{x^*}(B_1).$$

(2.b.iii) $A \cap (B_1 \setminus f'^{-1}(B_2)) \subset A \cap (B_1 \setminus f_{x^*}^{-1}(B_2))$:

Let $x \in A \cap (B_1 \setminus f'^{-1}(B_2))$.
Since $x \in A \cap (B_1 \setminus f'^{-1}(B_2))$, then $x \in A$ and $x \in B_1 \setminus f'^{-1}(B_2)$.
Since $x \in B_1 \setminus f'^{-1}(B_2)$, then $x \in B_1$ and $x \notin f'^{-1}(B_2)$.
Since $x \notin f'^{-1}(B_2)$, then, for all $x' \in B_2$,

$$\begin{aligned}(x, x') &\notin f' \\ &= \cup_{f \in \mathcal{F}^*} f.\end{aligned}$$

Since, for all $x' \in B_2$, $(x, x') \notin \cup_{f \in \mathcal{F}^*} f$, then, for all $x' \in B_2$ and all $f \in \mathcal{F}^*$,

$$(x, x') \notin f.$$

Since, for all $x' \in B_2$ and all $f \in \mathcal{F}^*$, $(x, x') \notin f$, then, for all $f \in \mathcal{F}^*$,

$$x \notin f^{-1}(B_2).$$

[20] Otherwise, $A \subset B_1 \setminus f'^{-1}(B_2) \subset B_1$ is finite, $\alpha \in K^A$, $\alpha \neq 0$ and $0 = \sum_{x' \in A} \alpha_{x'} x'$, and hence B_1 would not be linearly independent!!

Since $x \in B_1$ and, for all $f \in \mathcal{F}^*$, $x \notin f^{-1}(B_2)$, then, for all $f \in \mathcal{F}^*$,

$$x \in B_1 \setminus f^{-1}(B_2).$$

Since, for all $f \in \mathcal{F}^*$, $x \in B_1 \setminus f^{-1}(B_2)$, and $f_{x^*} \in \mathcal{F}^*$, then

$$x \in B_1 \setminus f_{x^*}^{-1}(B_2).$$

Since $x \in A$ and $x \in B_1 \setminus f_{x^*}^{-1}(B_2)$, then $x \in A \cap (B_1 \setminus f_{x^*}^{-1}(B_2))$.
Since, for all $x \in A \cap (B_1 \setminus f'^{-1}(B_2))$, $x \in A \cap (B_1 \setminus f_{x^*}^{-1}(B_2))$, then

$$A \cap (B_1 \setminus f'^{-1}(B_2)) \subset A \cap (B_1 \setminus f_{x^*}^{-1}(B_2)).$$

(2.b.iv) $A \subset f_{x^*}(B_1) \cup (B_1 \setminus f_{x^*}^{-1}(B_2))$:

Since $A \subset f'(B_1) \cup (B_1 \setminus f'^{-1}(B_2))$,
(2.b.ii) $A \cap f'(B_1) \subset A \cap f_{x^*}(B_1)$, and
(2.b.iii) $A \cap (B_1 \setminus f'^{-1}(B_2)) \subset A \cap (B_1 \setminus f_{x^*}^{-1}(B_2))$, then

$$\begin{aligned} A &= \\ A \cap (f'(B_1) \cup (B_1 \setminus f'^{-1}(B_2)) &= \\ (A \cap f'(B_1)) \cup (A \cap (B_1 \setminus f'^{-1}(B_2))) &\subset (A \cap f_{x^*}(B_1)) \cup (A \cap (B_1 \setminus f_{x^*}^{-1}(B_2))) \\ &= A \cap (f_{x^*}(B_1)) \cup (B_1 \setminus f_{x^*}^{-1}(B_2))) \\ &\subset f_{x^*}(B_1) \cup (B_1 \setminus f_{x^*}^{-1}(B_2)). \end{aligned}$$

(2.b.v) $f_{x^*} \notin \mathcal{F}^*$!! :

Since (2.b.iv) $A \subset f_{x^*}(B_1) \cup (B_1 \setminus f_{x^*}^{-1}(B_2))$ is finite, $\alpha \in K^A$, $\alpha \neq 0$, and

$$0 = \sum_{x' \in A} \alpha_{x'} x',$$

then $f_{x^*}(B_1) \cup (B_1 \setminus f_{x^*}^{-1}(B_2))$ is not linearly independent.
Since $f_{x^*}(B_1) \cup (B_1 \setminus f_{x^*}^{-1}(B_2))$ is not linearly independent, then $f_{x^*} \notin \mathcal{F}$.
Since $f_{x^*} \notin \mathcal{F}$ and $\mathcal{F}^* \subset \mathcal{F}$, then $f_{x^*} \notin \mathcal{F}^*$!!

Therefore $f'(B_1) \cup (B_1 \setminus f'^{-1}(B_2))$ is linearly independent.

Since (2.a) f' is injective and (2.b) $f'(B_1) \cup (B_1 \setminus f'^{-1}(B_2))$ is linearly independent, then

$$f' \in \mathcal{F}.$$

D. There exists a maximal f^* of $\mathcal{F}$ w.r.t. $\subset$:

Since, for all chain $\mathcal{F}^* \subset \mathcal{F}$ with respect to $\subset$, there exists an upper bound f' of $\mathcal{F}^*$ with respect to $\subset$ and $f' \in \mathcal{F}$, then[21] there exists $f^* \in \mathcal{F}$ such that, for all $f' \in \mathcal{F}$ such that $f^* \subset f'$ $f^* = f'$.

---

[21] By Zorn's Lemma.

E. $f^* \in B_2^{B_1}$ **and** f^* **is injective:**

Since $f^* \in \mathcal{F}$ and $\mathcal{F} \subset B_2^{B_1}$, then $f^* \in B_2^{B_1}$.
Since $f^* \in \mathcal{F}$ and, for all $f \in \mathcal{F}$, f is injective, then f^* is injective.

F. ${f^*}^{-1}(B_2) = B_1$**:**

Assume that ${f^*}^{-1}(B_2) \not\supset B_1$.

(1) $B_2 \cup (B_1 \setminus {f^*}^{-1}(B_2))$ is not linearly independent:

Since ${f^*}^{-1}(B_2) \not\supset B_1$, then $B_1 \setminus {f^*}^{-1}(B_2) \neq \phi$.
Let $x \in B_1 \setminus {f^*}^{-1}(B_2)$.
Since $x \in B_1 \setminus {f^*}^{-1}(B_2)$, then $x \in B_1$.
Since $x \in B_1$ and B_1 is a basis, then $x \neq 0$.
Since $x \neq 0$ and B_2 is a basis, then there exist a unique $A \subset B_2$ finite and a unique $\alpha \in (K \setminus \{0\})^A$ such that

$$x = \sum_{x' \in A} \alpha_{x'} x'.$$

Since $A \subset B_2$ and $x \in B_1 \setminus {f^*}^{-1}(B_2)$, then $A \cup \{x\} \subset B_2 \cup (B_1 \setminus {f^*}^{-1}(B_2))$.
Since A is finite, then $A \cup \{x\}$ is finite.
Let $\alpha' \in K^{A \cup \{x\}}$ be such that

$$\alpha'_{x'} = \begin{cases} \alpha_{x'} & \forall x' \in A \\ -1 & \text{if } x' = x. \end{cases}$$

Since $x = \sum_{x' \in A} \alpha_{x'} x'$, $\alpha'_x = -1$, and, for all $x' \in A$, $\alpha'_{x'} = \alpha_{x'}$, then

$$0 = \sum_{x' \in A \cup \{x\}} \alpha'_{x'} x'.$$

Since $A \cup \{x\} \subset B_2 \cup (B_1 \setminus {f^*}^{-1}(B_2))$ is finite, $\alpha' \in K^{A \cup \{x\}}$, $\alpha' \neq 0$, and $0 = \sum_{x' \in A \cup \{x\}} \alpha'_{x'} x'$, then $B_2 \cup (B_1 \setminus {f^*}^{-1}(B_2))$ is not linearly independent.

(2) f^* is not a maximal of $\mathcal{F}$ with respect to $\subset$!! :

Since
(i) $f^*(B_1) \cup (B_1 \setminus {f^*}^{-1}(B_2))$ is linearly independent and
(ii) $B_2 \cup (B_1 \setminus {f^*}^{-1}(B_2))$ is not linearly independent, then

$$f^*(B_1) \neq B_2.$$

Since $f^*(B_1) \neq B_2$, then $f^*(B_1) \not\supset B_2$.
Since $f^*(B_1) \not\supset B_2$, then $B_2 \setminus f^*(B_1) \neq \phi$.
Let $x_1 \in B_1 \setminus {f^*}^{-1}(B_2)$.
Let $x_2 \in B_2 \setminus f^*(B_1)$.
Either (2.1) $f^*(B_1) \cup \{x_2\} \cup (B_1 \setminus {f^*}^{-1}(B_2))$ is linearly independent,
or (2.2) $f^*(B_1) \cup \{x_2\} \cup (B_1 \setminus {f^*}^{-1}(B_2))$ is not linearly independent.

(2.1) Assume that $f^*(B_1) \cup \{x_2\} \cup (B_1 \setminus f^{*-1}(B_2))$ is linearly independent.

Let $\tilde{f} \in B_2^{B_1}$ be such that $\tilde{f}^{-1}(B_2) = f^{*-1}(B_2) \cup \{x_1\}$ and

$$\tilde{f}(x) = \begin{cases} f^*(x) & \forall x \in f^{*-1}(B_2) \\ x_2 & \text{if } x = x_1. \end{cases}$$

(2.1.a) $\tilde{f}$ is injective:

Since f^* is injective and $x_2 \notin f^*(B_1)$, then $\tilde{f}$ is injective.

(2.1.b) $\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2))$ is linearly independent:

Since $x_1 \in B_1 \setminus f^{*-1}(B_2)$,
$x_2 \in B_2 \setminus f^*(B_1)$,
$\tilde{f}(x_1) = x_2$ and,
for all $x \in f^{*-1}(B_2)$, $\tilde{f}(x) = f^*(x)$, then

$$\tilde{f}(B_1) = f^*(B_1) \cup \{x_2\}$$
$$\tilde{f}^{-1}(B_2) = f^{*-1}(B_2) \cup \{x_1\}.$$

Since $\tilde{f}^{-1}(B_2) = f^{*-1}(B_2) \cup \{x_1\}$, then

$$B_1 \setminus \tilde{f}^{-1}(B_2) =$$
$$B_1 \setminus (f^{*-1}(B_2) \cup \{x_1\}) \subset B_1 \setminus f^{*-1}(B_2).$$

Since $\tilde{f}(B_1) = f^*(B_1) \cup \{x_2\}$ and $B_1 \setminus \tilde{f}^{-1}(B_2) \subset B_1 \setminus f^{*-1}(B_2)$, then

$$\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2)) \subset f^*(B_1) \cup \{x_2\} \cup (B_1 \setminus f^{*-1}(B_2)).$$

Since $\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2)) \subset f^*(B_1) \cup \{x_2\} \cup (B_1 \setminus f^{*-1}(B_2))$ and
$f^*(B_1) \cup \{x_2\} \cup (B_1 \setminus f^{*-1}(B_2))$ is linearly independent, then
$\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2))$ is linearly independent.

Since $\tilde{f}$ is injective and
$\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2))$ is linearly independent, then
$\tilde{f} \in \mathcal{F}$.

Since $\tilde{f} \in \mathcal{F}$, $f^* \subset \tilde{f}$, and $f^* \neq \tilde{f}$, then
f^* is not a maximal of $\mathcal{F}$ with respect to $\subset$!!

(2.2) Assume that $f^*(B_1) \cup \{x_2\} \cup (B_1 \setminus f^{*-1}(B_2))$ is not linearly independent.

Since $f^*(B_1) \cup \{x_2\} \cup (B_1 \setminus f^{*-1}(B_2))$ is not linearly independent, then there exist $A \subset f^*(B_1) \cup \{x_2\} \cup (B_1 \setminus f^{*-1}(B_2))$ finite and $\alpha \in K^A$ such that $0 = \sum_{x' \in A} \alpha_{x'} x'$ and $\alpha \neq 0$.

Since B_2 is linearly independent, then $A \cap (f^*(B_1) \cup \{x_2\})^C \neq \phi$.[22]

Let $x_2' \in A \cap (f^*(B_1) \cup \{x_2\})^C$.

[22] Otherwise $A \subset f^*(B_1) \cup \{x_2\} \subset B_2$ is finite, $\alpha \in K^A$, $0 = \sum_{x' \in A} \alpha_{x'} x'$, and $\alpha \neq 0$, and hence B_2 would not be linearly dependent!!

Let $\tilde{f} \in B_2^{B_1}$ be such that $\tilde{f}^{-1}(B_2) = {f^*}^{-1}(B_2) \cup \{x_1\}$ and

$$\tilde{f}(x) = \begin{cases} f^*(x) & \forall x \in {f^*}^{-1}(B_2) \\ x_2' & \text{if } x = x_1. \end{cases}$$

(2.2.a) $\tilde{f}$ is injective:

Since f^* is injective and $x_2' \notin f^*(B_1)$, then $\tilde{f}$ is injective.

(2.2.b) $\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2))$ is linearly independent:

Since $x_1 \in B_1 \setminus {f^*}^{-1}(B_2)$,
$x_2' \in B_2 \setminus f^*(B_1)$,
$\tilde{f}(x_1) = x_2'$ and,
for all $x \in {f^*}^{-1}(B_2)$, $\tilde{f}(x) = f^*(x)$, then

$$\tilde{f}(B_1) = f^*(B_1) \cup \{x_2'\}$$
$$\tilde{f}^{-1}(B_2) = {f^*}^{-1}(B_2) \cup \{x_1\}.$$

Since $\tilde{f}^{-1}(B_2) = {f^*}^{-1}(B_2) \cup \{x_1\}$, then

$$B_1 \setminus \tilde{f}^{-1}(B_2) =$$
$$B_1 \setminus ({f^*}^{-1}(B_2) \cup \{x_1\}) \subset B_1 \setminus {f^*}^{-1}(B_2).$$

Since $\tilde{f}(B_1) = f^*(B_1) \cup \{x_2'\}$ and
$B_1 \setminus \tilde{f}^{-1}(B_2) \subset B_1 \setminus {f^*}^{-1}(B_2)$, then

$$\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2)) \subset f^*(B_1) \cup \{x_2'\} \cup (B_1 \setminus {f^*}^{-1}(B_2)).$$

Since $x_2' \in A \cap B_1 \setminus {f^*}^{-1}(B_2)$, then $x_2' \in B_1 \setminus {f^*}^{-1}(B_2)$.
Since $\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2)) \subset f^*(B_1) \cup \{x_2'\} \cup (B_1 \setminus {f^*}^{-1}(B_2))$ and
$x_2' \in B_1 \setminus {f^*}^{-1}(B_2)$, then

$$\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2)) \subset f^*(B_1) \cup (B_1 \setminus {f^*}^{-1}(B_2)).$$

Since $\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2)) \subset f^*(B_1) \cup (B_1 \setminus {f^*}^{-1}(B_2))$ and
$f^*(B_1) \cup (B_1 \setminus {f^*}^{-1}(B_2))$ is linearly independent, then
$\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2))$ is linearly independent.

Since $\tilde{f}$ is injective and
$\tilde{f}(B_1) \cup (B_1 \setminus \tilde{f}^{-1}(B_2))$ is linearly independent, then
$\tilde{f} \in \mathcal{F}$.
Since $\tilde{f} \in \mathcal{F}$, $f^* \subset \tilde{f}$, and $f^* \neq \tilde{f}$, then
f^* is not a maximal of $\mathcal{F}$ with respect to $\subset$!!

Therefore ${f^*}^{-1}(B_2) = B_1$.
Q.E.D.