

## THE STANDARD DYNAMIC PROGRAMMING PROBLEM

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The standard dynamic programming problem is that of finding a sequence  $\{x_n\}_{n \in \mathbb{N}}$  in a metric space  $X$  such that solves

$$\begin{aligned} \max_{x \in X^{\mathbb{N}}} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) \\ \forall n \in \mathbb{N}, x_n \in \Gamma(x_{n-1}) \\ x_0 \text{ given,} \end{aligned}$$

where  $\Gamma$  is a correspondence from  $X$  to itself,  $h$  is a real-valued function defined on the graph of  $\Gamma$ , and  $\delta$  is a positive discount factor smaller than 1. Bellman's optimality principle makes reference to a result establishing that the solution  $V$  to the so-called Bellman equation

$$V(x_0) = \max_{x \in \Gamma(x_0)} \{h(x_0, x) + \delta V(x)\}$$

associated to the standard dynamic programming problem is actually the value function of the problem, i.e. the function that gives the maximum attained as a function of the given parameter  $x_0$ . Usually we are more interested in the sequence that solves the dynamic programming problem than in the maximum value that is attained itself. Nevertheless, Bellman's equation enhances the interest in the value function for the resolution of this problem. In effect, the knowledge of the value function, along with Bellman's equation, provides an iterative method to compute the sequence that solves the standard dynamic programming problem: once  $V$  is known it suffices to solve the optimization problem at the right-hand side of Bellman's equation to get the next term  $x_n$  as a function  $\phi(x_{n-1})$  (the so-called *policy function*) of the previous term  $x_{n-1}$ , starting at the given initial condition  $x_0$ .

The resolution of the problem is in this way postponed until one has found the value function, i.e. solved Bellman's equation. In this equation the unknown is a function, and neither the way of finding its solution nor even the existence of a solution is straightforward. The existence of a solution can be guaranteed imposing conditions on the elements intervening in the standard dynamic programming problem, namely the function  $h$ , the discounting parameter  $\delta$  and the correspondence  $\Gamma$

that determines what are the feasible sequences  $\{x_n\}_{n \in \mathbb{N}}$ . More specifically, if  $h$  is continuous and bounded,<sup>1</sup>  $\delta$  is positive and smaller than one, and  $\Gamma$  is continuous and compact-valued,<sup>2</sup> then the right-hand side of Bellman's equation is, according to Blackwell's condition, a function of  $V$  which happens to be a contraction of the space of bounded, continuous, real-valued functions defined on the set where  $\Gamma$  is non-empty-valued. This set of functions is complete when endowed with the sup metric, and in this metric space Banach's Fixed Point Theorem guarantees the existence of a fixed point of that function, which is what  $V$  precisely is in that equation.

Moreover, since the value function  $V$  is the fixed point of a contraction it would suffice to iterate the right-hand side of Bellman's equation starting from an arbitrary continuous, bounded, real-valued function  $V_0$ , defined on the domain of  $\Gamma$ , to be sure to converge to the solution  $V$ . Nonetheless, analytically this procedure is most of the times unfeasible and one has to resort to substitute numerical approximations to the true functions. In this way one can obtain numerical approximations to both the value function  $V$  and the policy function  $\phi$ . For practical purposes this may well be enough.

Such procedure can clearly be computationally highly consuming, and therefore it is useful to have less costly methods to obtain some qualitative knowledge about the long run behavior of the solution  $\{x_n\}_{n \in \mathbb{N}}$  as a function of the starting point  $x_0$ . That is what Euler's equation

$$D_2h(x_{n-1}, x_n) + \delta D_1h(x_n, h_{n+1}) = 0$$

can provide, whenever the conditions for making sense of it hold (like differentiability of  $h$  and conditions guaranteeing that, for the solution  $\{x_n\}_{n \in \mathbb{N}}$ ,  $x_n$  is always an interior point of  $\Gamma(x_{n-1})$ ). The rationale behind Euler's equation is that for  $\{x_n\}_{n \in \mathbb{N}}$  to solve the standard dynamic programming problem, no gain can be obtained by departing from it even in a single term. Hence that, every other term given, the derivative with respect to any given term  $x_n$  must be null at the solution. Euler's equation is for the dynamic programming problems the analogous of Fermat's rule for static programming problems, i.e. that every maximizer (or minimizer) must necessarily be a critical point (i.e. a point where all the derivatives are zero) of the function being maximized if this function is differentiable and the maximizer (or minimizer) is interior to the (maybe constrained) domain of the function.

In the same way in which adding more conditions (namely, concavity) to the function being maximized in a static programming problem guarantees that a critical point is a maximizer indeed, there are additional conditions that can be imposed on the standard dynamic programming problem that also guarantee that a solution to Euler's equation is an optimal path indeed. Quite naturally the concavity of  $h$  is going to be one of this conditions, but in the dynamic programming problem an additional condition is required, namely that the solution to Euler's equation  $\{x_n\}_{n \in \mathbb{N}}$  satisfies the so-called *transversality condition*

$$\lim_{n \rightarrow \infty} \delta^{n-1} D_1h(x_{n-1}, x_n)x_{n+1} = 0.$$

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<sup>1</sup>Or, more generally, satisfies some other condition that prevents it from "increasing too fast".

<sup>2</sup>Besides the trivial requirement for its values to be contained in its domain, for the problem to make sense, which is trivially guaranteed to hold if the correspondence is non-empty-valued everywhere.

## THEOREMS

### The Standard Dynamic Programming Problem.

Given

- (1)  $(X, d)$  a metric space,
- (2)  $\Gamma \in \mathcal{P}(X)^X$  non-empty-valued,<sup>3</sup>
- (3)  $h \in \mathbb{R}^{X \times X}$  such that  $h^{-1}(\mathbb{R}) \supset \text{Gr}\Gamma$ ,
- (4)  $\delta \in (0, 1)$ ,
- (5)  $x_0 \in X$ ,<sup>4</sup>

solve

$$\begin{aligned} & \max_{x \in X^{\mathbb{N}}} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) \\ & \forall n \in \mathbb{N}, x_n \in \Gamma(x_{n-1}) \\ & x_0 \text{ given.} \end{aligned}$$

**S1. Lemma.** *If  $(X, d)$  is a metric space,  $h \in \mathbb{R}^{X \times X}$  is bounded, and  $\delta \in (0, 1)$ , then, for all  $x_0 \in X$  and all  $x \in X^{\mathbb{N}}$  such that, for all  $n \in \mathbb{N}$ ,  $(x_{n-1}, x_n) \in h^{-1}(\mathbb{R})$ ,  $\sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n)$  is convergent.*

*Proof.* Let  $(X, d)$  be a metric space,  $h \in \mathbb{R}^{X \times X}$  be bounded, and  $\delta \in (0, 1)$ .

Let  $x_0 \in X$  and  $x \in X^{\mathbb{N}}$  be such that, for all  $n \in \mathbb{N}$ ,  $(x_{n-1}, x_n) \in h^{-1}(\mathbb{R})$ . Since  $h$  is bounded, then there exists  $K > 0$  such that, for all  $(x, x') \in h^{-1}(\mathbb{R})$ ,  $|h(x, x')| \leq K$ . Since, for all  $(x, x') \in h^{-1}(\mathbb{R})$ ,  $|h(x, x')| \leq K$ ,  $\delta \in (0, 1)$ , and  $x \in X^{\mathbb{N}}$  is such that, for all  $n \in \mathbb{N}$ ,  $(x_{n-1}, x_n) \in h^{-1}(\mathbb{R})$ , then, for all  $n \in \mathbb{N}$ ,

$$\delta^{n-1} |h(x_{n-1}, x_n)| \leq \delta^{n-1} K.$$

Since, for all  $n \in \mathbb{N}$ ,  $\delta^{n-1} |h(x_{n-1}, x_n)| \leq \delta^{n-1} K$  and  $\delta > 0$ , then, for all  $N \in \mathbb{N}$ ,

$$\begin{aligned} & \left| \sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n) \right| \leq \\ & \sum_{n=1}^N \delta^{n-1} |h(x_{n-1}, x_n)| \leq K \sum_{n=1}^N \delta^{n-1} \\ & \leq K \sum_{n=1}^{\infty} \delta^{n-1} \\ & = \frac{K}{1-\delta}. \end{aligned}$$

Since, for all  $N \in \mathbb{N}$ ,  $\left| \sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n) \right| \leq \frac{K}{1-\delta}$ , then  $\{\sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n)\}_{N \in \mathbb{N}}$  is bounded. Since it is bounded, then it has a convergent subsequence. Since, for

<sup>3</sup>More generally, such that, for all  $x \in X$  such that  $\Gamma(x) \neq \phi$  and for all  $x' \in \Gamma(x)$ ,  $\Gamma(x') \neq \phi$ .

<sup>4</sup>More generally,  $x_0 \in X$  such that  $\Gamma(x_0) \neq \phi$ .

all  $N, N' \in \mathbb{N}$  such that  $N < N'$ ,

$$\begin{aligned}
\left| \sum_{n=1}^{N'} \delta^{n-1} h(x_{n-1}, x_n) - \sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n) \right| &= \\
\left| \sum_{n=N+1}^{N'} \delta^{n-1} h(x_{n-1}, x_n) \right| &= \\
\delta^N \left| \sum_{n=1}^{N'-N} \delta^{n-1} h(x_{N+n-1}, x_{N+n}) \right| &\leq \delta^N \sum_{n=1}^{N'-N} \delta^{n-1} |h(x_{N+n-1}, x_{N+n})| \\
&\leq \delta^N K \sum_{n=1}^{N'-N} \delta^{n-1} \\
&= \delta^N K \frac{1 - \delta^{N'-N+1}}{1 - \delta} \\
&\leq \delta^N K \frac{1}{1 - \delta}
\end{aligned}$$

and  $\lim_{N \rightarrow \infty} \delta^N K \frac{1}{1 - \delta} = 0$ , then  $\{\sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n)\}_{n \in \mathbb{N}}$  is Cauchy. Since  $\{\sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n)\}_{n \in \mathbb{N}}$  is Cauchy and has a convergent subsequence, then it is convergent, i.e.  $\sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n)$  is convergent. Q.E.D.

## S2. Existence of a solution to the Standard Dynamic Programming Problem (1).

If

- (1)  $(X, d)$  is a metric space,
- (2)  $\Gamma \in \mathcal{P}(X)^X$  is compact-valued and non-empty-valued,<sup>5</sup>
- (3)  $h \in \mathbb{R}^{X \times X}$  is continuous, bounded, and such that  $h^{-1}(\mathbb{R}) \supset \text{Gr}_{\Gamma}$ .
- (4)  $\delta \in (0, 1)$ , and
- (5)  $V \in \mathbb{R}^X$  is continuous, bounded, and such that, for all  $x \in X$ ,<sup>6</sup>

$$V(x) = \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x')\},$$

then, for all  $x_0 \in X$ ,<sup>7</sup>

(1)

$$\begin{aligned}
\phi \neq \arg \max_{\{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}}} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) \\
\forall n \in \mathbb{N}, x_n \in \Gamma(x_{n-1}) \\
x_0 \text{ given.}
\end{aligned}$$

<sup>5</sup>More generally, such that, for all  $x \in X$  such that  $\Gamma(x) \neq \phi$  and for all  $x' \in \Gamma(x)$ ,  $\Gamma(x') \neq \phi$ .

<sup>6</sup>More generally, for all  $x \in X$  such that  $\Gamma(x) \neq \phi$ .

<sup>7</sup>More generally, for all  $x_0 \in X$  such that  $\Gamma(x_0) \neq \phi$ .

(2) and

$$V(x_0) = \max_{x \in X^{\mathbb{N}}} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n)$$

$$\forall n \in \mathbb{N}, x_n \in \Gamma(x_{n-1})$$

$x_0$  given.

*Proof.* Let  $(X, d)$  be a metric space,  $\Gamma \in \mathcal{P}(X)^X$  be compact-valued and non-empty-valued,  $h \in \mathbb{R}^{X \times X}$  be continuous, bounded, and such that  $h^{-1}(\mathbb{R}) \supset \text{Gr}_{\Gamma}$ ,  $\delta \in (0, 1)$ ,  $V \in \mathbb{R}^X$  be continuous, bounded and such that, for all  $x \in X$ ,  $V(x) = \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x')\}$ .

Let  $x_0 \in X$ .

(1) There exists  $x^* \in X^{\mathbb{N}}$  such that, for all  $n \in \mathbb{N}$ ,  $x_n^* \in \Gamma(x_{n-1}^*)$  and

$$V(x_0) = \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}^*, x_n^*),$$

where  $x_0^* = x_0$ .

In effect,

i) since  $h$  and  $V$  are continuous and  $\Gamma(x_0)$  is compact, then there exists  $x_1^* \in \Gamma(x_0)$  such that  $V(x_0) = h(x_0, x_1^*) + \delta V(x_1^*)$ ,

ii) since  $h$  and  $V$  are continuous and  $\Gamma(x_1^*)$  is compact, then there exists  $x_2^* \in \Gamma(x_1^*)$  such that  $V(x_1^*) = h(x_1^*, x_2^*) + \delta V(x_2^*)$ ,

iii) since  $h$  and  $V$  are continuous and  $\Gamma(x_2^*)$  is compact, then there exists  $x_3^* \in \Gamma(x_2^*)$  such that  $V(x_2^*) = h(x_2^*, x_3^*) + \delta V(x_3^*)$ , and so on.

Therefore, there exists  $x^* \in X^{\mathbb{N}}$  such that, for all  $n \in \mathbb{N}$ ,  $x_n^* \in \Gamma(x_{n-1}^*)$ , and

$$V(x_{n-1}^*) = h(x_{n-1}^*, x_n^*) + \delta V(x_n^*),$$

where  $x_0^* = x_0$ . Since  $h$  is bounded and  $\delta \in (0, 1)$ , then  $\sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}^*, x_n^*)$  is convergent. Since  $V$  is bounded, then  $\lim_{n \rightarrow \infty} \delta^n V(x_n^*) = 0$ . Since, for all  $N \in \mathbb{N}$ ,

$$V(x_0) = \sum_{n=1}^N \delta^{n-1} h(x_{n-1}^*, x_n^*) + \delta^N V(x_N^*),$$

$\sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}^*, x_n^*)$  is convergent, and  $\lim_{n \rightarrow \infty} \delta^n V(x_n^*) = 0$ , then

$$\begin{aligned} V(x_0) &= \\ \lim_{N \rightarrow \infty} V(x_0) &= \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N \delta^{n-1} h(x_{n-1}^*, x_n^*) + \delta^N V(x_N^*) \right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \delta^{n-1} h(x_{n-1}^*, x_n^*) + \lim_{N \rightarrow \infty} \delta^N V(x_N^*) \\ &= \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}^*, x_n^*). \end{aligned}$$

(2) For all  $x \in X^{\mathbb{N}}$  such that, for all  $n \in \mathbb{N}$ ,  $x_n \in \Gamma(x_{n-1})$ ,

$$\sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) \leq \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}^*, x_n^*)$$

where  $x_0^* = x_0$ .

In effect, since  $x_0 \in X$  and

*i*) for all  $x_1 \in \Gamma(x_0)$ ,  $h(x_0, x_1) + \delta V(x_1) \leq V(x_0)$ ,

*ii*) for all  $x_1 \in \Gamma(x_0)$ , and all  $x_2 \in \Gamma(x_1)$ ,  $h(x_1, x_2) + \delta V(x_2) \leq V(x_1)$ ,

*iii*) for all  $x_1 \in \Gamma(x_0)$ , all  $x_2 \in \Gamma(x_1)$ , and all  $x_3 \in \Gamma(x_2)$ ,  $h(x_2, x_3) + \delta V(x_3) \leq V(x_2)$ , and so on,

then, for all  $x \in X^{\mathbb{N}}$  such that, for all  $n \in \mathbb{N}$ ,  $x_n \in \Gamma(x_{n-1})$ , and all  $N \in \mathbb{N}$ ,

$$\begin{aligned} V(x_0) &\geq h(x_0, x_1) + \delta V(x_1) \\ &\geq h(x_0, x_1) + \delta(h(x_1, x_2) + \delta V(x_2)) \\ &\geq h(x_0, x_1) + \delta(h(x_1, x_2) + \delta(h(x_2, x_3) + \delta V(x_3))) \\ &\vdots \\ &\geq \sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n) + \delta^N V(x_N). \end{aligned}$$

Since, for all  $x \in X^{\mathbb{N}}$  such that, for all  $n \in \mathbb{N}$ ,  $x_n \in \Gamma(x_{n-1})$ , and all  $N \in \mathbb{N}$ ,

$$V(x_0) \geq \sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n) + \delta^N V(x_N),$$

and  $V(x_0) = \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}^*, x_n^*)$ , where  $x_0^* = x_0$ , then, for all  $x \in X^{\mathbb{N}}$  such that, for all  $n \in \mathbb{N}$ ,  $x_n \in \Gamma(x_{n-1})$ ,

$$\begin{aligned} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}^*, x_n^*) &= \\ V(x_0) &\geq \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n) + \delta^N V(x_N) \right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \delta^{n-1} h(x_{n-1}, x_n) + \lim_{N \rightarrow \infty} \delta^N V(x_N) \\ &= \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n), \end{aligned}$$

where  $x_0^* = x_0$ .

Q.E.D.

**S3. Existence of a solution to the standard dynamic programming problem (2).** *If*

- (1)  $(X, d)$  is a metric space,
- (2)  $\Gamma \in \mathcal{P}(X)^X$  is continuous, compact-valued, and non-empty-valued,<sup>8</sup>
- (3)  $h \in \mathbb{R}^{X \times X}$  is continuous, bounded, and such that  $h^{-1}(\mathbb{R}) \supset \text{Gr}_\Gamma$ , and
- (4)  $\delta \in (0, 1)$

then, for all  $x_0 \in X$ ,<sup>9</sup>

$$\begin{aligned} \phi \neq \arg \max_{x \in X^{\mathbb{N}}} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) \\ \forall n \in \mathbb{N}, x_n \in \Gamma(x_{n-1}) \\ x_0 \text{ given.} \end{aligned}$$

*Proof.* Let  $(X, d)$  be a metric space,  $\Gamma \in \mathcal{P}(X)^X$  be continuous, compact-valued, and non-empty-valued,  $h \in \mathbb{R}^{X \times X}$  be continuous, bounded, and such that  $h^{-1}(\mathbb{R}) \supset \text{Gr}_\Gamma$ , and  $\delta \in (0, 1)$ .

Let  $T: \mathbb{R}^X \rightarrow \mathbb{R}^X$  be such that, for all continuous, bounded  $V \in \mathbb{R}^X$ ,  $T(V)$  is such that, for all  $x \in X$ ,

$$T(V)(x) = \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x')\}.$$

Since  $\Gamma$  is continuous and compact-valued,  $h$  continuous and defined in  $\text{Gr}_\Gamma$ , and  $V$  is continuous, then<sup>10</sup>  $T(V)$  is continuous.

Since  $h$  is bounded, then there exists  $K' \geq 0$  such that, for all  $x \in X$  and all  $x' \in \Gamma(x)$ ,  $|h(x, x')| \leq K'$ . Since  $V$  is bounded, then there exists  $K'' \geq 0$  such that, for all  $x \in X$  and all  $x' \in \Gamma(x)$ ,  $|V(x')| \leq K''$ , i.e.  $|\delta V(x')| \leq \delta K''$ . Since, for all  $x \in X$  and all  $x' \in \Gamma(x)$ ,  $|h(x, x')| \leq K'$  and  $|\delta V(x')| \leq \delta K''$ , then there exists  $K \geq K'' + \delta K'$  such that, for all  $x \in X$ ,

$$\begin{aligned} |h(x, x') + \delta V(x')| &\leq |h(x, x')| + |\delta V(x')| \\ &\leq K'' + \delta K' \\ &\leq K. \end{aligned}$$

Since, for all  $x \in X$  and all  $x' \in \Gamma(x)$ ,

$$-K \leq h(x, x') + \delta V(x') \leq K$$

then, for all  $x \in X$  and all  $x' \in \Gamma(x)$ ,

$$-K \leq \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x')\} \leq K$$

i.e.

$$\begin{aligned} |T(V)(x)| &= \\ \left| \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x')\} \right| &\leq K. \end{aligned}$$

<sup>8</sup>More generally, such that, for all  $x \in X$  such that  $\Gamma(x) \neq \phi$  and for all  $x' \in \Gamma(x)$ ,  $\Gamma(x') \neq \phi$ .

<sup>9</sup>More generally, for all  $x_0 \in X$  such that  $\Gamma(x_0) \neq \phi$ .

<sup>10</sup>By the theorem of the maximum.

Since, for all  $x \in X$ ,  $|T(V)(x)| \leq K$ , then  $T(V)$  is bounded.

Since, for all  $V \in \mathbb{R}^X$  continuous and bounded,  $T(V) \in \mathbb{R}^X$  is continuous and bounded, then  $T$  transforms any continuous, bounded, real-valued function defined in  $X$  into a continuous, bounded, real-valued function defined in  $X$ .

Since, for all  $V, V' \in \mathbb{R}^X$  such that, for all  $x' \in X$ ,  $V(x') \leq V'(x')$ , then, for all  $x, x' \in X$ ,

$$h(x, x') + \delta V(x') \leq h(x, x') + \delta V'(x').$$

Since for all  $x, x' \in X$ ,  $h(x, x') + \delta V(x') \leq h(x, x') + \delta V'(x')$ , then, for all  $x \in X$ ,

$$\begin{aligned} T(V)(x) &= \\ \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x')\} &\leq \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V'(x')\} \\ &= T(V')(x). \end{aligned}$$

Since, for all  $V, c \in \mathbb{R}^X$  such that  $c$  is constant, and all  $x \in X$ ,

$$\begin{aligned} T(V + c)(x) &= \\ \max_{x' \in \Gamma(x)} \{h(x, x') + \delta(V + c)(x')\} &= \\ \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x') + \delta c\} &= \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x')\} + \delta c \\ &= T(V)(x) + \delta c, \end{aligned}$$

then, there exists  $\delta \in [0, 1)$  such that, for all  $x \in X$ ,

$$T(V + c)(x) \leq T(V)(x) + \delta c.$$

Therefore,  $T$  is a contraction.

Since the set of continuous, bounded, real-valued functions defined in  $X$  with the supremum metric is complete, and  $T$  is a contraction in this space, then there exists a unique  $V \in \mathbb{R}^X$  continuous and bounded, such that  $T(V) = V$ , i.e. such that, for all  $x \in X$ ,

$$V(x) = \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x')\}.$$

Since  $V \in \mathbb{R}^X$  is continuous, bounded, and such that, for all  $x \in X$ ,

$$V(x) = \max_{x' \in \Gamma(x)} \{h(x, x') + \delta V(x')\},$$

then, for all  $x_0 \in X$ ,

$$\begin{aligned} V(x_0) &= \max_{\{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}}} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) \\ &\quad \forall n \in \mathbb{N}, x_n \in \Gamma(x_{n-1}) \\ &\quad x_0 \text{ given.} \end{aligned}$$

Q.E.D.

#### S4. Necessity of Euler's equation.

If

- (1)  $(X, d)$  is  $(\mathbb{R}^n, d_2)$ ,
- (2)  $\Gamma \in \mathcal{P}(X)^X$  is non-empty-valued,<sup>11</sup>
- (3)  $\delta \in (0, 1)$ ,
- (4)  $h \in \mathbb{R}^{\text{Gr}\Gamma}$  is concave and differentiable,
- (5)  $x_0 \in X$ , and
- (6)  $x^* \in X^{\mathbb{N}}$  is such that, for all  $n \in \mathbb{N}$ ,  $x_n^* \in \text{Int}\Gamma(x_{n-1}^*)$ , and solves

$$\begin{aligned} \max_{x \in X^{\mathbb{N}}} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) \\ \forall n \in \mathbb{N}, x_n \in \Gamma(x_{n-1}) \\ x_0 \text{ given,} \end{aligned} \tag{1}$$

then, for all  $n \in \mathbb{N}$ ,

$$D_2 h(x_{n-1}^*, x_n^*) + \delta D_1 h(x_n^*, x_{n+1}^*) = 0.$$

*Proof.* Let  $(X, d)$  be  $(\mathbb{R}^n, d_2)$ ,  $\Gamma \in \mathcal{P}(X)^X$  be non-empty-valued,  $\delta \in (0, 1)$ ,  $h \in \mathbb{R}^{\text{Gr}\Gamma}$  be concave and differentiable,  $x_0 \in X$ , and  $x^* \in X^{\mathbb{N}}$  is such that, for all  $n \in \mathbb{N}$ ,  $x_n^* \in \text{Int}\Gamma(x_{n-1}^*)$ , and solves (1).

Let  $n \in \mathbb{N}$ ,  $x_0^* = x_0$ , and  $x \in X^{\mathbb{N}}$  be such that, for all  $n' \in \mathbb{N} \setminus \{n\}$ ,  $x_{n'} = x_{n'}^*$ ,  $x_n \in \Gamma(x_{n-1})$ , and  $x_{n+1} \in \Gamma(x_n)$ . Since  $x^*$  solves (1), then

$$\sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) \leq \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}^*, x_n^*),$$

that is to say,

$$\delta^{n-1} h(x_{n-1}, x_n) + \delta^n h(x_n, x_{n+1}) \leq \delta^{n-1} h(x_{n-1}^*, x_n^*) + \delta^n h(x_n^*, x_{n+1}^*),$$

or equivalently

$$\begin{aligned} h(x_{n-1}^*, x_n) + \delta h(x_n, x_{n+1}^*) = \\ h(x_{n-1}, x_n) + \delta h(x_n, x_{n+1}) \leq h(x_{n-1}^*, x_n^*) + \delta h(x_n^*, x_{n+1}^*). \end{aligned}$$

Therefore  $x_n^*$  solves

$$\max_{x_n \in \Gamma(x_{n-1}^*)} h(x_{n-1}^*, x_n) + \delta h(x_n, x_{n+1}^*).$$

Since moreover  $x_n^* \in \text{Int}\Gamma(x_{n-1}^*)$ , then

$$D_2 h(x_{n-1}^*, x_n^*) + \delta D_1 h(x_n^*, x_{n+1}^*) = 0.$$

Q.E.D.

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<sup>11</sup>More generally, such that, for all  $x \in X$  such that  $\Gamma(x) \neq \emptyset$  and for all  $x' \in \Gamma(x)$ ,  $\Gamma(x') \neq \emptyset$ .

**S5. Sufficiency of Euler's Equation, concavity, and the transversality condition.**

If

- (1)  $(X, d)$  is  $(\mathbb{R}_+^n, d_2)$ ,
- (2)  $\Gamma \in \mathcal{P}(X)^X$  is non-empty-valued,<sup>12</sup>
- (3)  $\delta \in (0, 1)$ ,
- (4)  $h \in \mathbb{R}^{\text{Gr}\Gamma}$  is differentiable, concave, and such that, for all  $(x, x') \in \text{Gr}\Gamma$ ,  $D_1h(x, x') \in \mathbb{R}_+^n$ ,
- (5)  $x_0 \in X$ , and
- (6)  $x^* \in X^{\mathbb{N}}$  is such that, for all  $n \in \mathbb{N}$ ,

$$D_2h(x_{n-1}^*, x_n^*) + \delta D_1h(x_n^*, x_{n+1}^*) = 0$$

with  $x_0^* = x_0$ , and

$$\lim_{n \rightarrow \infty} \delta^n D_1h(x_n^*, x_{n+1}^*)x_n^* = 0,$$

then  $x^*$  solves

$$\begin{aligned} \max_{x \in X^{\mathbb{N}}} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) \\ \forall n \in \mathbb{N}, x_n \in \Gamma(x_{n-1}) \\ x_0 \text{ given.} \end{aligned} \tag{1}$$

*Proof.* Let  $(X, d)$  be  $(\mathbb{R}^n, d_2)$ ,  $\Gamma \in \mathcal{P}(X)^X$  be non-empty-valued,  $\delta \in (0, 1)$ ,  $h \in \mathbb{R}^{\text{Gr}\Gamma}$  be differentiable, concave, and such that, for all  $(x, x') \in \text{Gr}\Gamma$ ,  $D_1h(x, x') \in \mathbb{R}_+^n$ ,  $x_0 \in \Gamma^{-1}(\mathcal{P}(X) \setminus \{\phi\})$ , and  $x^* \in X^{\mathbb{N}}$  be such that, for all  $n \in \mathbb{N}$ ,  $D_2h(x_{n-1}^*, x_n^*) + \delta D_1h(x_n^*, x_{n+1}^*) = 0$ , with  $x_0^* = x_0$ , and

$$\lim_{n \rightarrow \infty} \delta^{n-1} D_1h(x_{n-1}^*, x_n^*)x_{n+1}^* = 0.$$

Let  $x_0^* = x_0$  and  $x \in (\mathbb{R}_+^n)^{\mathbb{N}}$  be such that, for all  $n \in \mathbb{N}$ ,  $x_n \in \Gamma(x_{n-1})$ . Since  $h$  is differentiable and concave, for all  $n \in \mathbb{N}$ ,  $D_2h(x_{n-1}^*, x_n^*) + \delta D_1h(x_n^*, x_{n+1}^*) = 0$ , and  $\lim_{n \rightarrow \infty} \delta^{n-1} D_1h(x_{n-1}^*, x_n^*)x_{n+1}^* = 0$ , then

$$\begin{aligned} \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}^*, x_n^*) - \sum_{n=1}^{\infty} \delta^{n-1} h(x_{n-1}, x_n) = \\ \lim_{N \rightarrow \infty} \sum_{n=1}^N \delta^{n-1} \left( h(x_{n-1}^*, x_n^*) - h(x_{n-1}, x_n) \right) \geq \\ \lim_{N \rightarrow \infty} \sum_{n=1}^N \delta^{n-1} \left( D_1h(x_{n-1}^*, x_n^*)(x_{n-1}^* - x_{n-1}) + D_2h(x_{n-1}^*, x_n^*)(x_n^* - x_n) \right) = \end{aligned}$$

<sup>12</sup>More generally, such that, for all  $x \in X$  such that  $\Gamma(x) \neq \phi$  and for all  $x' \in \Gamma(x)$ ,  $\Gamma(x') \neq \phi$ .

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \left( D_1 h(x_0^*, x_1^*)(x_0^* - x_0) \right. \\
& \quad + \sum_{n=1}^N \delta^{n-1} (D_2 h(x_{n-1}^*, x_n^*) + \delta D_1 h(x_n^*, x_{n+1}^*))(x_n^* - x_n) \\
& \quad \left. + \delta^{N-1} D_2 h(x_{N-1}^*, x_N^*)(x_N^* - x_N) \right) \\
&= \lim_{N \rightarrow \infty} \delta^{N-1} D_2 h(x_{N-1}^*, x_N^*)(x_N^* - x_N) \\
&= \lim_{N \rightarrow \infty} \delta^N D_1 h(x_N^*, x_{N+1}^*)(x_N - x_N^*) \\
&= \lim_{N \rightarrow \infty} \delta^N D_1 h(x_N^*, x_{N+1}^*)x_N - \lim_{N \rightarrow \infty} \delta^N D_1 h(x_N^*, x_{N+1}^*)x_N^* \\
&= \lim_{N \rightarrow \infty} \delta^N D_1 h(x_N^*, x_{N+1}^*)x_N \\
&\geq 0.
\end{aligned}$$

Q.E.D.