Lecture Notes for Econ 898

THE THEOREM OF THE MAXIMUM

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The theorem of the maximum establishes some properties about how the maximizers and the maximum of a real function on a constrained domain behave as some parameters of the problem affecting both the function and its constrained domain vary. Intuitively, this theorem asserts that, as long as the maximand and its domain do not respond too much to changes in the parameters of the problem, then both the maximum and the set of maximizers do not respond too much to those changes either.

More precisely, the theorem of the maximum states that the continuity of a maximand f with respect to both some parameters x and its variables y, and the continuity and compact-valuedness of a correspondence Γ assigning a constrained domain $\Gamma(x)$ to each value for the parameters x, are enough to guarantee both the upper hemicontinuity and compact-valuedness of the maximizers $y^*(x)$ and the continuity of the maximum $f^*(x)$, with respect to the parameters x.

The upper hemicontinuity and compact-valuedness of the correspondence of maximizers y^* is inherited from those of Γ once one realizes that y^* is trivially $y^* \cap \Gamma$ and has a closed graph. In effect, y^* has a closed graph since any convergent sequence (x_n, y_n) in the graph of y^* has its limit (x, y) in the graph of y^* as well. Otherwise, there would be some other y' in $\Gamma(x)$ making f bigger than y. In that case, the lower hemicontinuity of Γ would guarantee the existence of some other sequence (x_n, y'_n) within its graph converging to (x, y'). As a consequence, since f attains a higher value at (x, y') than at (x, y), somewhere along the way the sequence (x_n, y'_n) should make f bigger that what (x_n, y_n) does, contradicting the fact that y_n maximizes f for the parameters x_n.

The continuity of the maximum value $f^*(x)$ with respect to x follows from the fact that the maxima $f^*(x_n)$ corresponding to any sequence x_n converging to any given x, converge themselves to the maximum value $f^*(x)$. In effect, $f^*(x_n)$ converges because its $\limsup$ and $\liminf$ exist and coincide with $f^*(x)$. In effect, $\limsup f^*(x_n)$ is the supremum of the cluster points of $\{f^*(x_n)\}_{n \in \mathbb{N}}$, which exists because there is at least one such cluster point, namely $f^*(x)$ itself, and the set of cluster points is bounded above by $f^*(x)$, which guarantees $\limsup f^*(x_n) = f^*(x)$ (and similarly for the $\liminf$).

More precisely, the set of cluster points of $f^*(x_n)$ is not empty because any sequence of maximizers y_n in each $y^*(x_n)$ must have, given the upper hemicontinuity and compact-valuedness of the correspondence of maximizers y^*, a subsequence $y_{h(n)}$ converging to some maximizer y of f in $y^*(x)$. The continuity of f implies then that the maxima $f^*(x_{h(n)})$ attained at those $y_{h(n)}$ must converge to the maximum $f^*(x)$ attained at y, making of $f^*(x)$ a cluster point of $f^*(x_n)$.

Moreover, no cluster point of $f^*(x_n)$ is bigger than $f^*(x)$. Otherwise, infinitely many terms of $f^*(x_n)$ would then cluster around some point above $f^*(x)$. These infinitely many terms would constitute a subsequence $f^*(x_{h(n)})$ to which there would be associated infinitely many maximizers y_n. Then the upper hemicontinuity and compact-valuedness of Γ, within whose graph the terms of the sequence $(x_{h(n)}, y_n)$ lie, would imply the existence of still another subsequence $(x_{h'(h(n))}, y_{h'(n)})$ converging to some (x, y) within the graph of Γ. Again, the continuity of f would make the values of f at $(x_{h'(h(n))}, y_{h'(n)})$ converge to its value at (x, y), but since all those values are above and bounded away from $f^*(x)$, then the value of f at (x, y) should also be above $f^*(x)$, contradicting the fact that this is the highest value f can take in $\Gamma(x)$. As a consequence, $\limsup f^*(x_n) = f^*(x)$ and similarly $\liminf f^*(x_n) = f^*(x)$, establishing the continuity of f^*.

A detailed statement and proof of these results follows.

S1. The theorem of the maximum.

If

1. (X, d_X) *and* (Y, d_Y) *are metric spaces,*
2. $\Gamma \in \mathcal{P}(Y)^X$ *is compact-valued and continuous,*
3. $f \in \mathbb{R}^{X \times Y}$ *is continuous and such that* $f^{-1}(\mathbb{R}) \supset \mathrm{Gr}_\Gamma$,
4. $y^* \in \mathcal{P}(Y)^X$ *is such that, for all* $x \in X$,

$$y^*(x) = \arg\max_{y \in \Gamma(x)} f(x, y),$$

and

5. $f^* \in \mathbb{R}^X$ *is such that, for all* $x \in X$,

$$f^*(x) = \max_{y \in \Gamma(x)} f(x, y)$$

then

1. y^* *is upper hemicontinuous and compact-valued, and*
2. f^* *is continuous.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $\Gamma \in \mathcal{P}(Y)^X$ be compact-valued and continuous, $f \in \mathbb{R}^{X \times Y}$ be continuous and such that $f^{-1}(\mathbb{R}) \supset \mathrm{Gr}_\Gamma$, $y^* \in \mathcal{P}(Y)^X$ be such that, for all $x \in X$, $y^*(x) = \arg\max_{y \in \Gamma(x)} f(x, y)$, and $f^* \in \mathbb{R}^X$ be such that, for all $x \in X$, $f^*(x) = \max_{y \in \Gamma(x)} f(x, y)$.

1. y^* has a closed graph, i.e. Gr_{y^*} is a closed set of $(X \times Y, d_{X \times Y})$.

   Let $(x_0, y_0) \in \mathrm{Cl}\ \mathrm{Gr}_{y^*}$. Since $(x_0, y_0) \in \mathrm{Cl}\ \mathrm{Gr}_{y*}$, then there exists $x \in X^{\mathbb{N}}$ and $y \in Y^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $(x_n, y_n) \in \mathrm{Gr}_{y^*}$ and $(x_0, y_0) = \lim_n (x_n, y_n)$. Since, for all $n \in \mathbb{N}$, $(x_n, y_n) \in \mathrm{Gr}_{y^*}$, then, for all $n \in \mathbb{N}$, $y_n \in y^*(x_n) \subset \Gamma(x_n)$. Since Γ is continuous, then Γ is upper hemicontinuous. Since Γ is compact-valued, upper hemicontinuous, $x_0 = \lim_n x_n$, and, for all $n \in \mathbb{N}$, $y_n \in \Gamma(x_n)$, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and such that

$y \circ h$ is convergent and $\lim_n y_{h(n)} \in \Gamma(x_0)$. Since $y_0 = \lim_n y_n$ and $y \circ h$ is convergent, then $y_0 = \lim_n y_{h(n)}$. Since $y_0 = \lim_n y_{h(n)}$ and $\lim_n y_{h(n)} \in \Gamma(x_0)$, then $y_0 \in \Gamma(x_0)$.

Assume that $y_0 \notin y^*(x_0)$. Since $y_0 \notin y^*(x_0)$, then there exists $y'_0 \in \Gamma(x_0)$ such that $f(x_0, y'_0) > f(x_0, y_0)$. Since Γ is lower hemicontinuous and $y'_0 \in \Gamma(x_0)$, then there exists $y' \in Y^{\mathbb{N}}$ convergent and such that $y'_0 = \lim_n y'_n$ and, for all $n \in \mathbb{N}$, $y'_n \in \Gamma(x_n)$. Since $x_0 = \lim_n x_n$ and $y'_0 = \lim_n y'_n$, then $(x_0, y'_0) = \lim_n (x_n, y'_n)$. Since $(x_0, y'_0) = \lim_n (x_n, y'_n)$ and f is continuous, then $\lim_n f(x_n, y'_n) = f(x_0, y'_0)$. Since $(x_0, y_0) = \lim_n (x_n, y_n)$, and f is continuous, then $\lim_n f(x_n, y_n) = f(x_0, y_0)$. Since $f(x_0, y'_0) > f(x_0, y_0)$, then there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $f(x_{n'}, y'_{n'}) > f(x_{n'}, y_{n'})$. Since, for all $n' > n$, $y'_{n'} \in \Gamma(x_{n'})$ and $f(x_{n'}, y'_{n'}) > f(x_{n'}, y_{n'})$, then, for all $n' > n$, $y_{n'} \notin y^*(x_{n'})$! Therefore, $y_0 \in y^*(x_0)$, i.e. $(x_0, y_0) \in \mathrm{Gr}^*_y$. Since, for all $(x_0, y_0) \in \mathrm{Cl}\ \mathrm{Gr}_{y^*}$, $(x_0, y_0) \in \mathrm{Gr}^*_y$, then Gr_{y^*} is a closed set of $(X \times Y, d_{X \times Y})$.

(2) y^* is upper hemicontinuous and compact-valued.

Since Γ is upper hemicontinuous and compact-valued, and y^* has a closed graph, then $y^* = y^* \cap \Gamma$ is upper hemicontinuous and compact-valued.

(3) f^* is continuous.

Let $x_0 \in X$ be such that $\Gamma(x_0) \neq \phi$, $x \in X^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $\Gamma(x_n) \neq \phi$, and $x_0 = \lim_n x_n$.

Then $\limsup f^*(x_n)$ exists,[1] since

i) $f^*(x)$ is a cluster point of $\{f^*(x_n)\}_{n \in \mathbb{N}}$: in effect, let $y \in Y^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $y_n \in y^*(x_n)$. Since y^* is upper hemicontinuous and compact-valued, $x_0 = \lim_n x_n$, and, for all $n \in \mathbb{N}$, $y_n \in y^*(x_n)$, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing such that $y_{h(n)}$ is convergent and $\lim_n y_{h(n)} \in y^*(x_0)$. Since $x_0 = \lim_n x_n$, then $x_0 = \lim_n x_{h(n)}$. Since $\lim_n y_{h(n)} \in y^*(x_0)$ and $y^*(x_0) \subset \Gamma(x_0)$, then $\lim_n y_{h(n)} \in \Gamma(x_0)$. Since $\lim_n y_{h(n)} \in \Gamma(x_0)$ and $x_0 = \lim_n x_{h(n)}$, then $(\lim_n x_{h(n)}, \lim_n y_{h(n)}) \in \mathrm{Gr}_\Gamma$. Since $(\lim_n x_{h(n)}, \lim_n y_{h(n)}) \in \mathrm{Gr}_\Gamma \subset f^{-1}(\mathbb{R})$, then

$$\begin{aligned}\lim_n (x_{h(n)}, y_{h(n)}) &= (\lim_n x_{h(n)}, \lim_n y_{h(n)}) \\ &\in f^{-1}(\mathbb{R}).\end{aligned}$$

Since, for all $n \in \mathbb{N}$, $y_n \in y^*(x_n)$, f is continuous, $\lim_n (x_{h(n)}, y_{h(n)}) \in f^{-1}(\mathbb{R})$, and $\lim_n y_{h(n)} \in y^*(x_0)$, then

$$\begin{aligned}\lim_n f^*(x_{h(n)}) &= \\ \lim_n f(x_{h(n)}, y_{h(n)}) &= \\ f(\lim_n (x_{h(n)}, y_{h(n)})) &= f(\lim_n x_{h(n)}, \lim_n y_{h(n)}) \\ &= f(x_0, \lim_n y_{h(n)}) \\ &= f^*(x_0).\end{aligned}$$

Since there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and such that $f^*(x_{h(n)})$ is convergent and $f^*(x) = \lim_n f^*(x_{h(n)})$, then $f^*(x)$ is a cluster point of $f^*(x_n)$.

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[1] $\limsup f^*(x_n)$ is the supremum of the set of cluster points of $\{f^*(x_n)\}_{n \in \mathbb{N}}$.

ii) No cluster point of $\{f^*(x_n)\}_{n\in\mathbb{N}}$ is bigger than $f^*(x_0)$: in effect, assume that there exists $a \in \mathbb{R}$ such that $f^*(x_0) < a$ and a is a cluster point of $\{f^*(x_n)\}_{n\in\mathbb{N}}$. Since a is a cluster point of $\{f^*(x_n)\}_{n\in\mathbb{N}}$, then, for all $\varepsilon > 0$, $(f^* \circ x)^{-1}(B_\varepsilon(a))$ is not finite. Let $\varepsilon' \in (0, a - f^*(x_0))$, then $(f^* \circ x)^{-1}(B_{\varepsilon'}(a))$ is not finite. Since $(f^* \circ x)^{-1}(B_{\varepsilon'}(a))$ is not finite, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing such that, for all $n \in \mathbb{N}$, $h(n)$ is the n-th smallest integer of $(f^* \circ x)^{-1}(B_{\varepsilon'}(a))$. Since, for all $n \in \mathbb{N}$, $h(n) \in (f^* \circ x)^{-1}(B_\varepsilon(a))$, then, $(f^* \circ x \circ h)(\mathbb{N}) \subset B_{\varepsilon'}(a)$. Since $(f^* \circ x \circ h)(\mathbb{N}) \subset B_{\varepsilon'}(a)$ and $\varepsilon' \in (0, a - f^*(x_0))$, then, for all $n \in \mathbb{N}$, $f^*(x_0) - a < -\varepsilon' < f^*(x_{h(n)}) - a$. Since for all $n \in \mathbb{N}$, $y^*(x_{h(n)}) \neq \phi$, then, for all $n \in \mathbb{N}$, there exists $y_n \in y^*(x_{h(n)})$. Since Γ is upper hemicontinuous and compact-valued, $\{x_{h(n)}\}_{n\in\mathbb{N}}$ is convergent and such that $x_0 = \lim_n x_{h(n)}$, and, for all $n \in \mathbb{N}$, $y_n \in y^*(x_{h(n)}) \subset \Gamma(x_{h(n)})$, then there exists $h' \in \mathbb{N}^{\mathbb{N}}$ increasing and such that $\{y_{h'(n)}\}_{n\in\mathbb{N}}$ is convergent and $\lim_n y_{h'(n)} \in \Gamma(x_0)$. Since $\lim_n y_{h'(n)} \in \Gamma(x_0)$, then $(x_0, \lim_n y_{h'(n)}) \in \mathrm{Gr}_\Gamma \subset f^{-1}(\mathbb{N})$. Since for all $n \in \mathbb{N}$, $a - \varepsilon' < f^*(x_{h(n)})$, then $a - \varepsilon' \leq \lim_n f^*(x_{h(n)})$. Since $a - \varepsilon' \leq \lim_n f^*(x_{h(n)})$, then $a - \varepsilon' \leq \lim_n f^*(x_{h(h'(n))})$. Since $f^*(x_0) - a < -\varepsilon'$, for all $n \in \mathbb{N}$, $a - \varepsilon' \leq \lim_n f^*(x_{h(h'(n))})$, $y_{h'(n)} \in y^*(x_{h(h'(n))})$, $(x_0, \lim_n y_{h'(n)}) \in f^{-1}(\mathbb{N})$, and f is continuous,

$$\begin{aligned}
f(x_0, \lim_n y_{h'(n)}) &= \\
f(\lim_n x_{h(h'(n))}, \lim_n y_{h'(n)}) &= \\
\lim_n f(x_{h(h'(n))}, y_{h'(n)}) &= \\
\lim f^*(x_{h(h'(n))}) &\geq a - \varepsilon' \\
&> f^*(x_0),
\end{aligned}$$

but, since $\lim_n y_{h'(n)} \in \Gamma(x_0)$, then $f(x_0, \lim_n y_{h'(n)}) \leq f^*(x_0)$!

Since $f^*(x_0)$ is the smallest upper bound to the set of cluster points of $\{f^*(x_n)\}_{n\in\mathbb{N}}$, then

$$f^*(x_0) = \limsup f^*(x_n)$$

and similarly for $f^*(x_0) = \liminf f^*(x_n)$.

Since $\lim_n f^*(x_n) = f^*(\lim_n x_n)$, then f^* is continuous at x.

Q.E.D.