Lecture Notes for Econ 898

CORRESPONDENCES BETWEEN METRIC SPACES

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A function between metric spaces can actually be identified to a singleton-valued correspondence. From this point of view, the notion of continuity of a function can be extended to correspondences, but with a difference. The difference is that there are several ways in which this can be done.

In effect, on the one hand, the continuity of a function f at some point x, i.e. the inclusion of the image of some open neighborhood A of the point x in any open neighborhood B of its image $f(x)$, may well be restated as the inclusion of the image $f(x')$ of any point x' in some open neighborhood A of x in any open neighborhood B containing $f(x)$. Note that, expressed in this way, this requirement makes equally sense for nonsingleton-valued correspondences. On the other hand, the same condition of continuity of a function at some point can be restated as well as the nonempty intersection of the image $f(x')$ of any point x' in some open neighborhood A of x with any open B that intersects its image $f(x)$. Similarly, this requirement makes sense as well for nonsingleton-valued correspondences.

When applied to correspondences in general, the first kind of continuity is called **upper hemicontinuity**, while the second one is known as **lower hemicontinuity**. Therefore, if a correspondence is upper hemicontinuous at some point x and its image $\Gamma(x)$ is contained in some open B, so must it be for nearby points x', while if it is lower hemicontinuous at that point x and its image $\Gamma(x)$ intersects some open B, so must it be for nearby points x'. Obviously these notions of hemicontinuity coincide and are actually identical to the notion of continuity for functions when these are considered as singleton-valued correspondences. Analogously, a general correspondence is said to be continuous iff it is both upper and lower hemicontinuous.[1]

Not surprisingly, upper hemicontinuous and lower hemicontinuous correspondences from a space X to a space Y have properties that remind those of continuous functions. For instance, they are completely characterized by the way they transform open and closed sets of Y when considering their inverse images. The difference with functions one needs to be aware of is that the notion of inverse image of a set in Y by a function, when considered as a singleton-valued correspondence,

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[1] Note that this notion of continuity of set-valued functions has nothing to do with their continuity as functions, which would require to metrize the power set of the space containing the values of the correspondence, and this we do not want to do it now.

can be translated in two different ways for general correspondences. In effect, for any B in Y, its **upper inverse** $\Gamma_+^{-1}(B)$ is the set of points whose images are contained in B, while its **lower inverse** $\Gamma_-^{-1}(B)$ is the set of points whose images intersect B. These two inverses coincide for singleton-valued correspondences, but need not coincide for general correspondences.[2]

With this notions of inverse image at hand, the upper hemicontinuous correspondences happen to be the only ones whose upper inverse of every open is open, and the only ones whose lower inverse of every closed is closed. As for lower hemicontinuous correspondences, the roles of open and closed sets are exchanged (or, equivalently, those of the upper and lower inverses) and hence they are the only ones whose upper inverse of every closed is closed, and the only ones whose lower inverse of every open is open.

Also, similarly to how continuous functions transform compacts into compacts, upper hemicontinuous correspondences transform compacts into compacts *whenever compact-valued* (note that this additional requirement was trivially fulfilled by singleton valued correspondences). More precisely, whenever A is compact, the the union of all the values $\Gamma(x)$ of points x in A, say $\Gamma_*(A)$, is compact as well. Again, note that $\Gamma_*(A)$ is not strictly speaking the image of A by Γ, i.e. $\Gamma(A)$, which would rather be a subset of the power set of the space containing the values of the correspondence. In the case of singleton-valued correspondences, i.e. functions, there is an obvious identification of both concepts.

Very much like the continuity of functions, the upper and lower hemicontinuity of correspondences can be characterized by means of sequences. There is a complete characterization of the lower hemicontinuity of a correspondence that requires to be able to approach any point (x, y) in the graph of the correspondence from within it by means of a sequence (x_n, y_n), no matter how x is approached by x_n. Contrarily, the upper hemicontinuity can only be completely characterized by means of sequences whenever the correspondence is compact-valued. In effect, for a compact-valued correspondence Γ, its upper hemicontinuity is equivalent to the existence of an accumulation point in $\Gamma(x)$ of any sequence (x_n, y_n) within its graph and such that x_n converges to x. Typically, the upper hemicontinuous correspondences appearing as feasible domains in dynamic programming problems are compact-valued, hence the widespread use of this characterization almost as a definition of upper hemicontinuity.

Finally, the graph of a correspondence may be itself a closed set. A **closed graph correspondence** is equivalently said to be closed everywhere, which is clearly not the same as being closed-valued. Closed graph correspondences have the property of letting their intersections with upper hemicontinuous, compact-valued correspondences still be upper hemicontinuous and compact-valued themselves. This is a property that will deliver eventually the upper hemicontinuity and compact-valuedness of the correspondence of maximizers of a continuous function defined on the graph of a continuous, compact-valued correspondence in the Theorem of the Maximum.

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[2] Note that neither $\Gamma_+^{-1}(B)$ nor $\Gamma_-^{-1}(B)$ are related to the inverse image of B by Γ as a function, $\Gamma^{-1}(B)$, which is rather constituted by those x whose value $\Gamma(x)$ is exactly B.

DEFINITIONS

D1. *If* (X, d_X) *and* (Y, d_Y) *are metric spaces, and* $\Gamma \in \mathcal{P}(Y)^X$, *then*

(1) Γ *is* **upper hemicontinuous at** $x \in X$ *if, and only if,*

$$\forall B \subset Y \text{ open} \mid \Gamma(x) \subset B, \exists A \subset X \text{ open} \mid x \in A \text{ and } \forall x' \in A, \Gamma(x') \subset B,$$

(*Note that if* $\Gamma(x) = \phi$, *upper hemicontinuity at* x *requires* Γ *to be empty-valued in an open neighbourhood of* x *also.*)

(2) Γ *is* **lower hemicontinuous at** $x \in X$ *if, and only if,*

$$\forall B \subset Y \text{ open}, \mid \Gamma(x) \cap B \neq \phi, \exists A \subset X \text{ open} \mid x \in A \text{ and } \forall x' \in A, \Gamma(x') \cap B \neq \phi$$

(*Note that if* $\Gamma(x) = \phi$, Γ *is trivially lower hemicontinuous at* x.)

(3) Γ *is* **continuous at** $x \in X$ *if, and only if,* Γ *is both upper and lower hemicontinuous at* x.

D2. *If* (X, d_X) *and* (Y, d_Y) *are metric spaces, and* $\Gamma \in \mathcal{P}(Y)^X$, *then*

(1) Γ *is* **upper hemicontinuous** *if, and only if, for all* $x \in X$, Γ *is upper hemicontinuous at* x,
(2) Γ *is* **lower hemicontinuous** *if, and only if, for all* $x \in X$, Γ *is lower hemicontinuous at* x,
(3) Γ *is* **continuous** *if, and only if, for all* $x \in X$, Γ *is both upper and lower hemicontinuous at* x.

D3. Closed graph correspondence. *If* (X, d_X) *and* (Y, d_Y) *are metric spaces, and* $\Gamma \in \mathcal{P}(Y)^X$, *then* Γ *has a closed graph if, and only if,* Gr_Γ *is a closed set of* $(X \times Y, d_{X \times Y})$.

D4. *If* (X, d_X) *and* (Y, d_Y) *are metric spaces, and* $\Gamma \in \mathcal{P}(Y)^X$, *then*

(1) *for all* $A \subset X$, *its* **image** *by* Γ *is*

$$\Gamma_*(A) = \bigcup_{x \in A} \Gamma(x),$$

(2) *for all* $B \subset Y$, *its* **upper inverse image** *by* Γ *is*

$$\Gamma_+^{-1}(B) = \{x \in X \mid \Gamma(x) \subset B\}.$$

(3) *for all* $B \subset Y$, *its* **lower inverse image** *by* Γ *is*

$$\Gamma_-^{-1}(B) = \{x \in X \mid \Gamma(x) \cap B \neq \phi\}.$$

THEOREMS

Upper hemicontinuous correspondences.

S1. A correspondence is upper (lower) hemicontinuous iff the upper (lower) inverse of every open is open, and iff the lower (upper) inverse of every closed is closed.

If (X, d_X) *and* (Y, d_Y) *are metric spaces, and* $\Gamma \in \mathcal{P}(Y)^X$, *then*

1. Γ *is upper hemicontinuous if, and only if, for all* $B \subset Y$ *open,* $\Gamma_+^{-1}(B)$ *is open,*
2. Γ *is upper hemicontinuous if, and only if, for all* $B \subset Y$ *closed,* $\Gamma_-^{-1}(B)$ *is closed.*
3. Γ *is lower hemicontinuous if, and only if, for all* $B \subset Y$ *closed,* $\Gamma_+^{-1}(B)$ *is closed,*
4. Γ *is lower hemicontinuous if, and only if, for all* $B \subset Y$ *open,* $\Gamma_-^{-1}(B)$ *is open.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, and $\Gamma \in \mathcal{P}(Y)^X$.

1. Assume that Γ is upper hemicontinuous. Let $B \subset Y$ be open.

   If $\Gamma_+^{-1}(B) = \phi$, then $\Gamma_+^{-1}(B)$ is open.

   If $\Gamma_+^{-1}(B) \neq \phi$, then there exists $x \in \Gamma_+^{-1}(B)$. Since $x \in \Gamma_+^{-1}(B)$, then $\Gamma(x) \subset B$. Since $\Gamma(x) \subset B$, B is open, and Γ is upper hemicontinuous, then there exists $A \subset X$ open such that $x \in A$ and, for all $x' \in A$, $\Gamma(x') \subset B$. Since $x \in A$ and A is open, then there exists $\delta > 0$ such that $B_\delta(x) \subset A$. Since, for all $x' \in A$, $\Gamma(x') \subset B$, and $B_\delta(x) \subset A$, then, for all $x' \in B_\delta(x)$, $\Gamma(x') \subset B$. Since, for all $x' \in B_\delta(x)$, $\Gamma(x') \subset B$, then $B_\delta(x) \subset \Gamma_+^{-1}(B)$. Since for all $x \in \Gamma_+^{-1}(B)$, there exists $\delta > 0$ such that $B_\delta(x) \subset \Gamma_+^{-1}(B)$, then $\Gamma_+^{-1}(B)$ is open.

   Conversely, assume that Γ is such that, for all $B \subset Y$ open, $\Gamma_+^{-1}(B)$ is open. Let $x \in X$ and $B \subset Y$ be open and such that $\Gamma(x) \subset B$. Since $\Gamma(x) \subset B$, then $x \subset \Gamma_+^{-1}(B)$. Since, for all $B \subset Y$ open, $\Gamma_+^{-1}(B)$ is open, and B is open, then $\Gamma_+^{-1}(B)$ is open. Since $x \subset \Gamma_+^{-1}(B)$ and $\Gamma_+^{-1}(B)$ is open, then there exists $\delta > 0$ such that, $B_\delta(x) \subset \Gamma_+^{-1}(B)$. Since, $B_\delta(x) \subset \Gamma_+^{-1}(B)$, then, for all $x' \in B_\delta(x)$, $x' \in \Gamma_+^{-1}(B)$. Since, for all $x' \in B_\delta(x)$, $x' \in \Gamma_+^{-1}(B)$, then, for all $x' \in B_\delta(x)$, $\Gamma(x') \subset B$. Since there exists $\delta > 0$ such that, for all $x' \in B_\delta(x)$, $\Gamma(x') \subset B$, then Γ is upper hemicontinuous at x. Since for all $x \in X$, Γ is upper hemicontinuous at x, then Γ is upper hemicontinuous.
2. Exercise.
3. Exercise.
4. Exercise.

Q.E.D.

S2. Upper hemicontinuous, compact-valued correspondences transform compacts into compacts. *If* (X, d_X) *to* (Y, d_Y) *are metric spaces, and* $\Gamma \in \mathcal{P}(Y)^X$ *is upper hemicontinuous and compact-valued, then, for all* $A \subset X$ *compact,* $\Gamma_*(A)$ *is compact.*

Proof. Let (X, d_X) to (Y, d_Y) be metric spaces, and $\Gamma \in \mathcal{P}(Y)^X$ be upper hemicontinuous and compact-valued.

Let $A \subset X$ be compact, and $\mathcal{B}$ be an open cover of $\Gamma_*(A)$. Since $\mathcal{B}$ is an open cover of $\Gamma_*(A)$, then $\Gamma_*(A) \subset \cup_{B\in\mathcal{B}}B$. Since, for all $x \in A$, $\Gamma(x) \subset \Gamma_*(A)$, and $\Gamma_*(A) \subset \cup_{B\in\mathcal{B}}B$, then, for all $x \in A$, $\Gamma(x) \subset \cup_{B\in\mathcal{B}}B$. Since Γ is compact-valued, then, for all $x \in X$, $\Gamma(x)$ is compact. Since, for all $x \in X$, $\Gamma(x)$ is compact, and $A \subset X$, then, for all $x \in A$, $\Gamma(x)$ is compact. Since, for all $x \in A$, $\Gamma(x)$ is compact and $\Gamma(x) \subset \cup_{B\in\mathcal{B}}B$, then, for all $x \in A$, there exists $\mathcal{B}_x \subset \mathcal{B}$ finite and such that, $\Gamma(x) \subset \cup_{B\in\mathcal{B}_x}B$. Since, for all $x \in A$ and all $B \in \mathcal{B}_x$, B is open, then, for all $x \in A$, $\cup_{B\in\mathcal{B}_x}B$ is open. Since Γ is upper hemicontinuous and, for all $x \in A$, $\cup_{B\in\mathcal{B}_x}B$ is open, then, for all $x \in A$, $\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)$ is open. Since, for all $x \in A$, $\Gamma(x) \subset \cup_{B\in\mathcal{B}_x}B$, then, for all $x \in A$, $x \in \Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)$. Since, for all $x \in A$, $x \in \Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)$, then

$$A \subset \cup_{x\in A}\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B).$$

Since, for all $x \in A$, $\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)$ is open, and $A \subset \cup_{x\in A}\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)$, then $\{\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)\}_{x\in A}$ is an open cover of A. Since $\{\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)\}_{x\in A}$ is an open cover of A and A is compact, then there exists $A' \subset A$ finite such that, $A \subset \cup_{x\in A'}\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)$. Since $A \subset \cup_{x\in A'}\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)$, then, for all $x \in A$, $x \in \cup_{x\in A'}\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)$. Since, for all $x \in A$, $x \in \cup_{x\in A'}\Gamma_+^{-1}(\cup_{B\in\mathcal{B}_x}B)$, then, for all $x \in A$, there exists $\xi(x) \in A'$ such that $x \in \Gamma_+^{-1}(\cup_{B\in\mathcal{B}_{\xi(x)}}B)$. Since, for all $x \in A$, there exists $\xi(x) \in A'$ such that $x \in \Gamma_+^{-1}(\cup_{B\in\mathcal{B}_{\xi(x)}}B)$, then, for all $x \in A$, there exists $\xi(x) \in A'$ such that $\Gamma(x) \subset \cup_{B\in\mathcal{B}_{\xi(x)}}B$. Since, for all $x \in A$, there exists $\xi(x) \in A'$ such that $\Gamma(x) \subset \cup_{B\in\mathcal{B}_{\xi(x)}}B$, then

$$\begin{aligned}\Gamma_*(A) &= \\ \cup_{x\in A}\Gamma(x) &\subset \cup_{\xi\in A'}(\cup_{B\in\mathcal{B}_\xi}B).\end{aligned}$$

Since A' is finite and, for all $\xi \in A'$, $\mathcal{B}_\xi$ is finite, then $\cup_{\xi\in A'}\mathcal{B}_\xi$ is a finite subcover of $\Gamma_*(A)$. Q.E.D.

S3. Complete characterization of upper hemicontinuous, compact-valued correspondences by sequences. *If* (X, d_X) *to* (Y, d_Y) *are metric spaces and* $\Gamma \in \mathcal{P}(Y)^X$ *is compact-valued, then* Γ *is upper hemicontinuous if, and only if, for all* $x \in X$ *and all* $s_X \in X^{\mathbb{N}}$ *convergent and such that* $x = \lim s_X$*, and all* $s_Y \in Y$ *such that, for all* $n \in \mathbb{N}$, $s_Y(n) \in \Gamma(s_X(n))$*, there exists* $h \in \mathbb{N}^{\mathbb{N}}$ *increasing such that* $s_Y \circ h$ *is convergent and* $\lim s_Y \circ h \in \Gamma(x)$.

Proof. Let (X, d_X) to (Y, d_Y) be metric spaces, $\Gamma \in \mathcal{P}(Y)^X$ be compact-valued, and $x \in X$.

Assume that Γ is upper hemicontinuous at x. Let $s_X \in X^{\mathbb{N}}$ be convergent and such that $x = \lim s_X$, and $s_Y \in Y^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s_Y(n) \in \Gamma(s_X(n))$. Since $s_X \in X^{\mathbb{N}}$ is convergent and $x = \lim s_X$, then $s_X(\mathbb{N}) \cup \{x\}$ is compact. Since $s_X(\mathbb{N}) \cup \{x\}$ is compact and Γ is compact-valued and upper hemicontinuous, then $\Gamma_*(s_X(\mathbb{N}) \cup \{x\})$ is compact. Since, for all $n \in \mathbb{N}$, $s_Y(n) \in \Gamma(s_X(n))$, then

$$\begin{aligned}s_Y(\mathbb{N}) &= \\ \cup_{n\in\mathbb{N}}\{s_Y(n)\} &\subset \cup_{n\in\mathbb{N}}\Gamma(s_X(n)) \\ &= \Gamma_*(s_X(\mathbb{N})) \\ &\subset \Gamma_*(s_X(\mathbb{N})) \cup \Gamma(x) \\ &= \Gamma_*(s_X(\mathbb{N}) \cup \{x\}).\end{aligned}$$

Since $s_Y(\mathbb{N}) \subset \Gamma_*(s_X(\mathbb{N}) \cup \{x\})$ and $\Gamma_*(s_X(\mathbb{N}) \cup \{x\})$ is compact, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing such that $s_Y \circ h$ is convergent and

$$\begin{aligned}\lim s_Y \circ h &\in \Gamma_*(s_X(\mathbb{N}) \cup \{x\}) \\ &= \Gamma_*(s_X(\mathbb{N})) \cup \Gamma(x).\end{aligned}$$

Assume that $\lim s_Y \circ h \notin \Gamma(x)$.

(1) On the one hand, since $\Gamma(x)$ is compact, and d_Y is continuous, then, for all $y \in Y$, there exists $f \in \mathbb{R}_+^Y$ such that, for all $y \in Y$, $f(y) = \min_{y' \in \Gamma(x)} d_Y(y, y')$. Since $\Gamma(x)$ is compact, then $\Gamma(x)$ is closed. Since $\lim s_Y \circ h \notin \Gamma(x)$ and $\Gamma(x)$ is closed, then $f(\lim s_Y \circ h) > 0$.

(Assume $f(\lim s_Y \circ h) = 0$. Since $f(\lim s_Y \circ h) = 0$, then, for all $\varepsilon > 0$, there exists $y' \in \Gamma(x)$ such that $d_Y(\lim s_Y \circ h, y') < \varepsilon$. Since, for all $\varepsilon > 0$, there exists $y' \in \Gamma(x)$ such that $d_Y(\lim s_Y \circ h, y') < \varepsilon$, then $\lim s_Y \circ h$ is an accumulation point of $\Gamma(x)$. Since $\lim s_Y \circ h$ is an accumultion point of $\Gamma(x)$ and $\Gamma(x)$ is closed, then $\lim s_Y \circ h \in \Gamma(x)$!)

Let $\varepsilon = f(\lim s_Y \circ h)$. Since $f(\lim s_Y \circ h) > \frac{\varepsilon}{2}$, then $\lim s_Y \circ h \notin f^{-1}[0, \frac{\varepsilon}{2}]$.

(2) On the other hand, since, for all $y \in \Gamma(x)$, $f(y) = 0$, then $\Gamma(x) \subset f^{-1}[0, \frac{\varepsilon}{2})$. Since $[0, \frac{\varepsilon}{2}) \subset \mathbb{R}_+$ is open and f is continuous, then $f^{-1}[0, \frac{\varepsilon}{2})$ is open. Since $f^{-1}[0, \frac{\varepsilon}{2})$ is open, $\Gamma(x) \subset f^{-1}[0, \frac{\varepsilon}{2})$, and Γ is upper hemicontinuous at x, then there exists $\delta > 0$ such that, for all $x' \in B_\delta(x)$, $\Gamma(x') \subset f^{-1}[0, \frac{\varepsilon}{2})$. Since $x = \lim s_X$, then there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $s_X(n') \in B_\delta(x)$. Since, for all $n' > n$, $s_X(n') \in B_\delta(x)$, then, for all $n' > n$, $\Gamma(s_X(n')) \subset f^{-1}[0, \frac{\varepsilon}{2})$. Since, for all $n' > n$, $s_Y(n') \in \Gamma(s_X(n'))$ and $\Gamma(s_X(n')) \subset f^{-1}[0, \frac{\varepsilon}{2})$, then, for all $n' > n$, $s_Y(n') \in f^{-1}[0, \frac{\varepsilon}{2})$. Since, h is increasing, then, for all $n' > h^{-1}(n)$, $h(n') > n$. Since, for all $n' > h^{-1}(n)$, $h(n') > n$, and $s_Y(n') \in f^{-1}[0, \frac{\varepsilon}{2})$ then, for all $n' > h^{-1}(n)$, $s_Y(h(n')) \in f^{-1}[0, \frac{\varepsilon}{2})$. Since, for all $n' > h^{-1}(n)$, $(s_Y \circ h)(n') \in f^{-1}[0, \frac{\varepsilon}{2})$, then $\lim s \circ h$ is an accumulation point of $f^{-1}[0, \frac{\varepsilon}{2})$. Since $\lim s \circ h$ is an accumulation point of $f^{-1}[0, \frac{\varepsilon}{2})$ and $f^{-1}[0, \frac{\varepsilon}{2}) \subset f^{-1}[0, \frac{\varepsilon}{2}]$, then $\lim s \circ h$ is an accumulation point, and hence a closure point, of $f^{-1}[0, \frac{\varepsilon}{2}]$. Since, f is continuous and $[0, \frac{\varepsilon}{2}]$ is closed, then $f^{-1}[0, \frac{\varepsilon}{2}]$ is closed. Since $f^{-1}[0, \frac{\varepsilon}{2}]$ is closed and $\lim s \circ h$ is a closure point of $f^{-1}[0, \frac{\varepsilon}{2}]$, then $\lim s \circ h \in f^{-1}[0, \frac{\varepsilon}{2}]$!

Therefore, $\lim s_Y \circ h \in \Gamma(x)$.

Conversely, assume that Γ is not upper hemicontinuous at x. Since Γ is not upper hemicontinuous at x, then there exists $\varepsilon > 0$ and $y \in Y$ such that, $\Gamma(x) \subset B_\varepsilon(y)$ and, for all $\delta > 0$, there exists $x' \in B_\delta(x)$ such that $\Gamma(x') \not\subset B_\varepsilon(y)$. Since for all $\delta > 0$, there exists $x' \in B_\delta(x)$ such that $\Gamma(x') \not\subset B_\varepsilon(y)$, then, for all $n \in \mathbb{N}$, there exists $x_n \in B_{\frac{1}{n}}(x)$ such that $\Gamma(x_n) \not\subset B_\varepsilon(y)$. Since, for all $n \in \mathbb{N}$, $\Gamma(x_n) \not\subset B_\varepsilon(y)$, then, for all $n \in \mathbb{N}$, there exists $y_n \in \Gamma(x_n)$ such that $y_n \notin B_\varepsilon(y)$. Let $s_X \in X^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s_X(n) = x_n$, and $s_Y \in Y^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s_Y(n) = y_n$. Since, for all $\delta > 0$ and all $n > [\frac{1}{\delta}]$, $s_X(n) \in B_\delta(x)$, then, for all $\delta > 0$, $s_X^{-1}(B_\delta(x)^C)$ is finite. Since, for all $\delta > 0$, $s_X^{-1}(B_\delta(x)^C)$ is finite, then s_X is convergent and $x = \lim s_X$. Since, for all $n \in \mathbb{N}$, $s_Y(n) \notin B_\varepsilon(y)$, then, for all $h \in \mathbb{N}^{\mathbb{N}}$ increasing and all $n \in \mathbb{N}$, $s_Y(h(n)) \notin B_\varepsilon(y)$. Since, for all $h \in \mathbb{N}^{\mathbb{N}}$ increasing and all $n \in \mathbb{N}$, $s_Y(h(n)) \notin B_\varepsilon(y)$, then $(s_Y \circ h)(\mathbb{N}) \subset B_\varepsilon(y)^C$. Since $B_\varepsilon(y)$ is open, then $B_\varepsilon(y)^C$ is closed. Since $B_\varepsilon(y)^C$ is closed, then $B_\varepsilon(y)^C$ contains all its closure points. Since $\lim s_Y \circ h$ (if it exists) is a closure point of $(s_Y \circ h)(\mathbb{N})$

and $(s_Y \circ h)(\mathbb{N}) \subset B_\varepsilon(y)^C$, then $\lim s_Y \circ h$ is a closure point of $B_\varepsilon(y)^C$. Since $\lim s_Y \circ h$ is a closure point of $B_\varepsilon(y)^C$ and $B_\varepsilon(y)^C$ contains all its closure points, then $\lim s_Y \circ h \in B_\varepsilon(y)^C$. Since $\lim s_Y \circ h \in B_\varepsilon(y)^C$ and $\Gamma(x) \subset B_\varepsilon(y)$ then $\lim s_Y \circ h \notin \Gamma(x)$. Q.E.D.

Lower hemicontinuous correspondences.

S4. Complete characterization of lower hemicontinuous correspondences by sequences. *If (X, d_X) and (Y, d_Y) are metric spaces, $\Gamma \in \mathcal{P}(Y)^X$, and $x \in X$, then Γ is lower hemicontinuous at x if, and only if, for all $s_X \in X^{\mathbb{N}}$ convergent and such that $x = \lim s_X$, and all $y \in \Gamma(x)$, there exists $s_Y \in Y^{\mathbb{N}}$ convergent and such that, for all $n \in \mathbb{N}$, $s_Y(n) \in \Gamma(s_X(n))$ and $y = \lim s_Y$.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $\Gamma \in \mathcal{P}(Y)^X$, and $x \in X$.

Assume that Γ is lower hemicontinuous at x, and let $s_X \in X^{\mathbb{N}}$ be convergent and such that $x = \lim s_X$, and $y \in \Gamma(x)$. Since $y \in \Gamma(x)$, then, for all $\varepsilon > 0$, $\Gamma(x) \cap B_\varepsilon(y) \neq \phi$. Since, for all $\varepsilon > 0$, $\Gamma(x) \cap B_\varepsilon(y) \neq \phi$, and Γ is lower hemicontinuous at x, then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in B_\delta(x)$, $\Gamma(x') \cap B_\varepsilon(y) \neq \phi$. Since, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in B_\delta(x)$, $\Gamma(x') \cap B_\varepsilon(y) \neq \phi$, then, for all $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that, for all $x' \in B_{\delta_n}(x)$, $\Gamma(x') \cap B_{\frac{1}{n}}(y) \neq \phi$, and $\delta_n > \delta_{n+1}$.

(Assume that there exists $n' \in \mathbb{N}$ such that $\delta_{n'} \leq \delta_{n'+1}$. Let, for all $n \in \mathbb{N}$, $\{\delta'_n\}_{n\in\mathbb{N}}$ be such that $\delta'_n = \delta_n$ iff $n \neq n'+1$, and $\delta'_{n'+1} < \delta_{n'}$. Since $\delta'_{n'+1} < \delta_{n'}$, then $B_{\delta'_{n'+1}}(x) \subset B_{\delta_{n'}}(x)$. Since, for all $x' \in B_{\delta_{n'}}(x)$, $\Gamma(x') \cap B_{\frac{1}{n'}}(y) \neq \phi$, and $B_{\delta'_{n'+1}}(x) \subset B_{\delta_{n'}}(x)$, then, for all $x' \in B_{\delta'_{n'+1}}(x)$, $\Gamma(x') \cap B_{\frac{1}{n'}}(y) \neq \phi$, and $\delta'_{n'} > \delta'_{n'+1}$.)

Let, for all $n' \in \mathbb{N}$, $I_{n'} = s_X^{-1}(B_{\delta_{n'}}(x) \cap B_{\delta_{n'+1}}(x)^C)$. Since $x = \lim s_X$, then, for all $n' \in \mathbb{N}$, $I_{n'}$ is finite. Since, for all $n' \in \mathbb{N}$ and all $x' \in B_{\delta_{n'}}(x)$, $\Gamma(x') \cap B_{\frac{1}{n'}}(y) \neq \phi$, and, for all $n' \in \mathbb{N}$ and all $n \in I_{n'}$, $x_n \in B_{\delta_{n'}}(x)$, then, for all $n' \in \mathbb{N}$ and all $n \in I_{n'}$, $\Gamma(x_n) \cap B_{\frac{1}{n'}}(y) \neq \phi$. Since, for all $n' \in \mathbb{N}$ and all $n \in I_{n'}$, $\Gamma(x_n) \cap B_{\frac{1}{n'}}(y) \neq \phi$, then, for all $n' \in \mathbb{N}$ and all $n \in I_{n'}$, there exists $y_n \in \Gamma(x_n) \cap B_{\frac{1}{n'}}(y)$. Let $s_Y \in Y^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s_Y(n) = y_n$. Since, for all $\varepsilon > 0$, there exists $n' \in \mathbb{N}$ such that $B_{\frac{1}{n'}}(y) \subset B_\varepsilon(y)$, and, for all $n'' > n'$ and all $n \in I_{n''}$,

$$\begin{aligned} y_n &\in B_{\frac{1}{n''}}(y) \\ &\subset B_{\frac{1}{n'}}(y) \\ &\subset B_\varepsilon(y), \end{aligned}$$

then $s_Y^{-1}(B_\varepsilon(y)^C) \subset \cup_{n \leq n'} I_n$. Since $s_Y^{-1}(B_\varepsilon(y)^C) \subset \cup_{n \leq n'} I_n$ and for all $n \in \mathbb{N}$, I_n is finite, then $s_Y^{-1}(B_\varepsilon(y)^C)$ is finite. Since, for all $\varepsilon > 0$, $s_Y^{-1}(B_\varepsilon(y)^C)$ is finite, then $y = \lim s_Y$.

Conversely, assume that Γ is not lower hemicontinuous at x. Since Γ is not lower hemicontinuous at x, then there exists $B \subset Y$ open and such that $\Gamma(x) \cap B \neq \phi$ and, for all $A \subset X$ open and such that $x \in A$, there exists $x' \in A$ such that $\Gamma(x') \cap B = \phi$. Since $\Gamma(x) \cap B \neq \phi$, then there exists $y \in \Gamma(x) \cap B$. Since $y \in \Gamma(x) \cap B$, then $y \in \Gamma(x)$ and $y \in B$. Since $y \in B$ and B is open, then there exists $\varepsilon > 0$ such that $B_\varepsilon(y) \subset B$. Since $y \in \Gamma(x)$ and $y \in B_\varepsilon(y)$, then $\Gamma(x) \cap B_\varepsilon(y) \neq \phi$. Since $B_\varepsilon(y)$ is open and $\Gamma(x) \cap B_\varepsilon(y) \neq \phi$, then, for all $A \subset X$ open and such that $x \in A$, there exists $x' \in A$ such that $\Gamma(x') \cap B_\varepsilon(y) = \phi$. Since, for all $A \subset X$ open and such that $x \in A$, there exists $x' \in A$ such that $\Gamma(x') \cap B_\varepsilon(y) = \phi$, then for all $n \in \mathbb{N}$, there exists $x_n \in B_{\frac{1}{n}}(x)$ such that $\Gamma(x_n) \cap B_\varepsilon(y) = \phi$. Let $s_X \in X^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s_X(n) = x_n$. Since, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, $\frac{1}{n} < \varepsilon$, then, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $B_{\frac{1}{n'}}(x) \subset B_\varepsilon(x)$. Since, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $B_{\frac{1}{n'}}(x) \subset B_\varepsilon(x)$, and $x_{n'} \in B_{\frac{1}{n'}}(x)$ then, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $x_{n'} \in B_\varepsilon(x)$. Since, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $x_{n'} \in B_\varepsilon(x)$, and $s_X(n') = x_{n'}$ then $x = \lim s_X$. Since, for all $n \in \mathbb{N}$, $\Gamma(s_X(n)) \cap B_\varepsilon(y) = \phi$, then, for all $s_Y \in Y^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $s_Y(n) \in \Gamma(s_X(n))$, $y \neq \lim s_Y$. Q.E.D.

S5. The intersection of an upper hemicontinuous, compact-valued correspondence and a closed graph correspondence is upper hemicontinuous and compact-valued. *If (X, d_X) and (Y, d_Y) are metric spaces, $\Gamma_1, \Gamma_2 \in \mathcal{P}(Y)^X$ are such that Γ_1 is upper hemicontinuous and compact-valued, Γ_2 has a closed graph, then $\Gamma_1 \cap \Gamma_2$ is upper hemicontinuous and compact-valued.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $\Gamma_1, \Gamma_2 \in \mathcal{P}(Y)^X$ be such that Γ_1 is upper hemicontinuous and compact-valued, Γ_2 has a closed graph.

(1) $\Gamma_1 \cap \Gamma_2$ is upper hemicontinuous.

In effect, let $x \in X$, $s_X \in X^{\mathbb{N}}$ be convergent and such that $x = \lim s_X$, and $s_Y \in Y^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s_Y(n) \in \Gamma_1(s_X(n)) \cap \Gamma_2(s_X(n))$. Since Γ_1 is compact-valued and upper hemicontinuous, and hence Γ_1 is upper hemicontinuous at x, $x = \lim s_X$, and s_Y is such that, for all $n \in \mathbb{N}$, $s_Y(n) \in \Gamma_1(s_X(n)) \cap \Gamma_2(s_X(n))$, and hence $s_Y(n) \in \Gamma_1(s_X(n))$, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing such that $s_Y \circ h$ is convergent and $\lim s_Y \circ h \in \Gamma_1(x)$. Since $\lim(s_X, s_Y \circ h)$ is a closure point of $(s_X, s_Y \circ h)(\mathbb{N})$ and $(s_X, s_Y \circ h)(\mathbb{N}) \subset \mathrm{Gr}_{\Gamma_2}$, then $\lim(s_X, s_Y \circ h)$ is a closure point of Gr_{Γ_2}. Since $\lim(s_X, s_Y \circ h)$ is a closure point of Gr_{Γ_2}, and Gr_{Γ_2} is a closed set of $(X \times Y, d_{X \times Y})$, then $\lim(s_X, s_Y \circ h) \in \mathrm{Gr}_{\Gamma_2}$. Since $\lim(s_X, s_Y \circ h) \in \mathrm{Gr}_{\Gamma_2}$, $\lim(s_X, s_Y \circ h) = (\lim s_X, \lim s_Y \circ h)$, and $x = \lim s_X$, then $(x, \lim s_Y \circ h) \in \mathrm{Gr}_{\Gamma_2}$. Since $(x, \lim s_Y \circ h) \in \mathrm{Gr}_{\Gamma_2}$, then $\lim s_Y \circ h \in \Gamma_2(x)$. Since $\lim s_Y \circ h \in \Gamma_1(x)$ and $\lim s_Y \circ h \in \Gamma_2(x)$, then $\lim s_Y \circ h \in \Gamma_1(x) \cap \Gamma_2(x)$. Therefore, since for all $s_X \in X^{\mathbb{N}}$ convergent and such that $x = \lim s_X$, and all $s_Y \in Y^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $s_Y(n) \in \Gamma_1(s_X(n)) \cap \Gamma_2(s_X(n))$, there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and such that $s_Y \circ h$ is convergent and $\lim s_Y \circ h \in \Gamma_1(x) \cap \Gamma_2(x)$, then $\Gamma_1 \cap \Gamma_2$ is upper hemicontinuous at x. Since, for all $x \in X$, $\Gamma_1 \cap \Gamma_2$ is upper hemicontinuous at x, then $\Gamma_1 \cap \Gamma_2$ is upper hemicontinuous.

(2) $\Gamma_1 \cap \Gamma_2$ is compact-valued.

In effect, let $x \in X$. Since Γ_1 is compact-valued, then $\Gamma_1(x)$ is compact. Let y be a closure point of $\Gamma_2(x)$. Since y is a closure point of $\Gamma_2(x)$, then there exists $s_Y \in \Gamma_2(x)^{\mathbb{N}}$ convergent and such that $y = \lim s_Y$. Let $s_X \in X^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s_X(n) = x$. Since, for all $n \in \mathbb{N}$, $s_X(n) = x$, then s_X is convergent and $x = \lim s_X$. Since, for all $n \in \mathbb{N}$, $(s_X(n), s_Y(n)) \in \mathrm{Gr}_{\Gamma_2}$, $y = \lim s_Y$, and $x = \lim s_X$, then (x, y) is a closure point of Gr_{Γ_2}. Since (x, y) is a closure point of Gr_{Γ_2} and Gr_{Γ_2} is closed, then $(x, y) \in \mathrm{Gr}_{\Gamma_2}$. Since $(x, y) \in \mathrm{Gr}_{\Gamma_2}$, then $y \in \Gamma_2(x)$. Since $\Gamma_2(x)$ contains all its closure points, then $\Gamma_2(x)$ is closed. Since $\Gamma_1(x)$ is compact and $\Gamma_2(x)$

is closed, then $\Gamma_1(x) \cap \Gamma_2(x)$ is compact.

Q.E.D.