Lecture Notes for Econ 898

FUNCTIONS BETWEEN METRIC SPACES

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When the sets between which a function is defined are endowed with metrics, then it can be given a sense to the notion of how sensitive are the changes in the values of the function to changes in its argument. A very crude way to limit the variability of a function is to require its range to be bounded. Such a function is (not surprisingly) said to be **bounded**. Nevertheless, bounded functions can still behave wildly within the limits of their bounds. In order to rule out wild reactions of the value of a function to small (i.e. nearby) changes in its argument, the imposition of local constraints is therefore needed. Thus a function whose values remain close to the value it takes at some point for nearby points of its domain is said to be continuous at that point. If the function happens to be continuous at every point of its domain, then it is said to be **continuous**.

Continuity is in itself a very useful property for a function to have. In effect, continuous functions have the remarkable property of transforming compacts into compacts. When the function is real valued this means that the range of the function is a closed and bounded set of real numbers. The boundedness of its range guarantees the existence of a supremum and an infimum to it. Its closedness then guarantees that these supremum and infimum are actually the maximum and minimum of the range. Thus continuous real valued functions with compact domains can be maximized and minimized, which is the bread and butter of economics.

Still, as for the variability issue, a continuous function may exhibit different degrees of local variability at different points of its domain (e.g. the exponential function, which responds very little to some perturbation to a very small negative real number —i.e. very large in absolute value— while its response to the *same* perturbation to its positive opposite is in comparison huge). Therefore, a stronger way of restricting the variability of a function is to require it to vary with the same, so to speak, degree everywhere it is defined. Such a continuous function is said to be **uniformly continuous**. Thus uniformly continuity imposes the same constraint on the local variability of a function across its entire domain.

Nonetheless, a uniformly continuous function may still allow for extremely fast local variability as long as this phenomenon remains constrained to only a few points of its domain, complying therefore with the global constraint (e.g. think of the cubic root at zero). In order to rule out such episodes of extremely fast variability, an outright bound needs to be imposed to the variability of the function relative to the variability of the argument, everywhere. The functions for which such a bound exists are said to be **Lipschitz**. Whenever a Lipschitz function maps a metric space into itself, a further constraint on its variability is to require this bound to be smaller than 1. This additional condition has the effect of making the images of any two points of the domain to be closer to each other than the points originally were. Any Lipschitz function that complies with this requirement is said to be a **contraction**. Contractions truly contract the space in the intuitive sense of the word, since they make all the points of the space become closer to each other after the transformation that the function represents takes place. Nonetheless, be aware of the fact that a function may contract the space without being a contraction if its average variability is always smaller than 1 but cannot be bounded away from 1.

Contractions of a given metric space are extremely useful if the the metric space happens to be complete. The reason is that the sequence obtained by iterating the contraction starting from any given point of the space, is a Cauchy sequence. Therefore, since the space is complete, then the sequence necessarily converges somewhere. That limit is necessarily a fixed point of the contraction, i.e. a point mapped to itself, and it follows easily from the very idea of a contraction that there cannot any two such fixed points (otherwise they should get closer to each other after the contraction takes place and, hence, at least one of them would not be a fixed point). The existence of such a fixed point of a contraction on a complete metric (a.k.a. Banach) space and its uniqueness is the claim made by **Banach's Fixed Point Theorem**, also referred to as the Contraction Mapping Theorem (there are several distinct arguments showing the existence of fixed points of functions in different settings, hence their labelling by their fathers' names, Banach, Brower, Kakutani, Tarski, Picard,...). The interest of Banach's fixed point theorem is that it allows to show the existence and uniqueness of the solution to some equation precisely as the fixed point in question if the solution is shown to be embedded in some complete metric space and the equation is equivalent to the solution being a fixed point of some contraction. Moreover, it provides a constructive way to compute it with any precision. Since the underlying metric space can be very general, the procedure can be used to solve some functional equations. This is precisely what we will do to solve the standard dynamic programming problem.

Definitions

D1. Limit of a function at an accumulation point of its domain. *If (X, d_X) and (Y, d_Y) are metric spaces, $f \in Y^X$, x is an accumulation point of $f^{-1}(Y)$, and $y \in Y$, then y is the limit of f at x, $\lim_x f$, if, and only if,*

$$\forall \varepsilon > 0, \exists \delta > 0 \mid f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(y).$$

D2. Continuous function at a point of its domain. *If (X, d_X) and (Y, d_Y) are metric spaces, $f \in Y^X$, and $x \in f^{-1}(Y)$, then f is continuous at x if, and only if,*

$$\forall \varepsilon > 0, \exists \delta > 0 \mid f(B_\delta(x)) \subset B_\varepsilon(f(x)).$$

D3. *If (X, d_X) and (Y, d_Y) are metric spaces, and $f \in Y^X$, then*

(1) *f is* **continuous** *if, and only if,*

$$\forall x \in f^{-1}(Y), \forall \varepsilon > 0, \exists \delta > 0 \mid f(B_\delta(x)) \subset B_\varepsilon(f(x)),$$

(2) *f is* **uniformly continuous** *if, and only if,*

$$\forall \varepsilon > 0, \exists \delta > 0 \mid \forall x \in f^{-1}(Y), f(B_\delta(x)) \subset B_\varepsilon(f(x)),$$

(3) *f is* **Lipschitz** *if, and only if,*

$$\exists \alpha \in \mathbb{R}_+ \mid \forall x, x' \in f^{-1}(Y), x = x' \text{ or } \frac{d_Y(f(x), f(x'))}{d_X(x, x')} \leq \alpha.$$

(4) *f is a* **contraction** *if, and only if,*

$$\exists \alpha \in [0, 1) \mid \ \forall x, x' \in f^{-1}(Y), x = x' \text{ or } \frac{d_Y(f(x), f(x'))}{d_X(x, x')} \leq \alpha.$$

D4. Bounded function. *If (X, d_X) and (Y, d_Y) are metric spaces, and $f \in Y^X$, then f is bounded if, and only if, $f(X)$ is a bounded set of (Y, d_Y),*

N5. Notation. *If (X, d_X) and (Y, d_Y) are metric spaces, and $f \in Y^X$, then*

(1) *$f \in \mathrm{C}(Y^X)$ if, and only if, f is continuous,*
(2) *$f \in \mathrm{B}(Y^X)$ if, and only if, f is bounded,*
(3) *$f \in \mathrm{CB}(Y^X)$ if, and only if, f is continuous and bounded.*

Examples

Continuity of real valued functions.

E1. *If (X, d) is a metric space, $\mathbb{R}$ is endowed with any metric d_p with $p \geq 1$, $f, g \in \mathbb{R}^X$ are continuous, and $f^{-1}(\mathbb{R}) = g^{-1}(\mathbb{R})$, then*

(1) *$f + g$ is continuous,*
(2) *fg is continuous,*
(3) *αf is continuous, for all $\alpha \in \mathbb{R}$,*
(4) *$|f|$ is continuous,*
(5) *$\max\{f, g\}$ and $\min\{f, g\}$ are continuous,*
(6) *$\frac{f}{g}$ is continuous at every $x \in g^{-1}(\mathbb{R})$ such that $g(x) \neq 0$.*

Proof. Exercise.

Examples of continuous functions.

E2. *If $f \in \mathbb{R}^{\mathbb{R}}$ is such that $f^{-1}(\mathbb{R}) = \mathbb{R}_{++}$ and $f(x) = \frac{1}{x}$ for all $x > 0$, and $\mathbb{R}$ is endowed with the metric d_1, then f is continuous.*

Proof. Let $x > 0$ and $\varepsilon > 0$. If $x' > 0$ and $|x' - x| < \frac{\varepsilon x^2}{1 + \varepsilon x}$, then $x - x' < \frac{\varepsilon x^2}{1 + \varepsilon x}$ and hence $x - \frac{\varepsilon x^2}{1 + \varepsilon x} < x'$. Since $0 < x - \frac{\varepsilon x^2}{1 + \varepsilon x}$, then

$$\frac{1}{x'} < \frac{1}{1 - \frac{\varepsilon x^2}{1 + \varepsilon x}}.$$

Moreover, since

$$|x' - x| < \frac{\varepsilon x^2}{1 + \varepsilon x},$$

then

$$\begin{aligned} \left| \frac{1}{x'} - \frac{1}{x} \right| &= \\ \frac{|x' - x|}{x'x} &< \frac{\frac{\varepsilon x^2}{1 + \varepsilon x}}{x'x} \\ &< \frac{\frac{\varepsilon x^2}{1 + \varepsilon x}}{x\left(x - \frac{\varepsilon x^2}{1 + \varepsilon x}\right)} \\ &= \varepsilon. \end{aligned}$$

Therefore for all $x > 0$ and all $\varepsilon > 0$, there exists $\delta > 0$, namely $\delta = \frac{\varepsilon x^2}{1 + \varepsilon x}$, such that $f(B_\delta(x)) \subset f(B_\varepsilon(f(x)))$, i.e. f is continuous for all $x > 0$, and hence continuous. Q.E.D.

The next proposition is a consequence as well of the fact that any the distance from a given point to any other point is a continuous function for any metric. Here the point given is implicitly the constant sequence of zeros.

E3. *If $f \in \mathbb{R}^{B(\mathbb{R}^{\mathbb{N}})}$ is such that, for all $x \in \mathbb{R}^{\mathbb{N}}$, $f(x) = \sup_{n \in \mathbb{N}} |x(n)|$, and both $B(\mathbb{R}^{\mathbb{N}})$ and $\mathbb{R}$ are endowed with the metric d_∞, then f is continuous.*

Proof. Exercise.

E4. The Riemann integral is continuous. *If $R \in \mathbb{R}^{C[0,1]}$ is such that $R(f) = \int_0^1 f$ for all $f \in \mathbb{R}^{C[0,1]}$, and both $\mathbb{R}^{C[0,1]}$ and $\mathbb{R}$ are endowed with the metric d_∞, then R is continuous.*

Proof. Let $f, g \in \mathbb{R}^{C[0,1]}$. Since

$$\begin{aligned} (f - g)(x) &\leq |(f - g)(x)| \\ &\leq \sup_{[0,1]} |f - g| \end{aligned}$$

for all $x \in [0, 1]$, then

$$\begin{aligned} \int_0^1 (f - g) &\leq \int_0^1 \sup_{[0,1]} |f - g| \\ &= \sup_{[0,1]} |f - g|. \end{aligned}$$

Similarly,

$$-\int_0^1 (f-g) \leq \sup_{[0,1]} |f-g|,$$

and hence

$$\Big|\int_0^1 (f-g)\Big| \leq \sup_{[0,1]} |f-g|.$$

Therefore, for all $g \in C[0,1]$ and all $\varepsilon > 0$, if $f \in C[0,1]$ is such that $d_\infty(f,g) < \varepsilon$, then

$$\begin{aligned} d_\infty(R(f), R(g)) &= \\ |R(f) - R(g)| &= \\ \Big|\int_0^1 f - \int_0^1 g\Big| &= \\ \Big|\int_0^1 (f-g)\Big| &\leq \sup_{[0,1]} |f-g| \\ &= d_\infty(f,g) \\ &< \varepsilon. \end{aligned}$$

Thus, for all $g \in C[0,1]$ and all $\varepsilon > 0$, there exists $\delta > 0$, namely $\delta = \varepsilon$, such that $R(B_\delta(g)) \subset B_\varepsilon(R(g))$, and hence R is continuous at g for all $g \in C[0,1]$. Therefore R is continuous. Q.E.D.

E6. Continuity of every function defined on a discrete metric space. *If (X, d_X) is a discrete metric space, (Y, d_Y) is a metric space and $f \in X^Y$, then f is continuous.*

Proof. Exercise

E7. Continuity of the distance to a set. *If (X, d) is a metric space, $A \subset X$ is nonempty and $f \in \mathbb{R}^X$ is such that $f(x) = \inf_{x' \in A} d(x, x')$, then f is continuous.*

Proof. Exercise

Uniform continuity and continuity.

E9. Continuous but not uniformly continuous function. *If $f \in \mathbb{R}^{\mathbb{R}}$ is such that $f(x) = \frac{1}{x}$ and $\mathbb{R}$ is endowed with the absolute value metric, then f is continuous but not uniformly continuous.*

Proof. Exercise

Examples of Lipschitz functions.

E10. *The function $f \colon \ell_\infty \to \mathbb{R}_+$ such that $f(x) = \sup_{n \in \mathbb{N}} |x(n)|$ is Lipschitz.*

Proof. Exercise.

E11. *The function $R \colon C[0,1] \to \mathbb{R}$ such that $R(f) = \int_0^1 f$ is Lipschitz.*

Proof. Exercise.

E12. *The distance to a nonempty set is a Lipschitz function.*

Proof. Exercise.

Sufficient characterizations of Lipschitz functions and contractions.

E13. A differentiable real-valued function on the real line is Lipschitz iff it has a derivative bounded in absolute value. *If f is a differentiable real valued function on the real line,*[1] *then f is a Lipschitz if, and only if, there exists $K \in \mathbb{R}_+$ such that, for all $x \in \mathbb{R}$, $|f'(x)| \leq K$.*

Proof. Let f be a differentiable real valued function on the real line.

Assume that there exists $K \in \mathbb{R}_+$ such that, for all $x \in \mathbb{R}$, $|f'(x)| \leq K$ and let $a, b \in \mathbb{R}$ be such that $a < b$.

Since f is differentiable, then f is continuous on $[a, b]$ and differentiable on (a, b). Thus there exists $x \in (a, b)$ such that

$$f(b) - f(a) = f'(x)(b - a).$$

Therefore, if $f(a) \leq f(b)$, then $f'(x) \geq 0$ and

$$\begin{aligned} d_1(f(a), f(b)) = \\ |f(a) - f(b)| = \\ f(b) - f(a) &= f'(x)(b - a) \\ &= |f'(x)||b - a| \\ &\leq K|b - a| \\ &= K d_1(a, b). \end{aligned}$$

If, on the contrary, $f(a) \geq f(b)$, then $f'(x) \leq 0$ and

$$\begin{aligned} d_1(f(a), f(b)) = \\ |f(a) - f(b)| = \\ f(a) - f(b) &= -f'(x)(b - a) \\ &= |f'(x)||b - a| \\ &\leq K|b - a| \\ &= K d_1(a, b) \end{aligned}$$

as well. Therefore f is a Lipschitz.

Conversely, assume that

$$\forall K \in \mathbb{R}_+, \exists x \in \mathbb{R} \mid |f'(x)| > K.$$

Since, for all $x, x' \in \mathbb{R}$ distinct,

$$|f'(x)| \leq \left| f'(x) - \frac{f(x) - f(x')}{x - x'} \right| + \left| \frac{f(x) - f(x')}{x - x'} \right|,$$

[1] That is to say a function f from the real vector space $\mathbb{R}$ endowed with any norm to itself.

then, for all $x, x' \in \mathbb{R}$ distinct,

$$\left|\frac{f(x)-f(x')}{x-x'}\right| \geq |f'(x)| - \left|f'(x) - \frac{f(x)-f(x')}{x-x'}\right|.$$

Since for all $\alpha \in \mathbb{R}_+$ and all $K \in (\alpha, +\infty)$

(1) there exists $x \in \mathbb{R}$ such that $|f'(x)| > K$, and
(2) there exists $\varepsilon > 0$ such that $\alpha = K - \varepsilon$, and for such ε

$$\exists \delta > 0 \mid \forall x' \in B_\delta(x) \setminus \{x\}, \left|f'(x) - \frac{f(x)-f(x')}{x-x'}\right| < \varepsilon,$$

then there exists $x' \in B_\delta(x) \setminus \{x\}$ such that

$$\begin{aligned}\left|\frac{f(x)-f(x')}{x-x'}\right| &\geq |f'(x)| - \left|f'(x) - \frac{f(x)-f(x')}{x-x'}\right| \\ &> |f'(x)| - \varepsilon \\ &> K - \varepsilon \\ &= \alpha,\end{aligned}$$

i.e. it is not true that

$$\exists \alpha \in \mathbb{R}_+ \mid \forall x, x' \in \mathbb{R}, x = x' \text{ or } \left|\frac{f(x)-f(x')}{x-x'}\right| \leq \alpha.$$

Q.E.D.

From the previous fact, the following sufficient characterization of contractions of the real line follows.

E14. A differentiable real-valued function is a contraction iff it has a derivative bounded away from 1 in absolute value. *If f is a differentiable real valued function on the real line, then f is a contraction if, and only if, there exists $K \in [0,1)$ such that, for all $x \in \mathbb{R}$, $|f'(x)| \leq K$.*

E15. Blackwell's sufficient characterization of a contraction.

If

(1) *(X, d_X) is a metric space,*
(2) *$S \subset \mathbb{R}^X$ is such that, for all $f \in S$,*

i) f is bounded,
ii) $f^{-1}(\mathbb{R}) = X' \subset X$,
iii) and for all $c \in S$ constant, $f + c \in S$, and

(3) *$T \in S^S$ is such that*[2]

i) if $f, g \in S$ and $f \leq_{\mathbb{R}^X} g$, then $T(f) \leq_{\mathbb{R}^X} T(g)$, and
ii) there exists $\delta \in [0,1)$ such that, for all $f \in S$ and all $c \in \mathbb{R}^X$ constant,

$$T(f+c) \leq_{\mathbb{R}^X} T(f) + \delta c,$$

[2]From now on, $\leq_{\mathbb{R}^X}$ stands for the partial order in $\mathbb{R}^X$ defined by $f \leq_{\mathbb{R}^X} g$ if, and only if, for all $x \in X$, $f(x) \leq g(x)$.

then T is a contraction.

Proof. Let $f, g \in S$. Since, for all $x \in X'$,

$$\begin{aligned} f(x) - g(x) &\leq |f(x) - g(x)| \\ &\leq \sup_{x \in X'} |f(x) - g(x)| \\ &= d_\infty(f, g) \end{aligned}$$

then $f - g \leq_{\mathbb{R}^X} d_\infty(f, g)$, i.e. $f \leq_{\mathbb{R}^X} g + d_\infty(f, g)$. Hence

$$\begin{aligned} T(f) &\leq_{\mathbb{R}^X} T(g + d_\infty(f, g)) \\ &\leq_{\mathbb{R}^X} T(g) + \delta d_\infty(f, g) \end{aligned}$$

i.e. $T(f) - T(g) \leq_{\mathbb{R}^X} \delta d_\infty(f, g)$.

Similarly, since, for all $x \in X'$,

$$\begin{aligned} g(x) - f(x) &\leq |f(x) - g(x)| \\ &\leq \sup_{x \in X'} |f(x) - g(x)| \\ &= d_\infty(f, g) \end{aligned}$$

then $g - f \leq_{\mathbb{R}^X} d_\infty(f, g)$, i.e. $g \leq_{\mathbb{R}^X} f + d_\infty(f, g)$. Hence

$$\begin{aligned} T(g) &\leq_{\mathbb{R}^X} T(f + d_\infty(f, g)) \\ &\leq_{\mathbb{R}^X} T(f) + \delta d_\infty(f, g) \end{aligned}$$

i.e. $-\delta d_\infty(f, g) \leq_{\mathbb{R}^X} T(f) - T(g)$. Thus $|T(f) - T(g)| \leq_{\mathbb{R}^X} \delta d_\infty(f, g)$, i.e. for all $x \in X$, $|T(f)(x) - T(g)(x)| \leq \delta d_\infty(f, g)$ and hence

$$d_\infty(T(f), T(g)) = \sup_{x \in X} |T(f)(x) - T(g)(x)| \leq \delta d_\infty(f, g)$$

and therefore T is a contraction. Q.E.D.

Examples of contractions.

E16. *If $f\colon \mathbb{R} \to \mathbb{R}$ is such that, for all $x \in \mathbb{R}$, $f(x) = \dfrac{x}{|x| + 1}$, then f is NOT a contraction.*

Proof. Exercise.

E17. *If $\mathbb{R}^{[a,b]}$ is endowed with d_∞, $h\colon [a, b] \times [a, b] \to \mathbb{R}$ is continuous,[3] $\delta \in [0, 1)$, and $T\colon C[a, b] \to \mathbb{R}^{[a,b]}$ is such that, for all $x \in [a, b]$,*

$$T(f)(x) = \max_{x' \in [a,b]} (h(x, x') + \delta f(x')),$$

then T is a contraction.

Proof. Since

(1) $(\mathbb{R}^{[a,b]}, d_\infty)$ is a metric space,

(2) $C[a, b] \subset \mathbb{R}^{[a,b]}$ is such that

i) for all $f \in C[a, b]$, f is bounded,

ii) for all $f \in C[a, b]$, $f^{-1}(\mathbb{R}) = [a, b]$

iii) for all $f \in C[a, b]$ and all $c \in C[a, b]$ constant, $f + c \in C[a, b]$, and

(3) for all $f \in C[a, b]$, $T(f) \in C[a, b]$,[4]

[3] With respect to the usual metrics.

[4] To be proved by means of the Theorem of the Maximum.

(4) if $f, g \in C[a,b]$ and $f \leq_{\mathbb{R}^{[a,b]}} g$, (i.e. for all $x' \in [a,b]$, $f(x') \leq g(x')$), and $\delta \geq 0$, then, for all $x, x' \in [a,b]$,

$$h(x,x') + \delta f(x') \leq h(x,x') + \delta g(x')$$

and hence, for all $x \in [a,b]$,

$$\begin{aligned} T(f)(x) &= \\ \max_{x' \in [a,b]} (h(x,x') + \delta f(x')) &\leq \max_{x' \in [a,b]} (h(x,x') + \delta g(x')) \\ &= T(g)(x) \end{aligned}$$

i.e. $T(f) \leq_{\mathbb{R}^{[a,b]}} T(g)$, and

(5) there exists $\delta \in [0,1)$ such that, for all $f \in C[a,b]$ and all $c \in C[a,b]$ constant, and for all $x \in [a,b]$,

$$\begin{aligned} T(f+c)(x) &= \\ \max_{x' \in [a,b]} \big(h(x,x') + \delta(f(x') + c)\big) &= \max_{x' \in [a,b]} \big(h(x,x') + \delta f(x')\big) + \delta c \\ &= T(f)(x) + \delta c \end{aligned}$$

i.e. such that $T(f+c) \leq_{\mathbb{R}^{[a,b]}} T(f) + \delta c$,

then T is a contraction. Q.E.D.

E18. *If $\mathbb{R}^{[0,1]}$ is endowed with d_∞, $h\colon [0,1] \times [0,1] \to \mathbb{R}_+$ is continuous,*[5] *$\lambda \in [0, (\max_{[0,1]^2} h)^{-1})$, and $T\colon C[0,1] \to \mathbb{R}^{[0,1]}$ is such that, for all $x \in [0,1]$,*

$$T(f)(x) = \lambda \int_0^1 h(x,x') f(x') dx',$$

then T is a contraction.

Proof. Exercise.

Fixed points of contractions in complete metric spaces as solutions to functional equations.

E19. Bellman's functional equation. *If $h\colon [a,b] \times [a,b] \to \mathbb{R}$ is continuous, $\delta \in [0,1)$, and $T\colon C[a,b] \to \mathbb{R}^{[a,b]}$ is such that, for all $x \in [a,b]$,*

$$T(f)(x) = \max_{x' \in [a,b]} (h(x,x') + \delta f(x'))$$

then there exist $f \in C[a,b]$ such that, for all $x \in [a,b]$,

$$f(x) = \max_{x' \in [a,b]} (h(x,x') + \delta f(x')).$$

Proof. Since $h\colon [a,b] \times [a,b] \to \mathbb{R}$ is continuous, $\delta \in [0,1)$, and $T\colon C[a,b] \to \mathbb{R}^{[a,b]}$ is such that, for all $x \in [a,b]$,

$$T(f)(x) = \max_{x' \in [a,b]} (h(x,x') + \delta f(x')),$$

then T is a contraction. Since $C[a,b]$ with d_∞ is a complete metric space, then there exists a unique $f \in C[a,b]$ such that $T(f) = f$, i.e. such that, for all $x \in [a,b]$,

$$f(x) = \max_{x' \in [a,b]} (h(x,x') + \delta f(x')).$$

Q.E.D.

[5] With respect to the usual metrics.

Uniqueness of limits of functions.

The next proposition is the foundation of the notion of limit of a function at some point. It establishes that the values taken by the function around a given point in the domain cannot get arbitrarily close to more than one point, if ever they get arbitrarily close to any. That point, in the case it exists, is then said to be the limit of the function at the given point of the domain. Notice in the statement below that it only makes sense to talk about the limit of a function at a point that is an accumulation point of its domain.

S1. The limit of a function at an accumulation point of its domain is unique. *If (X, d_X) and (Y, d_Y) are metric spaces, $f \in Y^X$, $x \in X$ is an accumulation point of $f^{-1}(Y)$, $y \in Y$ is such that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(y)$, and $y' \in Y$ is such that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(y')$, then $y = y'$.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $f \in Y^X$, $x \in X$ be an accumulation point of $f^{-1}(Y)$, $y \in Y$ be such that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(y)$, and $y' \in Y$ be such that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(y')$.

Assume that $y \neq y'$. Since $y \neq y'$, then $d_Y(y, y') > 0$. Let $\varepsilon = \frac{1}{2} d_Y(y, y')$. Therefore,

$$\exists \delta_1 > 0 \mid f(B_{\delta_1} \setminus \{x\}) \subset B_\varepsilon(y),$$

and

$$\exists \delta_2 > 0 \mid f(B_{\delta_2}(x) \setminus \{x\}) \subset B_\varepsilon(y').$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(y)$$

and

$$f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(y'),$$

i.e.

$$f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(y) \cap B_\varepsilon(y').$$

Since

$$B_\varepsilon(y) \cap B_\varepsilon(y') = \phi$$

because of the triangular inequality, then

$$f(B_\delta(x) \setminus \{x\}) = \phi$$

But

$$B_\delta(x) \setminus \{x\} \cap f^{-1}(Y) \neq \phi$$

because x is an accumulation point of $f^{-1}(Y)$ and hence

$$f(B_\delta(x) \setminus \{x\}) \neq \phi!$$

Thus, it is not true that $y \neq y'$. Q.E.D.

Complete Characterization of the limit of a function by means of sequences.

The next proposition characterizes the limit of a function at some point by means of sequences. It basically says that any sequence in the domain convergent to the point must be transformed by the function in a convergent sequence too. The limit of the latter is then the limit of the function at that point.

S2. Complete characterization of the limit by sequences. *If (X, d_X) and (Y, d_Y) are metric spaces, $f \in Y^X$, x is an accumulation point of $f^{-1}(Y)$, and $y \in Y$, then $y = \lim_x f$ if, and only if, for all convergent $s \in (f^{-1}(Y) \setminus \{x\})^{\mathbb{N}}$ such that $x = \lim s$, $f \circ s$ is convergent and such that $y = \lim f \circ s$.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $f \in Y^X$, x be an accumulation point of $f^{-1}(Y)$, and $y \in Y$.

Assume that $y = \lim_x f$. Let $\varepsilon > 0$. Since $y = \lim_x f$, then x is an accumulation point of $f^{-1}(Y)$. Since x is an accumulation point of $f^{-1}(Y)$, then there exists $s \in (f^{-1}(Y) \setminus \{x\})^{\mathbb{N}}$ convergent and such that $x = \lim s$. Since $y = \lim_x f$, then there exists $\delta > 0$ such that, for all $x' \in f^{-1}(Y)$ such that $d_X(x', x) < \delta$, $d_Y(f(x'), y) < \varepsilon$. Since $x = \lim s$, then there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $d_X(s(n), x) < \delta$. Since, for all $n' > n$, $s(n) \in f^{-1}(Y)$ and $d_X(s(n), x) < \delta$, and, for all $x' \in f^{-1}(Y)$ such that $d_X(x', x) < \delta$, $d_Y(f(x'), y) < \varepsilon$, then, for all $n' > n$, $d_Y(f(s(n')), y) < \varepsilon$. Since, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $d_Y((f \circ s)(n'), y) < \varepsilon$, then $y = \lim f \circ s$.

Conversely, assume that $y \neq \lim_x f$. Since $y \neq \lim_x f$, then

$$\exists \varepsilon > 0 \mid \forall \delta > 0, f(B_\delta(x) \setminus \{x\}) \not\subset B_\varepsilon(y).$$

Since, for all $\delta > 0$, $f(B_\delta(x) \setminus \{x\}) \not\subset B_\varepsilon(y)$, then, for all $n \in \mathbb{N}$, $f(B_{\frac{1}{n}}(x) \setminus \{x\}) \not\subset B_\varepsilon(y)$. Since, for all $n \in \mathbb{N}$, $f(B_{\frac{1}{n}}(x) \setminus \{x\}) \not\subset B_\varepsilon(y)$, then, for all $n \in \mathbb{N}$, there exists $x_n \in B_{\frac{1}{n}}(x) \setminus \{x\} \cap f^{-1}(Y)$ such that $f(x_n) \in B_\varepsilon(y)^C$. Let $s \in (f^{-1}(Y) \setminus \{x\})^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s(n) = x_n$.

Let $r > 0$. Since, for all $n' > \frac{1}{r}$, $B_{\frac{1}{n'}}(x) \subset B_r(x)$, and, for all $n' \in \mathbb{N}$, $x_{n'} \in B_{\frac{1}{n'}}(x)$, then for all $n' > \frac{1}{r}$, $x_{n'} \in B_r(x)$. Since, for all $r > 0$, there exists $n = [\frac{1}{r}] \in \mathbb{N}$ such that, for all $n' > n$, $s(n') = x_{n'} \in B_r(x)$, then $x = \lim s$.

Since, for all $n \in \mathbb{N}$, $(f \circ s)(n) = f(x_n) \in B_\varepsilon(y)^C$, then $(f \circ s)^{-1}(B_\varepsilon(y)^C)$ is not finite. Since there exists $\varepsilon > 0$ such that $(f \circ s)^{-1}(B_\varepsilon(y)^C)$ is not finite, then $y \neq \lim f \circ s$. Q.E.D.

On continuity of functions between metric spaces.

A function between two metric spaces is said to be continuous at some point of its domain if any ball containing the value of the function at that point includes all the values taken by the function at points of the domain within some distance of the given point. As a consequence of the definition, any function is automatically continuous at any isolated point of its domain, as the next proposition shows.

S3. Every function is continuous at every isolated point of its domain. *If (X, d_X) and (Y, d_Y) are metric spaces, $f \in Y^X$, and x is an isolated point of $f^{-1}(Y)$, then f is continuous at x.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $f \in Y^X$, and x be an isolated point of $f^{-1}(Y)$.

Let $\varepsilon > 0$. Since x is an isolated point of $f^{-1}(Y)$, then there exists $\delta > 0$ such that $B_\delta(x) \cap f^{-1}(Y) = \{x\}$. Since $B_\delta(x) \cap f^{-1}(Y) = \{x\}$, then $f(B_\delta(x)) = f(\{x\}) = \{f(x)\}$, Since $f(B_\delta(x)) = \{f(x)\}$ and $\{f(x)\} \subset B_\varepsilon(f(x))$, then $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Since, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, then f is continuous at x. Q.E.D.

A function is continuous at a point then if either the point is isolated or, if it is not, the function has a limit at that point that coincides with the value taken by the function there.

S4. Complete characterization of continuity by limits. *If (X, d_X) and (Y, d_Y) are metric spaces, $f \in Y^X$, and $x \in f^{-1}(Y)$, then f is continuous at x if, and only if, either x is an isolated point of $f^{-1}(Y)$, or x is an accumulation point of $f^{-1}(Y)$ and $f(x) = \lim_x f$.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $f \in Y^X$, and $x \in f^{-1}(Y)$.

Assume that f is continuous at x. Since $x \in f^{-1}(Y)$, then x is a closure point of $f^{-1}(Y)$. Since x is a closure point of $f^{-1}(Y)$, then either x is an isolated point of $f^{-1}(Y)$, or x is an accumulation point of $f^{-1}(Y)$.

(1) Assume that x is an isolated point of $f^{-1}(Y)$. Since x is an isolated point of $f^{-1}(Y)$, then, trivially, either x is an isolated point of $f^{-1}(Y)$, or x is an accumulation point of $f^{-1}(Y)$ and $f(x) = \lim_x f$.

(2) Assume that x is an accumulation point of $f^{-1}(Y)$. Since f is continuous at x, then

$$\forall \varepsilon > 0, \exists \delta > 0 \mid f(B_\delta(x)) \subset B_\varepsilon(f(x)).$$

Since, for all $\delta > 0$, $B_\delta(x) \backslash \{x\} \subset B_\delta(x)$, then, for all $\delta > 0$, $f(B_\delta(x) \backslash \{x\}) \subset f(B_\delta(x))$. Since, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, and $f(B_\delta(x) \setminus \{x\}) \subset f(B_\delta(x))$, then

$$\forall \varepsilon > 0, \exists \delta > 0 \mid f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(f(x)).$$

Since x is an accumulation point of $f^{-1}(Y)$, and for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(f(x))$, then, $f(x) = \lim_x f$. Since x is an accumulation point of $f^{-1}(Y)$ and $f(x) = \lim_x f$, then, trivially, either x is an isolated point of $f^{-1}(Y)$, or x is an accumulation point of $f^{-1}(Y)$ and $f(x) = \lim_x f$.

Conversely, assume that either x is an isolated point of $f^{-1}(Y)$, or x is an accumulation point of $f^{-1}(Y)$ and $f(x) = \lim_x f$.

(1) If x is an isolated point of the domain of f, then f is continuous at x.

(2) If x is an accumulation point of $f^{-1}(Y)$ and $f(x) = \lim_x f$, then

$$\forall \varepsilon > 0, \exists \delta > 0 \mid f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(f(x)).$$

Since, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(f(x))$, and $f(x) \in B_\varepsilon(f(x))$, then

$$\forall \varepsilon > 0, \exists \delta > 0 \mid f(B_\delta(x)) \subset B_\varepsilon(f(x)).$$

Since, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, then f is continuous at x.

Q.E.D.

The next proposition follows naturally from the definition of continuity and the characterization of the limit of a function by means of sequences.

S5. A function is continuous at a point iff the sequence of images of the terms of any sequence converging to the point converges itself to the image of the point. *If (X, d_X) and (Y, d_Y) are metric spaces, $f \in Y^X$, and $x \in f^{-1}(Y)$, then f is continuous at x if, and only if, for all convergent $s \in (f^{-1}(Y))^{\mathbb{N}}$ such that $x = \lim s$, $f \circ s$ is convergent and $f(x) = \lim f \circ s$.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $f \in Y^X$, and $x \in f^{-1}(Y)$.

Assume that f is continuous at x. Let $s \in (f^{-1}(Y))^{\mathbb{N}}$ be convergent and such that $x = \lim s$. Since f is continuous at x, then either x is an isolated point of $f^{-1}(Y)$, or x is an accumulation point of $f^{-1}(Y)$ and $f(x) = \lim_x f$.

(1) If x is an isolated point of $f^{-1}(Y)$, then there exists $\delta > 0$ such that $B_\delta(x) \cap f^{-1}(Y) = \{x\}$. Since $x = \lim s$, then $s^{-1}(B_\delta(x)^C)$ is finite. Let $n = \max s^{-1}(B_\delta(x)^C)$. Since $n = \max s^{-1}(B_\delta(x)^C)$, then, for all $n' > n$, $s(n') \in B_\delta(x) \cap f^{-1}(Y)$. Since, for all $n' > n$, $s(n') \in B_\delta(x) \cap f^{-1}(Y)$, and $B_\delta(x) \cap f^{-1}(Y) = \{x\}$, then, for all $n' > n$, $s(n') = x$. Since, for all $n' > n$, $s(n') = x$, then, for all $n' > n$, $(f \circ s)(n') = f(x)$. Since for all $n' > n$, $(f \circ s)(n') = f(x)$, then, for all $\varepsilon > 0$, $\max(f \circ s)^{-1}(B_\varepsilon(f(x))^C) \leq n$. Since, for all $\varepsilon > 0$, $(f \circ s)^{-1}(B_\varepsilon(f(x))^C)$ is finite, then $f \circ s$ is convergent and $f(x) = \lim f \circ s$.

(2) If x is an accumulation point of $f^{-1}(Y)$ and $f(x) = \lim_x f$, then

$$\forall \varepsilon > 0, \exists \delta > 0 \mid f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(f(x)).$$

Since, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x) \setminus \{x\}) \subset B_\varepsilon(f(x))$, and $f(x) \in B_\varepsilon(f(x))$, then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Let $\varepsilon > 0$ and $\delta > 0$ be such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Since $x = \lim s$, then there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $s(n') \in B_\delta(x)$. Since, for all $n' > n$, $s(n') \in B_\delta(x)$, then, for all $n' > n$, $f(s(n')) \in f(B_\delta(x))$. Since for all $n' > n$, $(f \circ s)(n') \in f(B_\delta(x))$, and $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, then, for all $n' > n$, $(f \circ s)(n') \in B_\varepsilon(f(x))$. Since for all $n' > n$, $(f \circ s)(n') \in B_\varepsilon(f(x))$, then $(f \circ s)^{-1}(B_\varepsilon(f(x))^C)$ is finite. Since, for all $\varepsilon > 0$, $(f \circ s)^{-1}(B_\varepsilon(f(x))^C)$ is finite, then $f \circ s$ is convergent and $x = \lim f \circ s$.

Conversely, assume that for all convergent $s \in (f^{-1}(Y))^{\mathbb{N}}$ such that $x = \lim s$, $f \circ s$ is convergent and $f(x) = \lim f \circ s$.

Since $x \in f^{-1}(Y)$, then x is a closure point of $f^{-1}(Y)$. Since x is a closure point of $f^{-1}(Y)$, then either x is an isolated point of $f^{-1}(Y)$, or x is an accumulation point of $f^{-1}(Y)$.

(1) If x is an isolated point of $f^{-1}(Y)$, then f is continuous at x.

(2) If x is an accumulation point of $f^{-1}(Y)$, assume that f is not continuous at x. Since f is not continuous at x, then

$$\exists \varepsilon > 0 \mid \forall \delta > 0, f(B_\delta(x)) \not\subset B_\varepsilon(f(x)).$$

Since for all $\delta > 0$, $f(B_\delta(x)) \not\subset B_\varepsilon(f(x))$, then, for all $n \in \mathbb{N}$, $f(B_{\frac{1}{n}}(x)) \not\subset B_\varepsilon(f(x))$. Since, for all $n \in \mathbb{N}$, $f(B_{\frac{1}{n}}(x)) \not\subset B_\varepsilon(f(x))$, then, for all $n \in \mathbb{N}$, there exists $x_n \in B_{\frac{1}{n}}(x) \cap f^{-1}(Y)$ such that $f(x_n) \notin B_\varepsilon(f(x))$. Let $s \in (f^{-1}(Y))^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s(n) = x_n$. Since, for all $\delta > 0$, there exists $[\frac{1}{\delta}] \in \mathbb{N}$ such that, for all $n > [\frac{1}{\delta}]$, $\delta > \frac{1}{n}$ and $s(n) \in B_{\frac{1}{n}}(x)$, then, for all $\delta > 0$ and all $n > [\frac{1}{\delta}]$, $s(n) \in B_\delta(x)$. Since, for all $\delta > 0$, there exists $[\frac{1}{\delta}] \in \mathbb{N}$ such that, for all $n > [\frac{1}{\delta}]$, $s(n) \in B_\delta(x)$, then $x = \lim s$.

Since, for all $n \in \mathbb{N}$, $f(s(n)) \notin B_\varepsilon(f(x))$, then $f(x) \neq \lim f \circ s$.

Since, if f is not continuous at x, then there exists $s \in (f^{-1}(Y))^{\mathbb{N}}$ convergent and such that $x = \lim s$ and $f(x) \neq \lim f \circ s$, then, if, for all convergent $s \in (f^{-1}(Y))^{\mathbb{N}}$ such that $x = \lim s$, $f \circ s$ is convergent and $f(x) = \lim f \circ s$, then f is continuous at x.

Q.E.D.

A function is said to be continuous if it is so at every point of its domain. Next we have a characterization of continuous function by means of closed sets. It is followed by a characterization by open sets that is obtained by complementarity.

S6. A function is continuous iff the inverse image of any closed set is closed. *If (X, d_X) and (Y, d_Y) are metric spaces, and $f \in Y^X$ is such that $f^{-1}(Y) = X$, then f is continuous if, and only if, for all $B \subset Y$ closed, $f^{-1}(B)$ is closed.*

Proof. **As an exercise, to make sure you understand it, you can translate the unpalatable proof that follows into "common parlance" as I did with the its first (the *only if*) part in class. Enjoy!! ;-)**

Let (X, d_X) and (Y, d_Y) be metric spaces, and $f \in Y^X$ be such that $f^{-1}(Y) = X$.

Assume that f is continuous and let B be closed. Either $B \cap f(X) = \phi$, or $B \cap f(X) \neq \phi$.

(1) If $B \cap f(X) = \phi$, then $f^{-1}(B) = \phi$, and hence $f^{-1}(B)$ is closed.

(2) If $B \cap f(X) \neq \phi$, then $f^{-1}(B) \neq \phi$. Either $f^{-1}(B)$ has no accumulation point, or $f^{-1}(B)$ has an accumulation point.

i) If $f^{-1}(B)$ has no accumulation point, then $f^{-1}(B)$ contains all its accumulation points, and hence $f^{-1}(B)$ is closed.

ii) If $f^{-1}(B)$ has an accumulation point, let $x \in X$ be an accumulation point of $f^{-1}(B)$. Since $x \in X$ and $X = f^{-1}(Y)$, then $x \in f^{-1}(Y)$. Since f is continuous and $x \in f^{-1}(Y)$, then f is continuous at x. Since x is an accumulation point of $f^{-1}(B)$, then there exists a convergent $s \in (f^{-1}(B) \setminus \{x\})^{\mathbb{N}}$ such that $x = \lim s$. Since $s(\mathbb{N}) \subset f^{-1}(B) \setminus \{x\}$ and

$$\begin{aligned} f^{-1}(B) \setminus \{x\} \subset \\ f^{-1}(B) &\subset X \\ &= f^{-1}(Y), \end{aligned}$$

then $s(\mathbb{N}) \subset f^{-1}(B)$ and $s(\mathbb{N}) \subset f^{-1}(Y)$. Since $s(\mathbb{N}) \subset f^{-1}(Y)$, then $s \in (f^{-1}(Y))^{\mathbb{N}}$. Since f is continuous at x, $s \in (f^{-1}(Y))^{\mathbb{N}}$ is convergent, and $x = \lim s$, then $f(x) = \lim f \circ s$. Either $f(x) \in (f \circ s)(\mathbb{N})$, or $f(x) \notin (f \circ s)(\mathbb{N})$.

ii.a) Assume $f(x) \in (f \circ s)(\mathbb{N})$. Since $s(\mathbb{N}) \subset f^{-1}(B) \setminus \{x\}$ and $f^{-1}(B) \setminus \{x\} \subset f^{-1}(B)$, then $s(\mathbb{N}) \subset f^{-1}(B)$. Since $s(\mathbb{N}) \subset f^{-1}(B)$, then $f(s(\mathbb{N})) \subset B$, i.e. $(f \circ s)(\mathbb{N}) \subset B$. Since $f(x) \in (f \circ s)(\mathbb{N})$ and $(f \circ s)(\mathbb{N}) \subset B$, then $f(x) \in B$. Since $f(x) \in B$, then $x \in f^{-1}(B)$.

ii.b) Assume $f(x) \notin (f \circ s)(\mathbb{N})$. Since $f(x) \notin (f \circ s)(\mathbb{N})$ and $f(x) = \lim f \circ s$, then $f(x)$ is an accumulation point of $(f \circ s)(\mathbb{N})$. Since $s(\mathbb{N}) \subset f^{-1}(B)$, then $f(s(\mathbb{N})) \subset B$, i.e. $(f \circ s)(\mathbb{N}) \subset B$. Since $f(x)$ is an accumulation point of $(f \circ s)(\mathbb{N})$ and $(f \circ s)(\mathbb{N}) \subset B$, then $f(x)$ is an accumulation point of B. Since B is closed and $f(x)$ is an accumulation point of B, then $f(x) \in B$. Since $f(x) \in B$, then $x \in f^{-1}(B)$.

Therefore, $f^{-1}(B)$ contains all its accumulation points, i.e. $f^{-1}(B)$ is closed.

Conversely, assume that f is not continuous. Since f is not continuous, then there exists $x \in f^{-1}(Y)$ such that f is not continuous at x. Since f is not continuous at x, then there exists a convergent $s \in (f^{-1}(Y))^{\mathbb{N}}$ such that $x = \lim s$ and $f(x) \neq \lim f \circ s$. Since $f(x) \neq \lim f \circ s$, then there exists $\varepsilon > 0$ such that $(f \circ s)^{-1}(B_\varepsilon(f(x))^C)$ is not finite. Let $h \in \mathbb{N}^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $h(n)$ is the n-th smallest integer in $(f \circ s)^{-1}(B_\varepsilon(f(x))^C)$. Then h is increasing. Thus, although $\mathrm{Cl}((f \circ s \circ h)(\mathbb{N}))$ is trivially a closed set of (Y, d_Y), $f^{-1}(\mathrm{Cl}((f \circ s \circ h)(\mathbb{N})))$ is not a closed set of (X, d_X), since:

(1) x is an accumulation point of $f^{-1}(\mathrm{Cl}(f \circ s \circ h)(\mathbb{N}))$.

In effect, since $x = \lim s$, then $x = \lim s \circ h$. Moreover $x \notin (s \circ h)(\mathbb{N})$.

(Assume that $x \in (s \circ h)(\mathbb{N})$. Since $x \in (s \circ h)(\mathbb{N})$ then there exists $n \in \mathbb{N}$ such that $x = (s \circ h)(n)$. Since $x = s(h(n))$, then $f(x) = f(s(h(n)))$, i.e. $f(x) = (f \circ s)(h(n))$. Since $h(n) \in (f \circ s)^{-1}(B_\varepsilon(f(x))^C)$, then $(f \circ s)(h(n)) \in B_\varepsilon(f(x))^C$. Since $f(x) = (f \circ s)(h(n))$ and $(f \circ s)(h(n)) \in B_\varepsilon(f(x))^C$, then $f(x) \in B_\varepsilon(f(x))^C$.)

Since $x = \lim s \circ h$ and $x \notin (s \circ h)(\mathbb{N})$, then x is an accumulation point of $(s \circ h)(\mathbb{N})$. Moreover $(s \circ h)(\mathbb{N}) \subset f^{-1}(\mathrm{Cl}(f \circ s \circ h)(\mathbb{N}))$.

(Let $x' \in (s \circ h)(\mathbb{N})$. Since $x' \in (s \circ h)(\mathbb{N})$, then there exists $n \in \mathbb{N}$ such that $x' = (s \circ h)(n)$. Since $x' = (s \circ h)(n)$, then $f(x') = f((s \circ h)(n))$. Since $f(x') = f((s \circ h)(n))$, then $f(x') \in (f \circ s \circ h)(\mathbb{N})$. Since $f(x') \in (f \circ s \circ h)(\mathbb{N})$ and $(f \circ s \circ h)(\mathbb{N}) \subset \mathrm{Cl}((f \circ s \circ h)(\mathbb{N}))$, then $f(x') \in \mathrm{Cl}((f \circ s \circ h)(\mathbb{N}))$. Since $f(x') \in \mathrm{Cl}((f \circ s \circ h)(\mathbb{N}))$, then $x' \in f^{-1}(\mathrm{Cl}((f \circ s \circ h)(\mathbb{N})))$.)

Since x is an accumulation point of $(s \circ h)(\mathbb{N})$ and $(s \circ h)(\mathbb{N}) \subset f^{-1}(\mathrm{Cl}(f \circ s \circ h)(\mathbb{N}))$, then x is an accumulation point of $f^{-1}(\mathrm{Cl}(f \circ s \circ h)(\mathbb{N}))$.

(2) x is not in $f^{-1}(\mathrm{Cl}(f \circ s \circ h)(\mathbb{N}))$.

In effect, assume that $x \in f^{-1}(\mathrm{Cl}(f \circ s \circ h)(\mathbb{N}))$. Since $x \in f^{-1}(\mathrm{Cl}(f \circ s \circ h)(\mathbb{N}))$, then $f(x) \in \mathrm{Cl}(f \circ s \circ h)(\mathbb{N})$. Since $f(x) \in \mathrm{Cl}(f \circ s \circ h)(\mathbb{N})$, then either $f(x) \in (f \circ s \circ h)(\mathbb{N})$ or $f(x)$ is an accumulation point of $(f \circ s \circ h)(\mathbb{N})$. Nevertheless,

i) $f(x)$ is not an accumulation point of $(f \circ s \circ h)(\mathbb{N})$.

(In effect, since, for all $n \in \mathbb{N}$, $h(n) \in (f \circ s)^{-1}(B_\varepsilon(f(x))^C)$, then $(f \circ s \circ h)(\mathbb{N}) \subset B_\varepsilon(f(x))^C$. Since $(f \circ s \circ h)(\mathbb{N}) \subset B_\varepsilon(f(x))^C$, then $(f \circ s \circ h)(\mathbb{N}) \cap B_\varepsilon(f(x)) = \phi$. Since there exists $\varepsilon > 0$ such that $(f \circ s \circ h)(\mathbb{N}) \cap B_\varepsilon(f(x)) = \phi$, then $f(x)$ is not an accumulation point of $(f \circ s \circ h)(\mathbb{N})$.)

ii) $f(x) \notin (f \circ s \circ h)(\mathbb{N})$.

(Assume that $f(x) \in (f \circ s \circ h)(\mathbb{N})$. Since $f(x) \in (f \circ s \circ h)(\mathbb{N})$, then there exists $n \in \mathbb{N}$ such that $f(x) = (f \circ s \circ h)(n)$. Since $h(n) \in (f \circ s)^{-1}(B_\varepsilon(f(x)^C))$, then $(f \circ s \circ h)(n) \in B_\varepsilon(f(x)^C)$. Since $f(x) = (f \circ s \circ h)(n)$ and $(f \circ s \circ h)(n) \in B_\varepsilon(f(x)^C)$, then $f(x) \in B_\varepsilon(f(x))^C$!)

Q.E.D.

S7. A function is continuous iff the inverse image of any open set is open. *If (X, d_X) and (Y, d_Y) are metric spaces, and $f \in Y^X$ is such that $f^{-1}(Y) = X$, then f is continuous if, and only if, for all $B \subset Y$ open set of (Y, d_Y), $f^{-1}(B)$ is an open set of (X, d_X).*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f \in Y^X$ be such that $f^{-1}(Y) = X$.

Assume that f is continuous and let $B \subset Y$ be an open set of (Y, d_Y). Since B is an open set of (Y, d_Y), then B^C is a closed set of (Y, d_Y). Since B^C is a closed set of (Y, d_Y) and f is continuous, then $f^{-1}(B^C)$ is a closed set of (X, d_X). Since $f^{-1}(B^C)$ is a closed set of (X, d_X), then $(f^{-1}(B^C))^C$ is an open set of (X, d_X). Since,

(1) if $x \in (f^{-1}(B^C))^C$, i.e. $x \notin f^{-1}(B^C)$, then for all $y \in B^C$, $y \neq f(x)$, and hence $f(x) \in B$ i.e. $x \in f^{-1}(B)$, and
(2) if $x \in f^{-1}(B)$, then $f(x) \in B$, and hence, for all $y \in B^C$, $y \neq f(x)$, i.e. $x \notin f^{-1}(B^C)$, and thus $x \in (f^{-1}(B^C))^C$,

then $(f^{-1}(B^C))^C = f^{-1}(B)$. Since $(f^{-1}(B^C))^C = f^{-1}(B)$ and $(f^{-1}(B^C))^C$ is an open set of (X, d_X), then $f^{-1}(B)$ is an open set of (X, d_X). Therefore, for all $B \subset Y$ open set of (Y, d_Y), $f^{-1}(B)$ is an open set of (X, d_X).

Conversely, assume that for all $B' \subset Y$ open set of (Y, d_Y), $f^{-1}(B')$ is an open set of (X, d_X), and let B be a closed set of (Y, d_Y). Since B is a closed set of (Y, d_Y), then B^C is an open set of (Y, d_Y). Since B^C is an open set of (Y, d_Y) and, for all $B' \subset Y$ open set of (Y, d_Y), $f^{-1}(B')$ is an open set of (X, d_X), then $f^{-1}(B^C)$ is an open set of (X, d_X). Since $f^{-1}(B^C)$ is an open set of (X, d_X), then $(f^{-1}(B^C))^C$ is a closed set of (X, d_X). Since

(1) if $x \in (f^{-1}(B^C))^C$, then $x \notin f^{-1}(B^C)$, i.e. for all $y \in B^C$, $y \neq f(x)$, and hence $f(x) \in B$, i.e. $x \in f^{-1}(B)$.
(2) if $x \in f^{-1}(B)$, then $f(x) \in B$, and hence, for all $y \in B^C$, $y \neq f(x)$, i.e. $x \notin f^{-1}(B^C)$, and thus $x \in (f^{-1}(B^C))^C$,

then $(f^{-1}(B^C))^C = f^{-1}(B)$. Since $(f^{-1}(B^C))^C = f^{-1}(B)$ and $(f^{-1}(B^C))^C$ is a closed set of (X, d_X), then $f^{-1}(B)$ is a closed set of (X, d_X). Since, for all B closed set of (Y, d_Y), $f^{-1}(B)$ is a closed set of (X, d_X), then f is continuous. Q.E.D.

S8. Continuity of the restriction of a continuous function to a subspace. *If (X, d_X) and (Y, d_Y) are metric spaces, $f \in Y^X$ is continuous and $X' \subset X$, then $f|_{X'}$ is continuous on the metric subspace $(X', d|_{X'})$.*

Proof. Exercise.

The next proposition establishes that the composition of continuous functions delivers continuous function or, equivalently, that continuity is preserved by composition.

S9. The composition of continuous functions is a continuous function. *If (X, d_X), (Y, d_Y) and (Z, d_Z) are metric spaces, $f \in Y^X$ is continuous and $g \in Z^Y$ is continuous, then $g \circ f$ is continuous.*

Proof. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces, $f \in Y^X$ be continuous and $g \in Z^Y$ be continuous.

Let C be a closed set of (Z, d_Z). Since C is a closed set of (Z, d_Z) and g is continuous, then $g^{-1}(C)$ is a closed set of (Y, d_Y). Since $g^{-1}(C)$ is a closed set of (Y, d_Y) and f is continuous, then $f^{-1}(g^{-1}(C))$ is a closed set of (X, d_X). Since $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ and $f^{-1}(g^{-1}(C))$ is a closed set of (X, d_X), then $(g \circ f)^{-1}(C)$ is a closed set of (X, d_X) and therefore $g \circ f$ is continuous. Q.E.D.

On continuous functions.

Continuous functions transform compact sets into compact sets. This is at the foundation of the result establishing that real-valued continuous functions defined on compact sets attain a maximum value as well as a minimum one.

S1. Every continuous function transforms compacts into compacts. *If (X, d_X) to (Y, d_Y) are metric spaces, $f \in Y^X$ is continuous and such that $f^{-1}(Y)$ is a closed set of (X, d_X), and A is a compact set of (X, d_X), then $f(A)$ is a compact set of (Y, d_Y).*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $f \in Y^X$ be continuous and such that $f^{-1}(Y)$ is a closed set of (X, d_X), A be a compact set of (X, d_X), and $s \in (f(A)^{\mathbb{N}})$.

Since $s \in (f(A)^{\mathbb{N}})$, then, for all $n \in \mathbb{N}$, $s(n) \in f(A)$. Since for all $n \in \mathbb{N}$, $s(n) \in f(A)$, then for all $n \in \mathbb{N}$, there exists $x_n \in A \cap f^{-1}(Y)$ such that $s(n) = f(x_n)$. Let $s' \in (A \cap f^{-1}(Y))^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s'(n) = x_n$. Since $s'(\mathbb{N}) \subset A \cap f^{-1}(Y)$, then $s'(\mathbb{N}) \subset A$. Since $s'(\mathbb{N}) \subset A$ and A is a compact of (X, d_X), then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and such that $s' \circ h$ is convergent and $\lim s' \circ h \in A$. Either $\lim s' \circ h$ is an isolated point of $(s' \circ h)(\mathbb{N})$, or $\lim s' \circ h$ is an accumulation point of $(s' \circ h)(\mathbb{N})$.

(1) If $\lim s' \circ h$ is an isolated point of $(s' \circ h)(\mathbb{N})$, then $\lim s' \circ h \in (s' \circ h)(\mathbb{N})$. Since $\lim s' \circ h \in (s' \circ h)(\mathbb{N})$, $(s' \circ h)(\mathbb{N}) \subset s'(\mathbb{N})$, and $s'(\mathbb{N}) \subset A \cap f^{-1}(Y)$, then $\lim s' \circ h \in A \cap f^{-1}(Y)$.

(2) If $\lim s' \circ h$ is an accumulation point of $(s' \circ h)(\mathbb{N})$, since $(s' \circ h)(\mathbb{N}) \subset s'(\mathbb{N})$, and $s'(\mathbb{N}) \subset A \cap f^{-1}(Y)$, then $\lim s' \circ h$ is an accumulation point of $f^{-1}(Y)$, and since $f^{-1}(Y)$ is closed, then $\lim s' \circ h \in A \cap f^{-1}(Y)$.

Since $\lim s' \circ h \in A \cap f^{-1}(Y)$, then $\lim s' \circ h \in f^{-1}(Y)$. Since f is continuous, then f is continuous at $\lim s' \circ h$. Since f is continuous at $\lim s' \circ h$, then $f \circ s' \circ h$ is convergent and $f(\lim s' \circ h) = \lim f \circ s' \circ h$. Since $\lim s' \circ h \in A \cap f^{-1}(Y)$, then $\lim s' \circ h \in A$. Since $\lim s' \circ h \in A$, then $f(\lim s' \circ h) \in f(A)$. Since $f(\lim s' \circ h) = \lim f \circ s' \circ h$ and $f(\lim s' \circ h) \in f(A)$, then $\lim f \circ s' \circ h \in f(A)$. Since, for all $n \in \mathbb{N}$, $(f \circ s')(n) = s(n)$, then $f \circ s' = s$. Since $f \circ s' = s$, $f \circ s' \circ h$ is convergent and $\lim f \circ s' \circ h \in f(A)$, then $s \circ h$ is convergent and $\lim s \circ h \in f(A)$.

Therefore, every sequence $s \in (f(A))^{\mathbb{N}}$ has a convergent subsequence whose limit is in $f(A)$ too, i.e. $f(A)$ is compact. Q.E.D.

S2. Every continuous real-valued function on a compact domain has a maximum and a minimum. *If (X, d_X) is a metric space, and $f \in \mathbb{R}^X$ is continuous and such that $f^{-1}(Y)$ is compact, then there exist $\max f(X)$ and $\min f(X)$. (i.e. there is $x, x' \in f^{-1}(Y)$ such that for any $x'' \in f^{-1}(Y)$, $f(x') \le f(x'') \le f(x)$).*

Proof. Since $f^{-1}(Y)$ is compact and f is continuous, then $f(f^{-1}(Y)) = f(X)$ is a compact set of $(\mathbb{R}, d_2)$, i.e. a closed and bounded set.

Since $f(X)$ is bounded, its supremum y exists and since its supremum is an accumulation point of $f(X)$ (otherwise there would be an upper bound of $f(X)$ smaller than the smallest!) and this set is closed, then the supremum y of $f(X)$ is in $f(X)$, i.e. there is some $x \in f^{-1}(Y)$ such that $f(x) = y$. Since $f(x)$ is an upper bound of $f(X)$, then $\forall x' \in f^{-1}(Y)$, $f(x') \leq f(x)$. Q.E.D.

S3. Every continuous function is determined by its values at any dense subset of its domain. *If (X, d_X) and (Y, d_Y) are metric spaces, $f, g \in Y^X$ are continuous and such that $f^{-1}(Y) = g^{-1}(Y)$, $A \subset X$ is dense in $f^{-1}(Y) = g^{-1}(Y)$ and such that, for all $x \in A$, $f(x) = g(x)$, then $f = g$.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $f, g \in Y^X$ be continuous and such that $f^{-1}(Y) = g^{-1}(Y)$, $A \subset X$ be dense in $f^{-1}(Y) = g^{-1}(Y)$ and such that, for all $x \in A$, $f(x) = g(x)$.

Let $x \in f^{-1}(Y) = g^{-1}(Y)$. Since A is dense in $f^{-1}(Y) = g^{-1}(Y)$, then there exists a convergent sequence $s \in A^{\mathbb{N}}$ whose limit is x. Since f is continuous, then the sequence $f \circ s$ is convergent and its limit is $f(x)$. Since g is continuous, then the sequence $g \circ s$ is convergent and its limit is $g(x)$. Since $s(\mathbb{N}) \subset A$, then $f \circ s = g \circ s$. Therefore, for all $x \in f^{-1}(Y) = g^{-1}(Y)$, $f(x) = g(x)$, i.e. $f = g$. Q.E.D.

On uniformly continuous functions.

The notion of uniform continuity is stronger than that of continuity. To begin with, it only makes sense to talk about a function being uniformly continuous at its entire domain, not at any specific point of it. And the reason is that uniform continuity requires δ, that in general depends on both ε and x, to depend actually on ε only.

Being a more stringent notion, there are continuous functions that are not uniformly continuous, while uniform continuity implies continuity, as the next proposition shows.

S4. Every uniformly continuous function is a continuous. *If (X, d_X) and (Y, d_Y) are metric spaces, and $f \in Y^X$ is uniformly continuous, then f is continuous.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $f \in Y^X$ be uniformly continuous, and x be in $f^{-1}(Y)$.

Since f is uniformly continuous, then for all $s > 0$, there exists $r > 0$ such that, if $x' \in f^{-1}(Y)$ satisfies $d_X(x, x') < r$, then $d_Y(f(x), f(x')) < s$, i.e. $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Therefore, f is continuous at any $x \in f^{-1}(Y)$ and hence it is continuous. Q.E.D.

Next there is a sufficient condition for a function to be uniformly continuous, namely that the function does not stretch the distance between the points during the transformation.

S5. Sufficient condition for uniform continuity. *If (X, d_X) and (Y, d_Y) are metric spaces, $f \in Y^X$, and for all $x, x' \in f^{-1}(Y)$, $d_Y(f(x), f(x')) \leq d_X(x, x')$, then f is uniformly continuous.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, $f \in Y^X$ be such that

$$\forall x, x' \in f^{-1}(Y), d_Y(f(x), f(x')) \leq d_X(x, x').$$

For all $s > 0$, there exists a positive real number, the very s itself, such that if $x, x' \in f^{-1}(Y)$ satisfy $d_X(x, x') < s$, then $d_Y(f(x), f(x')) < s$, since $d_Y(f(x), f(x')) \leq d_X(x, x') < s$. Hence f is uniformly continuous Q.E.D.

On Lipschitz functions.

S6. Every Lipschitz function is uniformly continuous. *If (X, d_X) and (Y, d_Y) are metric spaces, and $f \in Y^X$ is Lipschitz, then f is uniformly continuous.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f \in Y^X$ not to be uniformly continuous.

Since $f \in Y^X$ is not uniformly continuous, then it is not true that

$$\forall \varepsilon > 0, \exists \delta > 0 \mid \forall x \in f^{-1}(Y), f(B_\delta(x)) \subset B_\varepsilon(f(x)),$$

that is to say,

$$\exists \varepsilon > 0 \mid \forall \delta > 0, \exists x \in f^{-1}(Y) \mid f(B_\delta(x)) \not\subset B_\varepsilon(f(x))$$

and hence there exists $x' \in B_\delta(x) \cap f^{-1}(Y)$ such that $f(x') \notin B_\varepsilon(f(x))$. Since $f(x') \notin B_\varepsilon(f(x))$, then $x' \neq x$ and $d_Y(f(x), f(x')) \geq \varepsilon$. Since $x' \in B_\delta(x)$ and $x' \neq x$, then $0 < d_X(x, x') < \delta$. Since $\varepsilon > 0$, $0 < d_X(x, x') < \delta$, and $0 < d_X(x, x') < \delta$, then

$$\begin{aligned} \frac{\varepsilon}{\delta} &< \frac{\varepsilon}{d_X(x, x')} \\ &\leq \frac{d_Y(f(x), f(x'))}{d_X(x, x')}. \end{aligned}$$

Since, for all $\alpha \in \mathbb{R}_{++}$, there exists $\delta > 0$ such that $\alpha = \frac{\varepsilon}{\delta}$, namely $\delta = \frac{\varepsilon}{\alpha}$, and for such δ there exists $x, x' \in f^{-1}(Y)$ distinct and such that

$$\alpha = \frac{\varepsilon}{\delta} < \frac{d_Y(f(x), f(x'))}{d_X(x, x')},$$

that is to say,

$$\forall \alpha \in \mathbb{R}_{++}, \exists x, x' \in f^{-1}(Y) \mid x \neq x' \wedge \alpha < \frac{d_Y(f(x), f(x'))}{d_X(x, x')},$$

moreover, clearly

$$\forall \alpha \in \{0\}, \exists x, x' \in f^{-1}(Y) \mid x \neq x' \wedge \alpha < \frac{d_Y(f(x), f(x'))}{d_X(x, x')},$$

and then it is not true that

$$\exists \alpha \in \mathbb{R}_+ \mid \forall x, x' \in f^{-1}(Y), x = x' \vee \frac{d_Y(f(x), f(x'))}{d_X(x, x')} \leq \alpha$$

i.e. f is not Lipschitz. Q.E.D.

S7. Every Lipschitz function is continuous. *If* (X, d_X) *and* (Y, d_Y) *are metric spaces, and* $f \in Y^X$ *is Lipschitz, then* f *is continuous.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f \in Y^X$ be Lipschitz.
Since f is Lipschitz, then f is uniformly continuous. Since f is uniformly continuous, then f is continuous. Q.E.D.

On contractions.

S8. Every contraction is Lipschitz. *If* (X, d_X) *is a metric space and* $f \in X^X$ *is a contraction, then* f *is a Lipschitz function.*

Proof. (Obvious from the definitions) Q.E.D.

S9. Every contraction is uniformly continuous. *If* (X, d_X) *is a metric space, and*$f \in X^X$ *is a contraction, then* f *is uniformly continuous.*

Proof. Let (X, d_X) be a metric space, and $f \in X^X$ be a contraction.
Since f is a contraction, then f is Lipschitz. Since f is Lipschitz, then f is uniformly continuous. Q.E.D.

S10. Every contraction is continuous. *If* (X, d_X) *is a metric space, and* $f \in X^X$ *is a contraction, then* f *is continuous.*

Proof. Let (X, d_X) be a metric space, and $f \in X^X$ be a contraction.
Since f is a contraction, then f is uniformly continuous. Since f is uniformly continuous, then f is continuous. Q.E.D.

The next proposition establishes that a contraction of a metric space cannot have more than one fixed point, if any. It thus establishes an upper bound to the number of fixed points that a contraction may have, but it says nothing about the existence of any such fixed point. In order to be have a result on the existence we will have to require the metric space to be complete, as it will be shown below.

S11. Uniqueness of the fixed point of a contraction of a space. *If* (X, d_X) *is a metric space,* $f \in X^X$ *is a contraction, and* $x, x' \in X$ *are such that* $x = f(x)$ *and* $x' = f(x')$*, then* $x = x'$*.*

Proof. Let (X, d_X) be a metric space, $f \in X^X$ be a contraction, and $x, x' \in X$ be such that $x = f(x)$ and $x' = f(x')$.
Assume that $x \neq x'$. Since $x, x' \in X$ are such that $x = f(x)$ and $x' = f(x')$, then $d_X(f(x), f(x')) = d_X(x, x')$. Since f is a contraction, then there exists $\alpha \in [0, 1)$ such that $d_X(f(x), f(x')) \leq \alpha d_X(x, x')$. Since $d_X(f(x), f(x')) = d_X(x, x')$ and $d_X(f(x), f(x')) \leq \alpha d_X(x, x')$, then $d_X(x, x') \leq \alpha d_X(x, x')$. Since $x \neq x'$, then $d_X(x, x') > 0$. Since $d_X(x, x') > 0$ and $d_X(x, x') \leq \alpha d_X(x, x')$, then $1 \leq \alpha$! Therefore, $x = x'$. Q.E.D.

S12. Banach Fixed point Theorem. Existence of a fixed point of a contraction of a complete metric space. *If* (X, d_X) *is a complete metric space and* $f \in X^X$ *is a contraction, then there exists* $x \in X$ *such that* $x = f(x)$*.*

Proof. Let (X, d_X) be a complete metric space, $f \in X^X$ be a contraction, and x_1 be any point in X. Either $f(x_1) = x_1$ or not.

(1) If $f(x_1) = x_1$, then x_1 is a fixed point of f.

(2) If $f(x_1) \neq x_1$, then let $\{x_n\}_{n\in\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $x_{n+1} = f(x_n)$. Since f is a contraction, then

$$\exists \alpha \in [0,1) \mid \forall x, x' \in X, d_X(f(x), f(x')) \leq \alpha d_X(x, x')$$

and hence, for all $n \in \mathbb{N}$,

$$\begin{aligned} d_X(x_{n+1}, x_n) &= \\ d_X(f(x_n), f(x_{n-1})) &\leq \alpha d_X(x_n, x_{n-1}) \\ &= \alpha d_X(f(x_{n-1}), f(x_{n-2})) \\ &\leq \alpha^2 d_X(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \alpha^{n-1} d_X(f(x_1), x_1). \end{aligned}$$

Therefore, for all $m, n \in \mathbb{N}$, if $m < n$,

$$\begin{aligned} d_X(x_n, x_m) &\leq d_X(x_n, x_{n-1}) + d_X(x_{n-1}, x_{n-2}) + \cdots + d_X(x_{m+1}, x_m) \\ &\leq d_X(f(x_1), x_1)(\alpha^{n-2} + \alpha^{n-3} + \cdots + \alpha^{m-1}) \\ &= d_X(f(x_1), x_1)\alpha^{m-1}(\alpha^{n-m-1} + \alpha^{n-m-2} + \cdots + \alpha + 1) \\ &= d_X(f(x_1), x_1)\alpha^{m-1}\frac{1-\alpha^{n-m}}{1-\alpha} \\ &\leq d_X(f(x_1), x_1)\alpha^{m-1}\frac{1}{1-\alpha} \end{aligned}$$

and if $m = n$, then $d_X(x_n, x_m) = 0 \leq d_X(f(x_1), x_1)\alpha^{m-1}\frac{1}{1-\alpha}$ as well. Since,

$$d_X(f(x_1), x_1)\alpha^{m-1}\frac{1}{1-\alpha} < \varepsilon$$

if

$$m > 1 + \frac{\ln\varepsilon + \ln(1-\alpha) - \ln d_X(f(x_1), x_1)}{\ln\alpha},$$

then there exists $N \in \mathbb{N}$, namely $N = [1 + \frac{\ln\varepsilon + \ln(1-\alpha) - \ln d_X(f(x_1), x_1)}{\ln\alpha}]$, such that, for all $m, n \in \mathbb{N}$ such that $N < m \leq n$,

$$\begin{aligned} d_X(x_n, x_m) &\leq d_X(f(x_1), x_1)\alpha^{m-1}\frac{1}{1-\alpha} \\ &< \varepsilon, \end{aligned}$$

i.e. $\{x_n\}$ is Cauchy.

Since (X, d_X) is a complete metric space and $\{x_n\}$ is Cauchy, then $\{x_n\}$ is convergent. Let $x = \lim x_n$. Since f is a contraction, then f is continuous. Since $x = \lim x_n$ and f is continuous and hence continuous at x, then $f(x) = \lim f(x_n)$. Since, for all $n \in \mathbb{N}$, $f(x_n) = x_{n+1}$, then $\{f(x_n)\}$ is a subsequence of $\{x_n\}$. Therefore, since $f(x)$ is the limit of a subsequence of $\{x_n\}$, and $\{x_n\}$ is Cauchy, then $f(x)$ is the limit of $\{x_n\}$. Since $x = \lim x_n$ and $f(x) = \lim x_n$ then $f(x) = x$.

Q.E.D.

S13. Sufficient conditions for the pointwise convergence of a sequence of contractions and their fixed points to a contraction and its fixed point. *If (X, d_X) is a complete metric space and for all $n \in \mathbb{N}$, $f_n \in X^X$ is a contraction of modulus $\alpha_n \leq \alpha$ with $\alpha < 1$, $x_n \in X$ being its unique fixed point, and $f \in X^X$ is such that $f(x) = \lim_{n\in\mathbb{N}} f_n(x)$ for all $x \in X$ then the sequence of fixed points $\{x_n\}_{n\in\mathbb{N}}$ is convergent and f is a contraction whose unique fixed point is $\lim_{n\in\mathbb{N}} x_n$.*

Proof. Exercise.

S14. Iterated contraction mapping theorem. *If (X, d_X) is a complete metric space, $f \in X^X$ and there exists $n \in \mathbb{N}$ such that f^n is a contraction, then there exists a unique fixed point of f.*

Proof. Since f^n is a contraction, then there exists $x \in X$ such that $f^n(x) = x$. Then

$$\begin{aligned} f(x) &= \\ f(f^n(x)) &= f^{n+1}(x) \\ &= f^n(f(x)). \end{aligned}$$

Hence $f(x)$ is a fixed point of f^n as well. Since the fixed point of f^n is unique, then $f(x) = x$.

Assume $x' \in X$ is such that $f(x') = x'$ as well. Then

$$\begin{aligned} f^n(x') &= f^{n-1}(f(x')) \\ &= f^{n-1}(x') \\ &= f^{n-2}(f(x')) \\ &= f^{n-2}(x') \\ &\vdots \\ &= f(x') \\ &= x', \end{aligned}$$

hence x' is a fixed point of f^n too. Since the fixed point of f^n is unique, then $x' = x$, and therefore x is the unique fixed point of f as well. Q.E.D.