*Lecture Notes for Econ 898*

# METRIC SPACES

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The defining characteristic of the mathematical object we call a set is embodied in the concept of membership. Every set partitions the universe of all existing things into two groups: those things that are members (i.e. elements) of the set and those things that are not. In particular, it is important to recognize that sets do not intrinsically involve any notion of distance between (i.e. nearness or farness of) their elements. Such a notion requires us to provide additional mathematical structure. In short, we must endow the set with something analogous to the concept of spatial distance, something with which we have presumably become quite familiar. In this context, and because of the analogy, a set that is endowed with such a notion of distance is often referred to as a **space**, and each of its elements is referred to as a **point** in that space.

The additional structure required to define a distance for a set (thus making it into a space) is the concept of a **metric**. A metric is a real-valued function with the following three properties: (1) it assigns a real value (distance) to every pairwise combination of elements in the set, (2) it assigns a zero to a pair of elements if and only if the elements of the pair are identical (i.e. the distance from any point to itself is always zero),[1] and (3) the distance between any two points is bounded above by the sum of the distances between each of the two points and any other (third) point.[2] A set endowed with a metric is called a **metric space**.

Having provided more mathematical structure to the original notion of a set by endowing it with a metric, we are now able to discuss a host of additional concepts. For example, we are now able to give a more precise sense to the idea of closeness between points and sets of points. This idea is formalized by a notion of neighborhood of a given point, namely a **ball** centered at the point and with a specified radius. It consists of the points of the space whose distance from the given point is less than the specified radius. Another fundamental concept that is available with the added structure of a metric space, from which all the rest can be

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CAVEAT: These handouts are inevitably under permanent construction. I thank the feedback from many students of previous years, and in particular I thank Graham Loftin for the editing and comments.

[1] Allowing this to happen also for distinct points leads to the weaker notion of **pseudometric**.

[2] The typical definition of a metric also requires the function to be $\mathbb{R}_+$-valued and symmetric, i.e. such that $d(x, x') = d(x', x)$ always. As a matter of fact, these last two requirements are redundant since they follow from the three properties above.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

derived, is that of **closure point**. A closure point of a given set is any point that has a point of the set within an arbitrarily small distance.

A closure point of a set may be or not a closure point of other sets. In particular, a closure point of a set may also be a closure point of either of two other sets of particular interest: (1) the original set but excluding the closure point of interest and (2) the complement of the original set.

(1) Any closure point of the original set that has other points of the set (i.e. points distinct from itself) within an arbitrarily small distance is known as an **accumulation point**, while any closure point for which this does not hold is known as an **isolated point**. Note that the isolated points of a set necessarily belong to the set, while an accumulation point may or may not belong to it.

(2) Any closure point of the original set that is also a closure point of the complement of that set is known as **boundary point**, while any closure point that is not a closure point of the complement is known as an **interior point**. Also note that (1) while the interior points of a set necessarily belong to the set, boundary points may or may not belong to the set and (2) while every interior point is necessarily an accumulation point and every isolated point is necessarily a boundary point, the converse statements are not true in either case.

Consistent with these definitions, the **closure**, **boundary**, and **interior** of a set are defined as those subsets comprised of the closure, boundary, and interior points of the set respectively. Of less common usage is the term **derived set** which refers to the accumulation points of a set.[3]

One convenient way of characterizing the closure and accumulation points of a set is through the use of sequences. Since closure points are characterized by having points of the set within an arbitrarily small distance, they can be approached arbitrarily closely by a sequence of elements extracted from the set. With this purpose in mind, we are naturally led to be particularly interested in those sequences that do not wander erratically in the space.

An increasingly stringent classification of such sequences starts with **bounded sequences**, sequences whose terms lie within a ball or, equivalently, whose range is a bounded set. A more stringent requirement is that the terms of the sequence not only stay within a given distance of a given point (as in the case of bounded sequences), but that they wander around with consistently smaller and smaller steps, so to speak. Sequences with this property are known as **Cauchy sequences**. An even more stringent requirement than being "Cauchy" is that the terms of the sequence eventually approach a given point arbitrarily closely. Such a sequence is said to be a **convergent sequence** and the[4] point to which the sequence converges is called its **limit**. Note that not every bounded sequence is Cauchy and that, likewise, not every Cauchy sequence is convergent. Indeed, whether a Cauchy sequence is convergent or not ultimately depends on the metric space in which the sequence is defined. Any metric space having the property that all Cauchy sequences are also convergent sequences is said to be a **complete metric space**.

As mentioned above, the closure points of a set can be characterized as those points that are the limit of some convergent sequence within the set. Note that,

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[3] I am not aware of any specific term used in reference to the isolated points of a set.

[4] It can be shown to be unique.

equivalently, the limit of a convergent sequence is characterized by having at most a finite number of terms of the sequence beyond any given distance, no matter how small. This is clearly a stronger requirement than that of having infinitely many terms of the sequence within an arbitrarily small distance of the point. As a matter of fact, there may be several points (known as **cluster points**) with this latter property, while we know that the limit of any convergent sequence is unique. The implication is that a cluster point of a sequence is not necessarily the limit of the sequence.

A fundamental class of sets of a metric space is the class of **open sets**, i.e. those sets whose points are all interior to the set or, equivalently, those sets which coincide with its interior. The class of **closed sets** constitutes somehow the complementary notion, since a closed set is a set whose complement is open. Equivalently, a closed set is a set that coincides with its closure. Note, however, that closedness is not the negation of openness, nor viceversa, since it is possible that a set be neither[5] open nor closed.

With the notions of open and closed sets in hand, we are able to discuss some properties that that apply to metric spaces. To start, a set is said to be a **dense** subset of another set if its closure coincides with the set that contains it, i.e. if for every point in the original set, a point in the subset can be found within an arbitrarily close distance. Trivially every set is dense in its closure. In some sense then a dense subset can be taken as a good approximation of the set, since one can approach arbitrarily close any point of the set by some point of the subset. This idea of approximating a set by using a dense subset of the set becomes of even greater interest when it happens that the dense subset is simpler in some sense than the set itself (e.g. by being at most countable while the set is of greater cardinality). Any metric space with a subset of points that is at the same time dense and countable, is known as a **separable metric space**. The leading example of separable space is any $\mathbb{R}^n$ with a non-discrete metric, since the points of $\mathbb{Q}^n$ with rational coordinates are simultaneously dense and countable for such a space. Any time we use a number in $\mathbb{Q}^n$ as a valid representation of real-world phenomena supposed to be described by real variables in $\mathbb{R}^n$,[6] we are relying on this property.

A second property of interest for sets associated with metric spaces is that of connectedness. A set is said to be a **connected set** if it does not consist of separate pieces that can each be contained in disjoint open sets. Accordingly, if the entire space is a connected set, then the metric space is said to be a **connected metric space**. Note that being connected is actually a less stringent requirement than demanding that the set not consist of separate pieces, in the intuitive sense that one can join any two points of a set by a continuous path. This stronger notion of connectedness is known as **path-connectedness**.

A third property of interest for sets associated with metric spaces is that of sequential compactness. A set is said to be **sequentially compact** when every sequence of points in the set has a subsequence that converges to an element of the set. In some spaces this is equivalent to the more general notion of compactness, which requires first that we define the concept of an open cover for a set. An **open cover** for a set is a collection of open sets whose union contains the set. According to the definition of compactness, a set is **compact** if it has a finite open cover. If

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[5] It is also possible for a set to be both open and closed. See, for example, E19.

[6] Which in our computation-intensive society is an all-pervasive phenomenon.

the entire space happens to be compact, then the space is said to be a **compact metric space**. Compactness and sequential compactness are equivalent notions in, for instance, $\mathbb{R}^n$ endowed with any metric, but they are not so in general.

The precise definitions of these concepts follow.

## BASIC NOTIONS ABOUT METRIC SPACES

**D1. Space.** *A space is a set. A point of a space is an element of the space.*

**D2. Metric.** *If $X$ is a space and $d \in \mathbb{R}^{X \times X}$, then $d$ is a metric on $X$ if, and only if,*

(1) *$d$ has a full domain,*
(2) *$d(x, x') = 0$ if, and only if,*

$$x = x',$$

(3) *for all $x, x', x'' \in X$,*

$$d(x', x'') \leq d(x, x') + d(x, x'').$$

**D3. Metric space.** *If $X$ is a set and $d$ is a metric on $X$, then $(X, d)$ is a metric space.*

**D4. Ball.** *If $(X, d)$ is a metric space, $x \in X$, and $r \in \mathbb{R}_{++}$, then $B_r(x) \subset X$ is the ball of radius $r$ and centered at $x$ if, and only if,*

(1) *for all $x' \in B_r(x)$,*

$$d(x, x') < r,$$

*and*

(2) *for all $A \subset X$ such that, for all $x' \in A$, $d(x, x') < r$,*

$$A \subset B_r(x).$$

**D5. Bounded set.** *If $(X, d)$ is a metric space and $A \subset X$, then $A$ is bounded if, and only if, there exist a ball $B_r(x)$ such that*

$$A \subset B_r(x).$$

## TYPES OF POINTS IN A METRIC SPACE, DEFINED IN RELATION TO A SET OF THE SPACE

**D6.** *If $(X, d)$ is a metric space and $A \subset X$, then*

(1) *$x$ is a* **closure point** *of $A$ if, and only if, for all $r \in \mathbb{R}_{++}$,*

$$A \cap B_r(x) \neq \phi,$$

(2) *$x$ is a* **boundary point** *of $A$ if, and only if, it is a closure point of both $A$ and $X \setminus A$,*
(3) *$x$ is an* **interior point** *of $A$ if, and only if, it is a closure point of $A$ but not of $X \setminus A$,*
(4) *$x$ is an* **accumulation point** *of $A$ if, and only if, it is a closure point of both $A$ and $A \setminus \{x\}$,*
(5) *$x$ is an* **isolated point** *of $A$ if, and only if, it is a closure point of $A$ but not of $A \setminus \{x\}$.*

**D7.** *If $(X, d)$ is a metric space and $A \subset X$, then*
(1) $\mathrm{Cl}A$ *is the* **closure** *of $A$ if, and only if,*
  (i) *for all $x \in \mathrm{Cl}A$, $x$ is a closure point of $A$, and*
  (ii) *for all $A' \subset X$ such that, for all $x \in A'$, $x$ is a closure point of $A$,*

$$A' \subset \mathrm{Cl}A,$$

(2) *$\partial A$ is the* **boundary** *of $A$ if, and only if,*
  (i) *for all $x \in \partial A$, $x$ is a boundary point of $A$, and*
  (ii) *for all $A' \subset X$ such that, for all $x \in A'$, $x$ is a boundary point of $A$,*

$$A' \subset \partial A,$$

(3) $\mathrm{Int}A$ *is the* **interior** *of $A$ if, and only if,*
  (i) *for all $x \in \mathrm{Int}A$, $x$ is an interior point of $A$, and*
  (ii) *for all $A' \subset X$ such that, for all $x \in A'$, $x$ is a interior point of $A$,*

$$A' \subset \mathrm{Int}A.$$

## Sequences in metric spaces

**D8.** *If $(X, d)$ is a metric space and $x \in X^{\mathbb{N}}$, then*
(1) *$x$ is a* **bounded sequence** *if, and only if, $x(\mathbb{N})$ is bounded,*
(2) *$x$ is a* **Cauchy sequence** *if, and only if, for all $r \in \mathbb{R}_{++}$, there exists $N \in \mathbb{N}$ such that, for all $n, m > N$,*

$$d(x_m, x_n) < r,$$

(3) *$x$ is a* **convergent sequence** *if, and only if, there exists $\bar{x} \in X$ such that, for all $r \in \mathbb{R}_{++}$, $x^{-1}(X \setminus B_r(\bar{x}))$ is finite.*

**D9.** *If $(X, d)$ is a metric space, $x \in X^{\mathbb{N}}$, and $\bar{x} \in X$ then*
(1) *$\bar{x}$ is a* **cluster point** *of $x$ if, and only if, for all $r \in \mathbb{R}_{++}$, $x^{-1}(B_r(\bar{x}))$ is not finite*
(2) *$\bar{x}$ is the*[7] **limit** *of $x$ if, and only if, for all $r \in \mathbb{R}_{++}$, $x^{-1}(X \setminus B_r(\bar{x}))$ is finite.*

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[7] It will be shown to be unique whenever it exists.

## TYPES OF SETS IN METRIC SPACES

**D10.** *If $(X, d)$ is a metric space and $A \subset X$, then*

1. *$A$ is* **open** *if, and only if,*
$$A \subset \mathrm{Int}A,$$
2. *$A$ is* **closed** *if, and only if,*
$$\mathrm{Cl}A \subset A,$$
3. *$A$ is* **dense** *if, and only if,*
$$X \subset \mathrm{Cl}A,$$
4. *$A$ is* **connected** *if, and only if, for all disjoint, open $C, C' \subset X$ such that $A \cap C \neq \phi$, and $A \cap C' \neq \phi$,*
$$A \not\subset C \cup C'$$
5. *$A$ is* **complete** *if, and only if, for all $x \in A^{\mathbb{N}}$ Cauchy, $x$ is convergent with limit in $A$,*
6. *$A$ is* **sequentially compact** *if, and only if, for all $x \in A^{\mathbb{N}}$, there exists a convergent subsequence of $x$ with limit in $A$.*

**D11. Open cover.** *If $(X, d)$ is a metric space, $A \subset X$, and $\mathcal{O} \subset \mathcal{P}(X)$, then $\mathcal{O}$ is an open cover of $A$ if, and only if,*

1. *for all $O \in \mathcal{O}$, $O$ is an open set in $(X, d)$,*
2. *$A \subset \cup_{O \in \mathcal{O}} O$.*

**D12.** *If $(X, d)$ is a metric space and $A \subset X$, then*

1. *$A$ is a* **compact set** *if, and only if, for all open cover $\mathcal{O}$ of $A$, there exists $\mathcal{O}' \subset \mathcal{O}$ finite such that $\mathcal{O}'$ is an open cover of $A$.*

## TYPES OF METRIC SPACES

**D13.** *If $(X, d)$ is a metric space, then*

1. *$(X, d)$ is a* **separable metric space** *if, and only if, there exists $A \subset X$ such that $A$ is a dense subset of $X$, and $A$ is countable,*
2. *$(X, d)$ is a* **connected metric space** *if, and only if, $X$ is a connected set of $(X, d)$,*
3. *$(X, d)$ is a* **compact metric space** *if, and only if, $X$ is a compact set of $(X, d)$,*
4. *$(X, d)$ is a* **complete metric space**[8] *if, and only if, $X$ is a complete set of $(X, d)$.*

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[8] Also known as Banach space.

## EXAMPLES

*Discrete metric spaces.*

The following is an example of a trivial way of endowing a set with the metric using the discrete metric. The discrete metric consists of indicating whether any two points in a space are the same point. A space endowed with the discrete metric is called (not surprisingly) a **discrete metric space**. Discrete metric spaces are very peculiar in that, for instance, their sets have no accumulation points and balls with greater radii can trivially be contained in balls with smaller radii.

**E1. Discrete metric space.** *If $d$ is a real valued function with domain $X \times X$ such that $d(x, x') = 0$ if $x = x'$ and $d(x, x') = 1$ otherwise, then $(X, d)$ is a discrete metric space.*

*Proof.* Exercise.

**E2.** *If $(X, d)$ is a discrete metric space, $A \subset X$ and $x \in X$, then there exists $r > 0$ such that $B_r(x) \cap A$ is finite.*

*Proof.* Let $A \subset X$ and $x \in X$. Since $B_{\frac{1}{2}}(x) = \{x\}$, then $B_{\frac{1}{2}}(x) \cap A \subset \{x\}$ and hence there exists $r > 0$ such that $B_r(x) \cap A$ is a finite set, i.e. $x$ is not an accumulation point of $A$. Q.E.D.

**E3.** *If $(X, d)$ is a discrete metric space $X$, then for all $r, s > 0$ such that $r < s$ and all $x \in X$, $B_s(x) \subset B_r(x)$.*

*$\mathbb{R}^n$ with the usual metrics $d_1$, $d_2$.*

Next come some simple metrics that we often use in $\mathbb{R}^n$, namely $d_1$, $d_2$[9] These metrics measure the distance from one point to another by taking, respectively, the sum of the coordinate-wise differences (in the case of $d_1$) and the Euclidean distance (in the case of $d_2$).

**E4. The metric space $(\mathbb{R}^n, d_1)$.** *Let the space be $\mathbb{R}^n$, for any $n \in \mathbb{N}$, and $d_1$ be such that $d_1(x, x') = \sum_1^n |x_i - x_i'|$. Then $(\mathbb{R}^n, d_1)$ is a metric space.*

*Proof.*

(1) Let $x, x' \in X$.

If $x = x'$, then $x_i = x_i'$ for all $i = 1, \ldots, n$ and hence $|x_i - x_i'| = 0$. Therefore $d_1(x, x') = \sum_1^n |x_i - x_i'| = 0$.

Conversely, assume $x \neq x'$. Then there exists $i_0 \in \{1, \ldots, n\}$ such that $x_{i_0} \neq x'_{i_0}$ and hence $|x_{i_0} - x'_{i_0}| > 0$. Since $|x_i - x_i'| \geq 0$ for all $i = 1, \ldots, n$, then $d_1(x, x') = \sum_1^n |x_i - x_i'| > 0$.

(2) Let $x, x', x'' \in X$. Then, for all $i = 1, \ldots, n$, $|x_i - x_i''| \leq |x_i - x_i'| + |x_i' - x_i''|$, since

$$-|x_i - x_i'| \leq x_i - x_i' \leq |x_i - x_i'|$$

and

$$-|x_i' - x_i''| \leq x_i' - x_i'' \leq |x_i' - x_i''|$$

and hence

$$-(|x_i - x_i'| + |x_i' - x_i''|) \leq x_i - x_i'' \leq |x_i - x_i'| + |x_i' - x_i''|$$

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[9] This metric is also known as the Euclidean metric.

i.e. $|x_i - x''_i| \leq |x_i - x'_i| + |x'_i - x''_i|$.

Therefore, $\sum_1^n |x_i - x''_i| \leq \sum_1^n |x_i - x'_i| + \sum_1^n |x'_i - x''_i|$ holds, i.e. $\sum_1^n |x_i - x''_i| \leq \sum_1^n |x'_i - x_i| + \sum_1^n |x'_i - x''_i|$, that is to say $d_1(x, x'') \leq d_1(x', x) + d_1(x', x'')$.

Q.E.D.

**E5. The metric space** $(\mathbb{R}^n, d_2)$**.** *Let the space be* $\mathbb{R}^n$*, for any* $n \in \mathbb{N}$*, and* $d_2$ *be such that* $d_2(x, x') = \sqrt{\sum_1^n (x_i - x'_i)^2}$*. Then* $(\mathbb{R}^n, d_2)$ *is a metric space.*

*Proof.*

(1) Let $x, x' \in X$.

Assume $x = x'$. Then $x_i = x'_i$ for all $i$ and $\sqrt{\sum_1^n (x_i - x'_i)^2} = 0$ for all $i$, i.e. $d_2(x, x') = 0$.

Conversely, assume $x \neq x'$. Then there exists $i_0$ such that $x_{x_0} \neq x'_{i_0}$ and implying that $(x_{i_0} - x'_{i_0}) > 0$. Since $(x_i - x'_i) \geq 0$ for all $i$, it follows that $\sqrt{\sum_1^n (x_i - x'_i)^2} > 0$, i.e. $d_2(x, x') \neq 0$.

(2) Let $x, x', x'' \in X$. Since

$$\begin{aligned}
d_2(x, x'')^2 &= \sum_1^n (x_i - x''_i)^2 \\
&= \sum_1^n \big((x_i - x'_i) + (x'_i - x''_i)\big)^2 \\
&= \sum_1^n \big((x_i - x'_i)^2 + 2(x_i - x'_i)(x'_i - x''_i) + (x'_i - x''_i)^2\big) \\
&= \sum_1^n (x_i - x'_i)^2 + 2\sum_1^n (x_i - x'_i)(x'_i - x''_i) + \sum_1^n (x'_i - x''_i)^2 \\
&\leq \sum_1^n (x_i - x'_i)^2 + 2\sqrt{\sum_1^n (x_i - x'_i)^2}\sqrt{\sum_1^n (x'_i - x''_i)^2} + \sum_1^n (x'_i - x''_i)^2 \\
&= \Big(\sqrt{\sum_1^n (x_i - x'_i)^2} + \sqrt{\sum_1^n (x'_i - x_i)^2}\Big)^2 \\
&= \Big(\sqrt{\sum_1^n (x'_i - x_i)^2} + \sqrt{\sum_1^n (x'_i - x_i)^2}\Big)^2 \\
&= \big(d_2(x', x) + d_2(x', x'')\big)^2
\end{aligned}$$

because of the Cauchy-Schwartz inequality, and $0 \leq d_2(x, x'')^2$, then

$$d_2(x, x'') \leq d_2(x, x') + d_2(x', x'').$$

Q.E.D.

**Exercise.** *Prove the Cauchy-Schwartz inequality in* $\mathbb{R}^n$*, for any* $n \in \mathbb{N}$*,*

$$\sum_{i=1}^{n} |x_i x'_i| \le \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} |x'_i|^2\Big)^{\frac{1}{2}}.$$

*Hint: Note that the following real polynomial in* $t$ *has at most one real root, and find out what restriction this imposes on its coefficients.*

$$P(t) \equiv \sum_{i=1}^{n} (|x_i|\, t + |x'_i|\,)^2.$$

*Hölder's and Minkowski's inequalities.*

Notice that the metrics introduced previously can be trivially written as follows

$$d_1(x,x') = \sum_{i=1}^{n} |x_i - x'_i| = \Big(\sum_{i=1}^{n} |x_i - x'_i|^1\Big)^{\frac{1}{1}}$$

$$d_2(x,x') = \sqrt{\sum_{i=1}^{n} (x_i - x'_i)^2} = \Big(\sum_{i=1}^{n} |x_i - x'_i|^2\Big)^{\frac{1}{2}}$$

In order to prove that the obvious generalization of the previous expressions to $d_p$ (for $p \ge 1$)

$$d_p(x,x') = \Big(\sum_{i=1}^{n} |x_i - x'_i|^p\Big)^{\frac{1}{p}}$$

is a metric, and in order to further present the general family of metric spaces $\ell_p^n$, we will first need generalizations of the Cauchy-Schwartz and triangular inequalities. These generalizations are known as Hölder's and Minkowski's inequalities, respectively. The next two results are themselves needed to prove Hölder's inequality. In it the expression $\int_a^b f$ stands for the Riemann integral of a real-valued function defined on the interval $[a,b]$.

**L1.** *If* $f$ *is a continuous real valued function defined on an interval* $[a,b]$ *of real numbers, then* $\int_a^b f \le (b-a) \max_{[a,b]} f$*.*

*Proof.* Exercise.

**L2.** *If* $f$ *is a continuous, increasing*[10] *real valued function defined on* $\mathbb{R}_+$ *such that* $f(0) = 0$*, and* $a, b \ge 0$*, then*

$$ab \le \int_0^a f + \int_0^b f^{-1}.$$

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[10] That is to say, $f(x) < f(x')$ whenever $x < x'$.

*Proof.* Either $f^{-1}(b) \leq a$ or $f^{-1}(b) > a$. If $f^{-1}(b) \leq a$, then $b \leq f(a)$, since $f$ is increasing. Since moreover

$$\begin{aligned} af(a) &= \int_0^a f + \int_0^{f(a)} f^{-1} \\ &= \int_0^a f + \int_0^b f^{-1} + \int_b^{f(a)} f^{-1}, \end{aligned}$$

that is to say,

$$ab = \int_0^a f + \int_0^b f^{-1} + \big(\int_b^{f(a)} f^{-1} - a(f(a) - b)\big),$$

because $f(0) = 0$, and $a = \max_{[b,f(a)]} f^{-1}$ because $f^{-1}$ is increasing, then it follows that $\int_b^{f(a)} f^{-1} \leq a(f(a) - b)$, and therefore

$$ab \leq \int_0^a f + \int_0^b f^{-1}.$$

If, on the contrary, $f^{-1}(b) > a$, then $b > f(a)$ because $f$ is increasing. Thus, since

$$\begin{aligned} bf^{-1}(b) &= \int_0^{f^{-1}(b)} f + \int_0^b f^{-1} \\ &= \int_0^a f + \int_a^{f^{-1}(b)} f + \int_0^b f^{-1} \end{aligned}$$

that is to say,

$$ab = \int_0^a f + \int_0^b f^{-1} + \big(\int_a^{f^{-1}(b)} f - b(f^{-1}(b) - a)\big),$$

and $b = \max_{[a,f^{-1}(b)]} f$ because $f$ is increasing, then $\int_a^{f^{-1}(b)} f^{-1} \leq b(f^{-1}(b) - a)$, and therefore

$$ab \leq \int_0^a f + \int_0^b f^{-1}.$$

Q.E.D.

Hölder's inequality, provided below as a generalization of the Cauchy-Schwartz inequality, is needed to establish the more important Minkowski's inequality that follows it.

**L3. Hölder's Inequality in $\mathbb{R}^n$.** *If $x, x' \in \mathbb{R}^n$ and $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\sum_1^n |x_i x_i'| \leq \Big(\sum_1^n |x_i|^p\Big)^{\frac{1}{p}} \Big(\sum_1^n |x_i'|^q\Big)^{\frac{1}{q}}.$$

*Proof.* Since $x, x' \in \mathbb{R}^n$ satisfy the inequality if, and only if, $\lambda x, \mu x' \in \mathbb{R}^n$ satisfy it too, for any $\lambda, \mu > 0$ (in effect, for any $\lambda, \mu > 0$,

$$\begin{aligned}\Big(\sum_1^n |\lambda x_i|^p\Big)^{\frac{1}{p}}\Big(\sum_1^n |\mu x_i'|^q\Big)^{\frac{1}{q}} &= \lambda\mu\Big(\sum_1^n |x_i|^p\Big)^{\frac{1}{p}}\Big(\sum_1^n |x_i'|^q\Big)^{\frac{1}{q}}\\ &\geq \lambda\mu\sum_1^n |x_i x_i'|\\ &= \sum_1^n |\lambda x_i \mu x_i'|\end{aligned}$$

if, and only if, the inequality holds for $x$ and $x'$, then we can assume that $x, x'$ are such that $\sum_1^n |x_i|^p = 1$ and $\sum_1^n |x_i'|^q = 1$ (if not, consider instead $\lambda x$ and $\mu x'$ with $\lambda = (\sum_1^n |x_i|^p)^{-1}$ and $\mu = (\sum_1^n |x_i'|^q)^{-1}$).

Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be such that $f(x) = x^{p-1}$, for all $x \in \mathbb{R}_+$. Then $f$ is continuous, increasing and such that $f(0) = 0$. Therefore, for all $i = 1, \ldots, n$,

$$\begin{aligned}|x_i x_i'| &\leq \int_0^{|x_i|} f + \int_0^{|x_i'|} f^{-1}\\ &= \int_0^{|x_i|} x^{p-1}dx + \int_0^{|x_i'|} y^{q-1}dy\\ &= \frac{1}{p}|x_i|^p + \frac{1}{q}|x_i'|^q,\end{aligned}$$

since $f^{-1}$ is such that $f^{-1}(y) = y^{q-1}$ for all $y \in \mathbb{R}_+$ (in effect, $f^{-1} \circ f = i$ since $f^{-1}(f(x)) = (x^{p-1})^{q-1} = x^{pq-p-q+1}$ for all $x \geq 0$ because $\frac{1}{p} + \frac{1}{q} = 1$). Hence

$$\begin{aligned}\sum_1^n |x_i x_i'| &\leq \frac{1}{p}\sum_1^n |x_i|^p + \frac{1}{q}\sum_1^n |x_i'|^q\\ &= \frac{1}{p} + \frac{1}{q}\\ &= 1\\ &= \Big(\sum_1^n |x_i|^p\Big)^{\frac{1}{p}}\Big(\sum_1^n |x_i'|^q\Big)^{\frac{1}{q}}.\end{aligned}$$

Q.E.D.

Minkowski's inequality is a generalization of the triangle inequality to a family of distances that generalize the usual Euclidean distance: instead of considering the square root (i.e. the $\frac{1}{2}$-th power) of the sum of the squares (i.e. the 2-nd power) of the coordinate-wise differences, an arbitrary $\frac{1}{p}$-th power of the sum of the $p$-th power of the coordinate-wise differences, for any $p \geq 1$ is considered.

**L4. Minkowski's inequality in $\mathbb{R}^n$.** *For all $x, x' \in \mathbb{R}^n$ and all $p \geq 1$,*

$$\Big(\sum_1^n |x_i + x_i'|^p\Big)^{\frac{1}{p}} \leq \Big(\sum_1^n |x_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_1^n |x_i'|^p\Big)^{\frac{1}{p}}.$$

*Proof.* If $p = 1$, since for all $i \in \{1, \ldots, n\}$,

$$|x_i + x'_i| \leq |x_i| + |x'_i|$$

because of the triangular inequality in the real line, then

$$\sum_{i=1}^{n} |x_i + x'_i| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |x'_i|.$$

If $p > 1$, since

$$\begin{aligned}(|x_i| + |x'_i|)^p &= (|x_i| + |x'_i|)^{p-1}(|x_i| + |x'_i|) \\ &= (|x_i| + |x'_i|)^{p-1}|x_i| + (|x_i| + |x'_i|)^{p-1}|x'_i|,\end{aligned}$$

then

$$\begin{aligned}\sum_1^n (|x_i| + |x'_i|)^p &= \sum_1^n (|x_i| + |x'_i|)^{p-1}|x_i| + \sum_1^n (|x_i| + |x'_i|)^{p-1}|x'_i| \\ &\leq \left(\sum_1^n (|x_i| + |x'_i|)^{(p-1)q}\right)^{\frac{1}{q}} \left[\left(\sum_1^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^n |x'_i|^p\right)^{\frac{1}{p}}\right] \\ &= \left(\sum_1^n (|x_i| + |x'_i|)^p\right)^{1-\frac{1}{p}} \left[\left(\sum_1^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^n |x'_i|^p\right)^{\frac{1}{p}}\right]\end{aligned}$$

where $q = p/(p-1)$, because of Hölder's inequality, i.e.

$$\left(\sum_1^n (|x_i| + |x'_i|)^p\right)^{\frac{1}{p}} \leq \left(\sum_1^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^n |x'_i|^p\right)^{\frac{1}{p}}$$

whence Minkowski's inequality follows because of the triangular inequality in the real line that implies

$$\left(\sum_1^n |x_i + x'_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_1^n (|x_i| + |x'_i|)^p\right)^{\frac{1}{p}}.$$

Q.E.D.

*The metric spaces* $\ell_p^n$.

With the previous inequalities we can provide the following example of a family of metrics $d_p$ defined on $\mathbb{R}^n$ that metrize this set, yielding a family of metric spaces that includes the Euclidean metric space $(\mathbb{R}^n, d_2)$ and the real line $(\mathbb{R}, d_1)$ as a particular cases. These metric spaces are known as $\ell_p^n$.

**E6. The metric space $\ell_p^n$.** *Let the space be $\mathbb{R}^n$, for any integer $n$, and let $d_p$, for any real $p \geq 1$, be such that $d_p(x,x') = (\sum_1^n |x_i - x_i'|^p)^{\frac{1}{p}}$. Then $(\mathbb{R}^n, d_p)$ is a metric space known as $\ell_p^n$.*

*Proof.*

(1) Let $x, x' \in X$.

Assume $x = x'$. Then $x_i = x_i'$ for all $i$. Hence $|x_i - x_i'| = 0$ for all $i$, i.e. $d_p(x,x') = \left(\sum_1^n |x_i - x_i'|^p\right)^{\frac{1}{p}} = 0$.

Conversely, assume $x \neq x'$. Then there exists $i_0$ such that $x_{x_0} \neq x_{i_0}'$ and thus $|x_{i_0} - x_{i_0}'| > 0$. Since $|x_i - x_i'| \geq 0$ for all $i$, then $\left(\sum_1^n |x_i - x_i'|^p\right)^{\frac{1}{p}} > 0$, i.e. $d_p(x,x') \neq 0$.

(2) Let $x, x', x'' \in X$. Then

$$\begin{aligned} d_p(x,x'') &= \left(\sum_1^n |x_i - x_i''|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_1^n |x_i - x_i' + x_i' - x_i''|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_1^n |x_i - x_i'|^p\right)^{\frac{1}{p}} + \left(\sum_1^n |x_i' - x_i''|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_1^n |x_i' - x_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^n |x_i' - x_i''|^p\right)^{\frac{1}{p}} \\ &= d_p(x',x) + d_p(x',x''). \end{aligned}$$

because of Minkowski's inequality.

Q.E.D.

*The metric space $\ell_p$.*

Hölder's and Minkowski's inequalities hold also for the $p$-summable real series $\sum_1^\infty |x_i|^p$, i.e. the limit of the sequence $\{\sum_1^n |x_i|^p\}_{n\in\mathbb{N}}$. Actually Minkowski's inequality for $p$-summable sequences follows from its version in $\mathbb{R}^n$. This allows us to show that the obvious generalization of the functions $d_p$ to convergent series is a metric, and in that way we can metrize some spaces of real sequences.

**S. Every $p$-summable real sequence converges to 0.** *If $x \in \mathbb{R}^\mathbb{N}$ is $p$-summable, then*

$$\lim_{n\to\infty} x_n = 0.$$

*Proof.* Let $x \in \mathbb{R}^\mathbb{N}$ be $p$-summable.

Assume that $\lim_{n\to\infty} x_n \neq 0$. Since $\lim_{n\to\infty} x_n \neq 0$, then there exists $r \in \mathbb{R}_{++}$ such that, $x^{-1}(B_r(0)^C)$ is countable. Since for all $n \in x^{-1}(B_r(0)^C)$, $r < |x_n|$ and $x^{-1}(B_r(0)^C)$ is countable, then

$$\sum_{n \in x^{-1}(B_r(0)^C)} |x_n| \notin \mathbb{R}.$$

Since $\sum_{n\in x^{-1}(B_r(0)^C)}|x_n|\notin\mathbb{R}$ and

$$\sum_{n\in x^{-1}(B_r(0)^C)}|x_n|\leq\sum_1^\infty|x_n|,$$

then

$$\sum_1^\infty|x_n|\notin\mathbb{R}.$$

Since $\sum_1^\infty|x_n|\notin\mathbb{R}$, then $x$ is not $p$-summable.

Therefore, since $x$ is $p$-summable, then

$$\lim_{n\to\infty}x_n=0.$$

Q.E.D.

**S. Every sequence equal to a $p$-summable sequence but for finitely many terms is $p$-summable as well.** *If $x,x'\in\mathbb{R}^{\mathbb{N}}$ are such that $x$ is $p$-summable and $N=\{n\in\mathbb{N}|x_n\neq x'_n\}$ is finite, then $x'$ is $p$-summable.*

*Proof.* Let $x,x'\in\mathbb{R}^{\mathbb{N}}$ be such that $x$ is $p$-summable and $N=\{n\in\mathbb{N}|x_n\neq x'_n\}$ is finite.

Since $N$ is finite, then

$$\sum_{n\in N}|x'_n|^p\in\mathbb{R}.$$

Since $x$ is $p$-summable, then

$$\sum_{n\in\mathbb{N}}|x_n|^p\in\mathbb{R}.$$

Since $\sum_{n\in\mathbb{N}}|x_n|^p\in\mathbb{R}$, then

$$\sum_{n\in\mathbb{N}\setminus N}|x_n|^p\in\mathbb{R}.$$

Since $\sum_{n\in N}|x'_n|^p\in\mathbb{R}$, $\sum_{n\in\mathbb{N}}|x_n|^p\in\mathbb{R}$, and $\sum_{n\in\mathbb{N}\setminus N}|x'_n|^p=\sum_{n\in\mathbb{N}\setminus N}|x_n|^p$, then

$$\begin{aligned}\sum_{n\in\mathbb{N}}|x'_n|^p&=\sum_{n\in\mathbb{N}\setminus N}|x'_n|^p+\sum_{n\in N}|x'_n|^p\\&=\sum_{n\in\mathbb{N}\setminus N}|x'_n|^p+\sum_{n\in N}|x_n|^p\\&\in\mathbb{R}.\end{aligned}$$

Q.E.D.

**S. Every $p$-summable real sequence is $p'$-summable, for all $p\leq p'$.** *If $x\in\mathbb{R}^{\mathbb{N}}$ is $p$-summable and $p\leq p'$, then $x$ is $p'$-summable.*

*Proof.* Let $x\in\mathbb{R}^{\mathbb{N}}$ is $p$-summable and $p\leq p'$.

Since $x$ is $p$-summable, then $\lim_{n\to\infty}x_n=0$. Since $\lim_{n\to\infty}x_n=0$, then $x^{-1}(B_1(0)^C)$ is finite. Let $x'\in\mathbb{R}^{\mathbb{N}}$ be such that,

(1) for all $n\in x^{-1}(B_1(0))$, $x'_n=x_n$
(2) for all $n\in x^{-1}(B_1(0)^C)$, $x'_n=0$.

Therefore, $x'(\mathbb{N}) \subset B_1(0)$ and, since $x$ is $p$-summable, then $x'$ is $p$-summable. Since $x'(\mathbb{N}) \subset B_1(0)$ and $p \leq p'$, then, for all $n \in \mathbb{N}$,

$$|x'_n|^{p'} < |x'_n|^p.$$

Since for all $n \in \mathbb{N}$, $|x'_n|^{p'} < |x'_n|^p$, and $x'$ is $p$-summable, then, for all $n' \in \mathbb{N}$,

$$\begin{aligned}\sum_1^{n'} |x'_n|^{p'} &< \sum_1^{n'} |x'_n|^p \\ &\leq \sum_1^{\infty} |x'_n|^p\end{aligned}$$

Since $\{\sum_1^{n'} |x'_n|^{p'}\}_{n\in\mathbb{N}}$ is non-decreasing and bounded above, then $\sum_1^\infty |x'_n|^{p'} \in \mathbb{R}$. Since $x'$ is $p'$-summable and $\{n \in \mathbb{N} | x'_n \neq x_n\}$ is finite, then $x$ is $p'$-summable. Q.E.D.

**L5. Minkowski's inequality for $p$-summable real sequences.** *For all $x, x' \in \mathbb{R}^\mathbb{N}$ and all $p \geq 1$ such that both $\sum_1^\infty |x_i|^p$ and $\sum_1^\infty |x'_i|^p$ are convergent series,*

$$\left(\sum_1^\infty |x_i + x'_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_1^\infty |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^\infty |x'_i|^p\right)^{\frac{1}{p}}.$$

*Proof.* Let $x, x' \in \mathbb{R}^\mathbb{N}$ and $p > 0$ be such that $\sum_1^\infty |x_i|^p$ and $\sum_1^\infty |x'_i|^p$ are convergent series.

Since, for all $n \in \mathbb{N}$, $0 \leq \sum_1^n |x_i|^p \leq \sum_1^\infty |x_i|^p$ and $0 \leq \sum_1^n |x'_i|^p \leq \sum_1^\infty |x'_i|^p$, then for all $n \in \mathbb{N}$, $(\sum_1^n |x_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}}$ and $(\sum_1^n |x'_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}}$. Then, for all $n \in \mathbb{N}$,

$$(\sum_1^n |x_i|^p)^{\frac{1}{p}} + (\sum_1^n |x'_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}},$$

Since, moreover, by Minkowski's inequality, for all $n \in \mathbb{N}$,

$$(\sum_1^n |x_i + x'_i|^p)^{\frac{1}{p}} \leq (\sum_1^n |x_i|^p)^{\frac{1}{p}} + (\sum_1^n |x'_i|^p)^{\frac{1}{p}},$$

then, for all $n \in \mathbb{N}$,

$$0 \leq (\sum_1^n |x_i + x'_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}}.$$

Hence for all $n \in \mathbb{N}$,

$$\sum_1^n |x_i + x'_i|^p \leq \left((\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}}\right)^p.$$

Since moreover $\sum_1^n |x_i+x'_i|^p$ is non-decreasing, then there exists $\sum_1^\infty |x_i+x'_i|^p$ and

$$\sum_1^\infty |x_i+x'_i|^p \leq \Big( (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}} \Big)^p.$$

i.e.

$$(\sum_1^\infty |x_i+x'_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}}.$$

Q.E.D.

The following result establishes that the difference between two $p$-summable real sequences is a $p$-summable real sequence as well. This result and Minkowski's inequality together allow us to show that the set of $p$-summable real sequences is a metric space with the metric $d_p$.

**L6.** *If $x, x' \in \mathbb{R}^{\mathbb{N}}$ and $p > 0$ are such that $\sum_1^\infty |x_i|^p$ and $\sum_1^\infty |x'_i|^p$ are convergent series, then $\sum_1^\infty |x_i - x'_i|^p$ is a convergent series.*

*Proof.* Let $x, x' \in \mathbb{R}^{\mathbb{N}}$ and $p > 0$ be such that $\sum_1^\infty |x_i|^p$ and $\sum_1^\infty |x'_i|^p$ are convergent series.

Since, for all $n \in \mathbb{N}$, $0 \leq \sum_1^n |x_i|^p \leq \sum_1^\infty |x_i|^p$ and $0 \leq \sum_1^n |x'_i|^p \leq \sum_1^\infty |x'_i|^p$, then for all $n \in \mathbb{N}$, $(\sum_1^n |x_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}}$ and $(\sum_1^n |x'_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}}$. Then, for all $n \in \mathbb{N}$,

$$(\sum_1^n |x_i|^p)^{\frac{1}{p}} + (\sum_1^n |x'_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}},$$

Since, moreover, by Minkowski's inequality, for all $n \in \mathbb{N}$,

$$\begin{aligned} &(\sum_1^n |x_i - x'_i|^p)^{\frac{1}{p}} = \\ &(\sum_1^n |x_i + (-x'_i)|^p)^{\frac{1}{p}} \leq (\sum_1^n |x_i|^p)^{\frac{1}{p}} + (\sum_1^n |-x'_i|^p)^{\frac{1}{p}} \\ &\qquad = (\sum_1^n |x_i|^p)^{\frac{1}{p}} + (\sum_1^n |x'_i|^p)^{\frac{1}{p}}, \end{aligned}$$

then, for all $n \in \mathbb{N}$,

$$0 \leq (\sum_1^n |x_i - x'_i|^p)^{\frac{1}{p}} \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}}.$$

Hence for all $n \in \mathbb{N}$,

$$\sum_1^n |x_i - x'_i|^p \leq \Big( (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} + (\sum_1^\infty |x'_i|^p)^{\frac{1}{p}} \Big)^p.$$

Since moreover $\sum_1^n |x_i - x'_i|^p$ is non-decreasing, then there exists $\sum_1^\infty |x_i - x'_i|^p$. Q.E.D.

As mentioned earlier, this result enables us to show that the space of $p$-sumable real sequences is a metric space with the metric $d_p$.

**E7. The metric space $\ell_p$.** *Let $X$ be the subset of sequences $x \in \mathbb{R}^{\mathbb{N}}$ such that $\sum_1^\infty |x_i|^p$ is a convergent series, and let $d_p$, for any real $p \geq 1$, be such that $d_p(x,x') = (\sum_1^\infty |x_i - x'_i|^p)^{\frac{1}{p}}$. Then $(X, d_p)$ is a metric space known as $\ell_p$.*

*Proof.*

(1) Let $x, x' \in X$.

Assume $x = x'$. Then $x_i = x'_i$ for all $i \in \mathbb{N}$. Hence $|x_i - x'_i| = 0$ for all $i \in \mathbb{N}$, i.e. $d_p(x,x') = \left(\sum_1^\infty |x_i - x'_i|^p\right)^{\frac{1}{p}} = 0$.

Conversely, assume $x \neq x'$. Then there exists $i_0 \in \mathbb{N}$ such that $x_{x_0} \neq x'_{i_0}$ and thus $|x_{i_0} - x'_{i_0}| > 0$. Since $|x_i - x'_i| \geq 0$ for all $i \in \mathbb{N}$, then $\left(\sum_1^\infty |x_i - x'_i|^p\right)^{\frac{1}{p}} \geq |x_{i_0} - x'_{i_0}| > 0$, i.e. $d_p(x,x') \neq 0$.

(2) Let $x, x', x'' \in X$. Then

$$\begin{aligned}
d_p(x,x'') &= \left(\sum_1^\infty |x_i - x''_i|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_1^\infty |x_i - x'_i + x'_i - x''_i|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_1^\infty |x_i - x'_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^\infty |x'_i - x''_i|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_1^\infty |x'_i - x_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^\infty |x'_i - x''_i|^p\right)^{\frac{1}{p}} \\
&= d_p(x',x) + d_p(x',x'').
\end{aligned}$$

because of Minkowski's inequality.

Q.E.D.

*The metric spaces $\ell_\infty^n$, $\ell_\infty$, and more generally the metric space of the real valued bounded functions with the sup metric.*

The next two examples consider the new metric $d_\infty$, a metric that is not unrelated to the previous metric $d_p$ as we will see. Applied to the spaces $\mathbb{R}^n$ and $\mathbb{R}^{\mathbb{N}}$ , this metric defines the distance between any two points as the supremum of the set of absolute values of coordinate-wise differences, and results in the metric spaces known as $\ell_\infty^n$ and $\ell_\infty$ respectively. These metric spaces are easily recognized as particular instances of some more general classes of metric spaces that are described in more detail below. As a consequence of this fact and in order to save effort, instead of providing proofs of these two special cases, the results will be shown to follow from a proposition establishing that the sup metric on real valued bounded functions is a metric indeed.

**E8. The metric space $\ell_\infty^n$.** *Let the space be $\mathbb{R}^n$, for any $n \in \mathbb{N}$, and $d_1$ be such that $d_\infty(x,x') = \max_i |x_i - x'_i|$. Then $(\mathbb{R}^n, d_\infty)$ is a metric space known as $\ell_\infty^n$.*

**E9. The metric space $\ell_\infty$.** *The set of real sequences $x \in \mathbb{R}^{\mathbb{N}}$ such that there exists* $\sup_n |x_n|$,[11] *is a metric space with the metric $d_\infty(x, x') = \sup_{n\in\mathbb{N}} |x_n - x'_n|$ denoted $\ell_\infty$*

Note in the example below that the domain $X$ of the functions is left unspecified. If $X$ happens to be some finite set, for example the integers $\{1, \ldots, n\}$, then the space is $\mathbb{R}^n$, the suprema are maxima indeed, and with this metric the metric space is known as $\ell^n_\infty$. If, alternatively, $X$ is $\mathbb{N}$, then the space is that of the bounded real valued sequences, which with this metric is known as $\ell_\infty$.

**E10. The metric space of bounded real functions with the sup metric.** *Let $b(\mathbb{R}^X)$ be the set of bounded real valued functions with full domain $X$ and $d_\infty$ be such that $d_\infty(f, g) = \sup_X |f(x) - g(x)|$, for any $f$, $g$ in $b(\mathbb{R}^X)$. Then $(b(\mathbb{R}^X), d_\infty)$ is a metric space.*

*Proof.* Exercise.

The object of the next example (the space of continuous real valued functions defined on a closed interval) is a particular case of the previous one. As an aside we should note that the example is more specific than necessary, since the domain of the space of functions can be any compact set of a metric space without changing the result . . . but that is still to come.

**E11. The metric space of $C[a, b]$ with $d_\infty$.** *Let $C[a, b]$ be the set of continuous real valued functions defined in the interval of the reals $[a, b]$ and $d$ be such that $d_\infty(f, g) = \max_{[a,b]} |f(x) - g(x)|$, for any $f, g \in C[a, b]$. Then $(C[a, b], d_\infty)$ is a metric space.*

*Proof.* For any $f \in C[a, b]$, there exist $x_0, x_1 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(x_1)$, for all $x \in [a, b]$, i.e. $f(x_0)$ and $f(x_1)$ are respectively lower and upper bounds to $f[a, b]$. Therefore every $f \in C[a, b]$ is bounded.

Moreover, for any $f, g \in C[a, b]$, $|f - g|$ is a continuous real valued function defined on the compact $[a, b]$. Hence there exists $\max_{[a,b]} |f(x) - g(x)|$ and it is $\sup_{[a,b]} |f(x) - g(x)|$, i.e. $d_\infty(f, g) = \sup_{[a,b]} |f(x) - g(x)|$.

The conclusion follows from the previous theorem. Q.E.D.

*The metric spaces $\ell^n_\infty$ and $\ell_\infty$ as limits of $\ell^n_p$ and $\ell_p$ as $p \to \infty$.*

**E12. Convergence of $d_p$ to $d_\infty$ in $\mathbb{R}^n$.**

*For all $x, x' \in \mathbb{R}^n$, $d_\infty(x, x') = \lim_{p\to\infty} d_p(x, x')$.*

*Proof.* Exercise. Let $x, x' \in \mathbb{R}^n$ be distinct (otherwise the result holds trivially). Then

$$\lim_{p\to\infty} \Big( \sum_{i=1}^{n} \big( \frac{|x_i - x'_i|}{\sup_j |x_j - x'_j|} \big)^p \Big)^{\frac{1}{p}} = 1$$

since, for all $\varepsilon > 0$, there exists $p' \geq 1$ such that, for all $p > p'$,

$$\left| \Big( \sum_{i=1}^{n} \big( \frac{|x_i - x'_i|}{\sup_j |x_j - x'_j|} \big)^p \Big)^{\frac{1}{p}} - 1 \right| < \varepsilon,$$

---

[11] That is to say, such that $x(\mathbb{N})$ is bounded.

i.e.

$$-\varepsilon < \Big(\sum_{i\notin I}\big(\frac{|x_i - x'_i|}{\sup_j |x_j - x'_j|}\big)^p + \sum_{i\in I} 1\Big)^{\frac{1}{p}} - 1 < \varepsilon,$$

where $I$ is the subset of $\{1,\ldots,n\}$ such that $|x_i - x'_i| = \sup_j |x_j - x'_j|$, that is to say

$$(-\varepsilon+1)^p - \#I < \sum_{i\notin I}\big(\frac{|x_i - x'_i|}{\sup_j |x_j - x'_j|}\big)^p < (\varepsilon+1)^p - \#I,$$

if $\varepsilon \leq 1$. In effect, while the the left-hand side of this inequality goes to $-\#I \leq -1$ and the the right-hand side goes to $\infty$ as $p \to \infty$, the expression in between goes to zero. Hence, the existence of a $p' > 1$ beyond which the inequality holds.

Therefore

$$\lim_{p\to\infty}\Big(\sum_{i=1}^{n} |x_i - x'_i|^p\Big)^{\frac{1}{p}} = \sup_i |x_i - x'_i|.$$

Q.E.D.

**E13. Convergence of $d_p$ to $d_\infty$ for the $p$-summable real sequences.** *For all* $x, x' \in \mathbb{R}^{\mathbb{N}}$ *such that, for all* $p \geq 1$, $\sum_{i=1}^{\infty} |x_i|^p$ *and* $\sum_{i=1}^{\infty} |x'_i|^p$ *are convergent series,* $d_\infty(x, x') = \lim_{p\to\infty} d_p(x, x')$.

*Proof.* Exercise.

**E14.** *In* $\mathbb{R}^2$ *a ball using the metric* $d_1$ *(i.e in the metric space* $\ell_1^2$*) is a "diamond", while using the metric* $d_2$ *(i.e in* $\ell_2^2$*) makes it a "disk" and using the metric* $d_\infty$ *(i.e in* $\ell_\infty^2$*) makes it a "square". Do you see a pattern emerge for this ball in the different metric spaces* $\ell_1^2$, $\ell_2^2$, $\ell_3^2$, $\ell_4^2$, *... ?*

*Sequences in* $\ell_p$ *and* $\ell_\infty$.

Next are some examples of sequences of sequences of real numbers that may or may not be convergent depending on the particular metric space we consider (e.g. $\ell_p$ or $\ell_\infty$). To illustrate the importance of the choice of metric in deciding matters of, for example, convergence, closedness, and compactness, example E.17 below shows how a point may be the limit of a sequence under one metric but not under another (different) metric.

**E15.** *The sequence* $x \in (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ *whose* $n$*-th term* $x_n \in \mathbb{R}^{\mathbb{N}}$ *is itself a sequence defined by*

$$x_n(n') = \begin{cases} 0 & \text{if } n' \neq n \\ \frac{1}{n} & \text{if } n' = n \end{cases}$$

*is a sequence in* $\ell_\infty$ *and in* $\ell_p$ *for all* $p \geq 1$ *and moreover converges to* $0 \in \mathbb{R}^{\mathbb{N}}$ *in each of these spaces.*

*Proof.* Exercise.

**E16.** *The sequence* $x \in (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ *whose* $n$*-th term* $x_n \in \mathbb{R}^{\mathbb{N}}$ *is itself a sequence defined by*

$$x_n(n') = \begin{cases} 0 & \text{if } n' \neq n \\ 1 & \text{if } n' = n \end{cases}$$

*is a sequence in* $\ell_\infty$ *and in* $\ell_p$ *for all* $p \geq 1$ *and moreover does not converge to* $0 \in \mathbb{R}^{\mathbb{N}}$ *in any of these spaces.*

*Proof.* Exercise.

**E17.** *The sequence $x \in (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ whose $n$-th term $x_n \in \mathbb{R}^{\mathbb{N}}$ is itself a sequence defined by*

$$x_n(n') = \begin{cases} \frac{1}{n} & \text{if } n' \leq n \\ 0 & \text{if } n' > n \end{cases}$$

*is a sequence in $\ell_\infty$ and in $\ell_p$ for all $p \geq 1$ and moreover converges to $0 \in \mathbb{R}^{\mathbb{N}}$ in $\ell_\infty$, but not in $\ell_1$.*

*Proof.* On the one hand, since $|x_n|(\mathbb{N}) = \{0, \frac{1}{n}\}$, $\sup |x_n|(\mathbb{N}) = \frac{1}{n}$ for all $n \in \mathbb{N}$, and hence $x$ is a sequence of $\ell_\infty$. Moreover, if $r > 0$, there exists $N(= [\frac{1}{r}]) \in \mathbb{N}$ such that if $n > N$, then $d_p(x_n, 0) = \frac{1}{n} < r$ (since $[\frac{1}{r}] \leq \frac{1}{r} < [\frac{1}{r}] + 1 \leq n$ again) i.e. the sequence $x$ converges to 0 in $\ell_\infty$.

On the other hand, since $\sum_{n'=1}^{\infty} |x_n(n')|^p = n\frac{1}{n^p}$, for all $n \in \mathbb{N}$ and for all $p \geq 1$, it follow that $x$ is a sequence in $\ell_p$.

Moreover, in $\ell_1$, $0 \in \mathbb{R}^{\mathbb{N}}$ is not the limit of $x$, since $d_1(x_n, 0) = \sum_{n'=1}^{\infty} |x_n(n') - 0| = n\frac{1}{n} = 1$. So, letting $r \leq 1$, if $n > N$ for all $N \in \mathbb{N}$, we have that $d_\infty(x_n, 0) = 1 \geq r$, i.e. the sequence $x$ does not converge to 0 in $\ell_1$.

(can it converge somewhere else? what about $\ell_p$ for $p > 1$?) Q.E.D.

*Open, closed, and bounded sets.*

**E18.** *Let $(X, d)$ be the reals with the absolute value metric (i.e. $\ell_1^1$) and $a < b$ be two real numbers. Then the interval $(a, b)$ is an open set in this metric space.*

*Proof.* Let $x \in (a, b)$ and $r = \min\{x - a, b - x\}$. Then $B_r(x) \subset (a, b)$, since

(1) if $x \leq \frac{a+b}{2}$, then $r = x - a$, and for any $x' \in B_r(x)$, i.e. such that $|x - x'| < x - a$ or, equivalently, such that $-x + a < x - x' < x - a$, then respectively $x' < 2x - a \leq b$ and $a < x'$, i.e. $x' \in (a, b)$;

(2) if $\frac{a+b}{2} < x$, then $r = b - x$, and for any $x' \in B_r(x)$, i.e. such that $|x - x'| < b - x$ or, equivalently, such that $x - b < x - x' < b - x$, then respectively $x' < b$ and $a < 2x - b < x'$, i.e. $x' \in (a, b)$ as well.

Q.E.D.

**E19.** *If $(X, d)$ is the discrete metric space for $X$, then every set in the space is both open and closed.*[12]

*Proof.* Let $A \subset X$ and $a \in A$ and $a' \in A^C$. On the one hand, since $B_{\frac{1}{2}}(a) = \{a\}$ and $a \in A$, then $B_{\frac{1}{2}}(a) \subset A$. Therefore $a$ is an interior point and thus $A$ is open.

On the other hand, since $B_{\frac{1}{2}}(a') = \{a'\}$ and $a' \in A^C$, then $B_{\frac{1}{2}}(a') \subset A^C$. Therefore $a'$ is an interior point of $A^C$ and thus $A^C$ is open. Hence $A$ is closed. Q.E.D.

**E20.** *Every finite set in a metric space is closed.*

*Proof.* Exercise.

*Examples relating to sequences.*

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[12] Hence no discrete metric space with more than one point is connected, i.e. if a discrete metric space is connected, then it only has one point (see below).

**E21.** *Every constant sequence is convergent to the constant term.*

*Proof.* Exercise.

**E22.** *The limit of a constant sequence is a closure point, but not an accumulation point, of the range of the sequence.*

*Proof.* Exercise.

**E23.** *A sequence in a discrete metric space is convergent iff beyond some term the sequence is constant.*

*Proof.* Exercise.

*Sequences in $\mathbb{R}$ with the usual metric.*

The linear order associated with the field of real numbers provides some structure that allows us to discuss some important properties of sequences that don't generalize to sets without this structure. The examples that follow illustrate the usefulness of some of these properties. An interesting starting point is to notice that if a sequence of real numbers heads steadily towards some bound, then it must be convergent.

**E24.** *Every bounded non-decreasing (non-incresing) sequence in $\mathbb{R}$ is a convergent sequence whose limit is the supremum (infimum) of the range of the sequence.*

*Proof.* Let $x \in \mathbb{R}^{\mathbb{N}}$ be non-decreasing and bounded, and let $r > 0$. Since $x$ is a bounded sequence, then $x(\mathbb{N})$ is a bounded set in $\mathbb{R}$. Hence its supremum $x' \in \mathbb{R}$ exists and $x(\mathbb{N}) \cap B_r(x') \neq \phi$. Let $x'' \in x(\mathbb{N}) \cap B_r(x')$. Since $x'' \in x(\mathbb{N})$, there exists $n \in \mathbb{N}$ such that $x(n) = x''$. Since $x$ is non-decreasing and $x'$ is the supremum of $x(\mathbb{N})$, then $x(n) \leq x(n') \leq x'$ for all $n' \in \mathbb{N}$ such that $n \leq n'$. Thus, since $x(n), x' \in B_r(x')$, then $x(n') \in B_r(x')$ for all $n' \in \mathbb{N}$ such that $n \leq n'$, i.e. $x(\mathbb{N}) \cap B_r(x')^C$ is finite. Therefore $x$ is convergent and $x'$ is its limit.

(Analogous proof for the other case.)

Q.E.D.

Another useful fact associated with real sequences is that one can always extract monotone sequences from any sequence of real numbers. If we combine this fact with the fact that the boundedness of a monotone sequence implies that it converges, we are able to show that it's possible to extract a convergent subsequence from any bounded sequence. This of course guarantees the existence of at least one cluster point for the sequence.

**E26.** *Every sequence in $\mathbb{R}$ has either a non-increasing or a non-decreasing subsequence, or both.*

*Proof.* Let $x \in \mathbb{R}^{\mathbb{N}}$ and $S \subset \mathbb{N}$ be such that $n \in S$ if, and only if, $x(n) \geq x(n')$ for all $n' > n$. Either $S$ is finite or not.

If $S$ is finite let $n$ be its maximum. Let $r \in \mathbb{N}^{\mathbb{N}}$ be such that $r(1) = n + 1$. Since $n + 1 \notin S$, then there exists $n_1 > n + 1$ such that $x(n + 1) < x(n_1)$. Let $r(2) = n_1$. Since $n_1 \notin S$, then there exists $n_2 > n + 1$ such that $x(n_1) < x(n_2)$. Let $r(3) = n_2$, and so on. Therefore $x(r(1)) \leq x(r(2)) \leq x(r(3)) \leq \dots$, i.e. the sequence $x \circ r$, a subsequence of $x$ since $r$ is increasing, is non-decreasing.

If, on the contrary, $S$ is not finite, then let $r \in \mathbb{N}^{\mathbb{N}}$ be such that $r(n)$ is the $n$-th smallest integer in $S$. This implies that the function $r$ is increasing, i.e. if $n < n'$, then $r(n) < r(n')$ and hence $x(r(n)) \geq x(r(n'))$. Therefore the sequence $x \circ r$, a subsequence of $x$ since $r$ is increasing, is non-increasing. Q.E.D.

**E27.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

*Proof.* Let $x \in \mathbb{R}^{\mathbb{N}}$ be bounded. Then either $x$ has a non-decreasing bounded subsequence and hence a convergent subsequence, or $x$ has a non-increasing bounded subsequence and hence a convergent subsequence. Q.E.D.

**E28.** *Every bounded sequence in $\mathbb{R}$ has a cluster point.*

*Proof.* Let $x \in \mathbb{R}^{\mathbb{N}}$ be bounded. Then there is a convergent subsequence of $x$ whose limit is a cluster point of the subsequence, and hence of the sequence too. Q.E.D.

The next proposition states that if from some term onwards, one convergent sequence of real numbers is always greater than another convergent sequence, then the same must be true for their limits.

**E29.** *If $x, y \in \mathbb{R}^{\mathbb{N}}$ are convergent to $x', y'$ respectively and there is some integer $n$ such that for all $n' > n$, $x(n') \leq y(n')$, then $x' \leq y'$.*

*Proof.* Let $x, y \in \mathbb{R}^{\mathbb{N}}$ be convergent to $x', y'$ respectively and such that there is $n \in \mathbb{N}$ for which if $n' > n$, then $x(n') \leq y(n')$, and let $r > 0$.

Since $x'$ is the limit of $x$, then $x^{-1}(B_r(x')^C)$ is finite, and similarly since $y'$ is the limit of $y$, then $y^{-1}(B_r(y')^C)$ is finite as well. Let $n_x = \max x^{-1}(x(B_r(x')^C)$ and $n_y = \max y^{-1}(y(B_r(y')^C)$.

Let $n_0 = \max\{n, n_x, n_y\}$. Then, for all $n' > n_0$, $x(n') \in B_r(x')$ and $y(n') \in B_r(y')$, i.e.

$$\begin{aligned} x' - r < x(n') < x' + r \\ y' - r < y(n') < y' + r. \end{aligned}$$

Since for any $n' > n_0$, $x' - y' = x' - x(n') + x(n') - y(n') + y(n') - y'$, and $x' - x(n') < r$, $y(n') - y' < r$ and $x(n') - y(n') \leq 0$, then

$$\begin{aligned} x' - y' &= x' - x(n') + x(n') - y(n') + y(n') - y' \\ &< 2r + x(n') - y(n') \\ &\leq 2r. \end{aligned}$$

Therefore $x' - y' < 2r$ for all $r > 0$ and hence $x' - y' \leq 0$ (in effect, assume $x' - y' > 0$ and let $r = \frac{1}{2}(x' - y')$, then $x' - y' < 2\frac{1}{2}(x' - y') = x' - y'!$), i.e. $x' \leq y'$. Q.E.D.

**E30.** *If $x, y \in \mathbb{R}^{\mathbb{N}}$ converge respectively to $x'$ and $y'$, then*

(1) *the sequence $x + y$ converges to $x' + y'$,*
(2) *the sequence $xy$ converges to $x'y'$,*

*Proof.*

(1) Let $x, y \in \mathbb{R}^{\mathbb{N}}$ and $r > 0$. Since $x$ converges to $x'$, we know that $x^{-1}(B_r(x')^C)$ is finite. Let $n_x$ be the greatest number in $x^{-1}(B_r(x')^C)$.

Similarly, since $y$ converges to $y'$, we know that $y^{-1}(B_r(y')^C)$ is finite. Let $n_y$ be the greatest number in $y^{-1}(B_r(y')^C)$.

Now, let $n = \max\{n_x, n_y\}$. Then for all $n' \geq n$,

$$\begin{aligned} x' - r < x(n') < x' + r \\ y' - r < y(n') < y' + r \end{aligned}$$

and hence

$$x' + y' - 2r < x(n') + y(n') < x' + y' + 2r$$

i.e. $x(n') + y(n') \in B_r(x' + y')$. Therefore $(x + y)^{-1}(B_r(x' + y')^C)$ is finite, for any $r > 0$, i.e. $x + y$ converges to $x' + y'$

(2) Let $x, y \in \mathbb{R}^{\mathbb{N}}$ and $r > 0$. Since $x$ is a convergent sequence, we know it is bounded. Now let $R \geq 0$ be such that $|x(n)| < R$ for all $n \in \mathbb{N}$. Either $y' \neq 0$ or $y' = 0$.

Assume $y' \neq 0$. Since $y$ converges to $y'$, then $y^{-1}(y(B_{\frac{r}{2R}}(y')^C)$ is finite. Let $n_y$ be the greatest number in $y^{-1}(B_{\frac{r}{2R}}(y'))$, i.e. such that for all $n' > n_y$, $|y(n') - y'| < \frac{r}{2R}$.

Similarly, since $x$ converges to $x'$, then $x^{-1}(B_{\frac{r}{2R}}(x')^C)$ is finite. Let $n_x$ be the greatest number in $x^{-1}(B_{\frac{r}{2R}}(x'))$, i.e. such that for all $n' > n_x$, $|x(n') - x'| < \frac{r}{2|y|}$.

Let $n = \max\{n_x, n_y\}$. Then for all $n' > n$,

$$\begin{aligned} |x(n)y(n) - x'y'| &= |x(n)y(n) - x(n)y' + x(n)y' - x'y'| \\ &\leq |x(n)||y(n) - y'| + |x(n) - x'||y'| \\ &< R\frac{r}{2R} + |y|\frac{r}{2|y|} \\ &= r. \end{aligned}$$

Therefore $(xy)^{-1}(B_r(x'y'))$ is finite for any $r > 0$, i.e. the sequence $xy$ converges to $x'y'$.

Q.E.D.

**E31.** *If* $y \in \mathbb{R}^{\mathbb{N}}$ *converges to a positive (negative) limit* $y'$, *then at most a finite number of terms of the sequence are not positive (negative).*

*Proof.* Since $y$ converges to $y'$ and $|y'| > 0$, then $y^{-1}(B_{|y'|}(y')^C)$ is finite. Let $n = \max y^{-1}(B_{|y'|}(y')^C)$. Then $y' - |y'| < y(n') < y' + |y'|$ for all $n' > n$. If $y' > 0$, then $y' - |y'| = 0$ and hence $0 < y(n')$ for all $n' > n$. If $y' < 0$, then $y' + |y'| = 0$ and hence $y(n') < 0$ for all $n' > n$. Q.E.D.

**E32.** *If* $y \in \mathbb{R}^{\mathbb{N}}$ *converges to* $y'$, *then the sequence* $|y|$ *converges to* $|y'|$.

*Proof.* Let $y \in \mathbb{R}^{\mathbb{N}}$ converge to $y'$ and $r > 0$. Either $y' > 0$, $y' < 0$ or $y' = 0$.

Assume $y' > 0$. Since $y$ converges to $y'$, there is $n_0 \in \mathbb{N}$ such that $y(n') > 0$ for all $n' > n_0$, and $y^{-1}(B_r(y')^C)$ is finite, i.e. there is $n_1 \in \mathbb{N}$ such that $y' - r < y(n') < y' + r$ for all $n' > n_1$. Let $n = \max\{n_0, n_1\}$. Then, for all $n' > n$, $|y'| - r < |y(n')| < |y'| + r$ . Therefore $|y|^{-1}(B_r(|y'|)^C)$ is finite for all $r > 0$, i.e. the sequence $|y|$ converges to $|y'|$.

Assume $y' < 0$. Since $y$ converges to $y'$, there is $n_0 \in \mathbb{N}$ such that $y(n') < 0$ for all $n' > n_0$, and $y^{-1}(B_r(y'))$ is finite, i.e. there is $n_1 \in \mathbb{N}$ such that $y' - r < y(n') <$

$y'+r$, i.e. $-y'-r < -y(n') < -y'+r$ for all $n' > n_1$. Let $n = \max\{n_0, n_1\}$. Then, for all $n' > n$, $|y'| - r < |y(n')| < |y'| + r$ . Therefore $|y|^{-1}(B_r(|y'|)^C)$ is finite for all $r > 0$, i.e. the sequence $|y|$ converges to $|y'|$.

Assume, finally, $y' = 0$. Since $y$ converges to $y'$, $y^{-1}(B_r(0))$ is finite, i.e. there is $n \in \mathbb{N}$ such that $-r < y(n') < r$, i.e. $-r < 0 \geq |y(n')| < r$ for all $n' > n_0$. Therefore $|y|^{-1}(B_r(|y'|)^C)$ is finite for all $r > 0$, i.e. the sequence $|y|$ converges to $|y'|$. Q.E.D.

**E33.** *If $y \in \mathbb{R}^{\mathbb{N}}$ converges to $y'$, for all $n \in \mathbb{N}$, $y(n) \neq 0$, and $y' \neq 0$, then the sequence $\frac{1}{y}$ converges to $\frac{1}{y'}$,*

*Proof.* Let $y \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ converge to $y' \neq 0$ and $r > 0$. On the one hand, since $y$ converges to $y'$, we know that $y^{-1}(B_{\frac{1}{2}|y'|^2 r}(y')^C)$ is finite. Let $n_0 = \max y^{-1}(B_{\frac{1}{2}|y'|^2 r}(y')^C)$, i.e.

$$|y' - y(n')| < \frac{|y'|^2 r}{2}$$

for all $n' > n_0$.

On the other hand, since $y$ converges to $y'$, we know that $|y|$ converges to $|y'|$, i.e. $|y|^{-1}(B_{\frac{1}{2}|y'|}(|y'|)^C)$ is finite. Let $n_1 = \max |y|^{-1}(B_{\frac{1}{2}|y'|}(|y'|)^C)$. Then $|y'| - \frac{1}{2}|y'| < |y(n')| < |y'| - \frac{1}{2}|y'|$, and more specifically $\frac{1}{2}|y'| < |y(n')|$, for all $n' > n_1$, i.e.

$$\frac{1}{|y(n')||y'|} < \frac{2}{|y'|^2}$$

for all $n' > n_1$.

Let $n = \max\{n_0, n_1\}$. Then

$$\frac{|y' - y(n')|}{|y(n')||y'|} < \frac{2}{|y'|^2}\frac{|y'|^2 r}{2}$$

or, equivalently,

$$|\frac{1}{y(n')} - \frac{1}{y'}| < r$$

for all $n' > n$. Therefore $(\frac{1}{y})^{-1}(B_r(\frac{1}{y'})^C)$ is finite for all $r > 0$, i.e. the sequence $\frac{1}{y}$ converges to $\frac{1}{y'}$. Q.E.D.

**E34.** *If a sequence $x \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ converges to $0$, then the sequence $\frac{1}{x}$ is not bounded.*

*Proof.* Let $x \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ converge to 0 and $r > 0$. Then $x^{-1}(B_{\frac{1}{r}}(0)^C)$ is finite, i.e. there is $n \in \mathbb{N}$ such that, for all $n' > n$, either $-\frac{1}{r} < x(n') < 0$ or $0 < x(n') < \frac{1}{r}$, i.e. either $\frac{1}{x(n')} < -r$ or $r < \frac{1}{x(n')}$. Therefore $\frac{1}{x}(\mathbb{N}) \not\subset B_r(0)$ for any $r > 0$, i.e. the sequence $\frac{1}{x}$ is not bounded. Q.E.D.

**E35.** *A sequence $x \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ converges to $0$ if, and only if, any ball contains only finitely many terms of the sequence $\frac{1}{x}$ or, equivalently, $(\frac{1}{x})^{-1}(B_r(0))$ is finite, for all $r > 0$.*

*Proof.* Let $x \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ converge to $0$ and $r > 0$. Then $x^{-1}(B_{\frac{1}{r}}(0)^C)$ is finite. Let $n = \max x^{-1}(B_{\frac{1}{r}}(0)^C)$. Then, for all $n' > n$, either $-\frac{1}{r} < x(n') < 0$ or $0 < x(n') < \frac{1}{r}$, i.e. either $\frac{1}{x(n')} < -r < 0$ or $0 < r < \frac{1}{x(n')}$. Therefore, $(\frac{1}{x})^{-1}(B_r(0))$ contains at most $n$ integers, for all $r > 0$.

Conversely, let $x \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ be such that $(\frac{1}{x})^{-1}(B_{\frac{1}{r}}(0))$ is finite, for all $r > 0$. Let $r > 0$ and $n = \max(\frac{1}{x})^{-1}(B_{\frac{1}{r}}(0))$. Then $-\frac{1}{r} < \frac{1}{x(n')} < \frac{1}{r}$, for all $n' > n$, i.e. either $-r < x(n') < 0$ or $0 < x(n') < r$, for all $n' > n$. Therefore $x^{-1}(x(\mathbb{N}) \cap B_r(0)^C)$ is finite, for all $r > 0$, i.e. the sequence $x$ converges to $0$. Q.E.D.

**E36.** *A sequence $x$ in the metric space $(\mathbb{R}^n, d_p)$ is convergent to $x'$ if, and only if, all its coordinate sequences $x_i$ are convergent to ${x'}_i$ respectively.*

*Proof.* Let $x$ converge to $x'$ in $(\mathbb{R}^n, d_p)$ and let $r > 0$. Then $x^{-1}(B_r(x')^C)$ is finite. Let $m = \max x^{-1}(B_r(x')^C)$, then, for all $i = 1, \ldots, n$,

$$\begin{aligned}
|x_i(m') - {x'}_i| &= \left(|x_i(m') - {x'}_i|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^{n} |x_i(m') - {x'}_i|^p\right)^{\frac{1}{p}} \\
&< r
\end{aligned}$$

for all $m' > m$. Therefore, for all $i = 1, \ldots, n$, $x_i^{-1}(B_r({x'}_i)^C)$ is finite for all $r > 0$, i.e. the sequence $x_i$ converges to $x'_i$.

Conversely, let $x$ be such that, for all $i = 1, \ldots, n$, the sequence $x_i$ converges to $x'_i$ and let $r > 0$. Then, for all $i = 1, \ldots, n$, $x_i^{-1}(B_r({x'}_i)^C)$ is finite and letting $m_i = \max x_i^{-1}(B_{\frac{r}{n^{\frac{1}{p}}}}({x'}_i)^C)$, then $|x_i(m') - x_i| < \frac{r}{n^{\frac{1}{p}}}$ for all $m' > m_i$. Let $m = \max\{m_1, \ldots, m_n\}$. Then, for all $i = 1, \ldots, n$,

$$\begin{aligned}
\left(|x_i(m') - {x'}_i|^p\right)^{\frac{1}{p}} &= |x_i(m') - x_i| \\
&< \frac{r}{n^{\frac{1}{p}}}
\end{aligned}$$

that is to say,

$$|x_i(m') - {x'}_i|^p < \frac{r^p}{n}$$

and hence

$$\sum_{i=1}^{n} |x_i(m') - {x'}_i|^p < r^p$$

that is to say

$$\left(\sum_{i=1}^{n} |x_i(m') - {x'}_i|^p\right)^{\frac{1}{p}} < r.$$

Therefore $x^{-1}(B_r(x')^C)$ is finite for all $r > 0$, where $x' = (x'_1, \ldots, x'_n)$, i.e. $x$ converges to $x'$. Q.E.D.

**E37.** *The sequences in the reals with the usual metric whose $n$-th terms are respectively $\frac{1}{n}$ and $\frac{1}{\sqrt{n}}$ are both convergent and their limits are $0$.*

*Proof.*

(1) Let $r \in \mathbb{R}_{++}$. If $n \in \mathbb{N}$ is such that $n > \frac{1}{r}$, then $\frac{1}{n} < r$, i.e. $\frac{1}{n} \in B_r(0)$. Therefore at most $[\frac{1}{r}]$ terms of the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ are not in $B_r(0)$ and hence it is convergent and $0$ is its limit.
(2) Let $r \in \mathbb{R}_{++}$. If $n \in \mathbb{N}$ is such that $n > \frac{1}{r^2}$, then $\frac{1}{\sqrt{n}} < r$, i.e. $\frac{1}{\sqrt{n}} \in B_r(0)$. Therefore at most $[\frac{1}{r^2}]$ terms of the sequence $\{\frac{1}{\sqrt{n}}\}_{n\in\mathbb{N}}$ are not in $B_r(0)$ and hence it is convergent and $0$ is its limit.

Q.E.D.

*Separable metric spaces.*

**E38.**

(1) *For all $n \in \mathbb{N}$ and all $p \geq 1$, $\ell_p^n$ is separable.*
(2) *For all $p \geq 1$, $\ell_p$ is separable.*
(3) *$\ell_\infty$ is not separable.*
(4) *The metric space $C[a,b]$ of continuous real valued functions on the interval $[a,b]$ of $\mathbb{R}$ with the sup metric $d_\infty$ is separable.*

*Proof.* Exercise.

**E39.** *A discrete metric space is separable iff it is countable.*

*Proof.* Exercise.

**E40.** *Any metric subspace of a separable metric space is separable itself.*

*Proof.* Exercise.

**E41.** *If the metric space $(X,d)$ is separable, then there exists a countable set $\mathcal{A}$ of open sets of $X$ such that for all $A \subset X$ open,*

$$A = \cup_{A'\in\mathcal{A}|A'\subset A} A'.$$

*Proof.* Exercise.

*Connected metric spaces.*

**E42.** *A discrete metric space is connected if, and only if, it has only one element.*

*Proof.* Let $(X,d)$ be a connected metric space and $x, x' \in X$. Assume $x \neq x'$. Since every set of a discrete metric space is both open and closed, then $\{x\}$ is non empty, open and $\{x\}^C$ is nonempty (since $x' \notin \{x\}$ because $x' \neq x$), open (since $\{x\}$ is closed as well) and $\{x\} \cup \{x\}^C = X$! Therefore $x = x'$ necessarily.

Conversely, if $X$ has only one element, then the only sets of the space are $\phi$ and $X$. The conclusion follows trivially. Q.E.D.

*Compact metric spaces.*

Every compact set is closed and bounded but the converse is not true in general, as the next example shows.

**E43.** *No discrete metric space $(X, d)$ for which $X$ is infinite can be compact, while every discrete metric space is both closed and bounded.*

*Proof.* Let $(X, d)$ be a discrete metric space such that $X$ is not finite. Since $X$ is not finite, there exists an injective sequence $s \in \mathbb{X}^{\mathbb{N}}$. Let $h \in \mathbb{N}^{\mathbb{N}}$ be increasing and $s \circ h$ be a subsequence of $s$. Since $s$ is injective and $h$ is increasing, then $s \circ h$ is injective too, i.e. $(s \circ h)(n) \neq (s \circ h)(n')$ if $n \neq n'$ and hence $d((s \circ h)(n), (s \circ h)(n')) = 1$ if $n \neq n'$. Let $x \in X$ and $0 < r < 1$. Either $(s \circ h)(\mathbb{N}) \cap B_r(x)$ is $\phi$ or not. If it is empty, $s \circ h$ does not converge to $x$. If it is not empty, and $(s \circ h)(n), (s \circ h)(n') \in (s \circ h)(\mathbb{N}) \cap B_r(x)$ then $d((s \circ h)(n), (s \circ h)(n')) < r < 1$ and hence $n = n'$, i.e. $(s \circ h)(n) = (s \circ h)(n')$. Therefore $(s \circ h)(\mathbb{N}) \cap B_r(x)$ is a singleton, i.e. $(s \circ h)^{-1}(B_r(x))$ is a singleton. Thus $(s \circ h)^{-1}(B_r(x)^C)$ is not finite and hence $(s \circ h)$ does not converge to $x$, i.e. $s$ has no convergent subsequence and thus $X$ is not compact.

Nevertheless, $X$ is clearly closed and bounded as well, since $X \subset B_r(x)$ for all $x \in X$ and $r > 1$. Q.E.D.

**E44.** *The metric space $(X, d)$ where $X$ is the set of all sequences $x_n \in \mathbb{R}^{\mathbb{N}}$ satisfying, for all $n \in \mathbb{N}$, $x_n(n') = 1$ if $n' = n$, 0 otherwise, is not compact even though $X$ is both closed and bounded.*

*Proof.* Let $x_n, x_{n'} \in X$. Then

$$\begin{aligned} d_p(x_n, x_{n'}) &= \big(\sum_{m=1}^{\infty} |x_n(m) - x_{n'}(m)|^p\big)^{\frac{1}{p}} \\ &= \begin{cases} 0 & \text{if } n = n' \\ 2^{\frac{1}{p}} & \text{if } n \neq n'. \end{cases} \end{aligned}$$

Therefore $(X, d_p)$ is a discrete metric space with $X$ not finite. The conclusion follows from the previous result. Q.E.D.

*Complete metric spaces.*

**E45.**

(1) *$\ell_1^1$ is a complete metric space.*
(2) *For all $p \geq 1$, $\ell_p^n$ is a complete metric space.*
(3) *For all $p \geq 1$, $\ell_p$ is a complete metric space.*
(4) *$\ell_\infty$ is a complete metric space.*
(5) *The space $C[0, 1]$ of continuous real valued functions defined on the interval $[0, 1]$, with the metric $d_1$, is not complete.*
(6) *The space $C[0, 1]$ of continuous real valued functions defined on the interval $[0, 1]$, with the metric $d_\infty$, is a complete metric space.*
(7) *The space $D[a, b]$ of differentiable real valued functions on an interval $[a, b]$ with the metric $d_\infty$ is complete.*
(8) *The space $C^1[a, b]$ of continuously differentiable real valued functions on an interval $[a, b]$ with the metric $d_\infty$ is not complete.*

*Proof.*

(1) Let $x \in \mathbb{R}^{\mathbb{N}}$ be a Cauchy sequence in $\ell_1^1$. Since it is a Cauchy sequence, then it is also a bounded sequence. Since every bounded sequence of real numbers has a convergent subsequence, then the sequence $x$ itself is necessarily convergent too and to the same limit.

(2) Let $x \in (\mathbb{R}^n)^{\mathbb{N}}$ be a Cauchy sequence in $\ell_p^n$. Then for any real $r > 0$, there is $K \in \mathbb{N}$ such that $d_p(x(k), x(k')) < r$, i.e.

$$\left(\sum_{i=1}^{n} |x_i(k) - x_i(k')|^p\right)^{\frac{1}{p}} < r$$

for all for all $k, k' > N$. Hence, for all $i = 1, \ldots, n$,

$$\begin{aligned} |x_i(k) - x_i(k')| = \\ \left(|x_i(k) - x_i(k')|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^{n} |x_i(k) - x_i(k')|^p\right)^{\frac{1}{p}} \\ &< r, \end{aligned}$$

that is to say each coordinate sequence $x^i \in \mathbb{R}^{\mathbb{N}}$ is a Cauchy sequence in $\ell_1^1$. Since $\ell_1^1$ is complete, then each coordinate sequence $x_i$ is convergent. If $a_i$ is the limit of $x_i$, for each $i = 1, \ldots, n$, then the sequence $x$ is convergent and $(a_1, \ldots, a_n)$ is its limit.

(3) Let $x \in (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ be a Cauchy sequence in $\ell_p$ and $r > 0$. Since $x$ is Cauchy, there exists $n \in \mathbb{N}$ such that, for all $n', n'' > n$, $d_p(x(n'), x(n'')) < \frac{1}{2}r$, i.e.

$$\left(\sum_{i=1}^{\infty} |x_i(n') - x_i(n'')|^p\right)^{\frac{1}{p}} < \frac{1}{2}r,$$

where the series is convergent because $x(n'), x(n'') \in \ell_p$, and hence both series $\sum_{i=1}^{\infty} |x_i(n')|^p$ and $\sum_{i=1}^{\infty} |x_i(n'')|^p$ are convergent, and because of Minkowski's inequality,

$$\left(\sum_{i=1}^{\infty} |x_i(n') - x_i(n'')|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |x_i(n')|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |x_i(n'')|^p\right)^{\frac{1}{p}}.$$

Since, for all $i \in \mathbb{N}$ and all $n', n'' > n$

$$\begin{aligned} |x_i(n') - x_i(n'')| = \\ \left(|x_i(n') - x_i(n'')|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^{\infty} |x_i(n') - x_i(n'')|^p\right)^{\frac{1}{p}} \\ &< \frac{1}{2}r \\ &< r, \end{aligned}$$

then, for all $i \in \mathbb{N}$, $x_i \in \mathbb{R}^{\mathbb{N}}$ is Cauchy in $\ell_1^1$. Since $\ell_1^1$ is complete, then, for all $i \in \mathbb{N}$, $x_i$ is convergent to a limit $a_i$.

Since, for all $k \in \mathbb{N}$ and all $n', n'' > n$,

$$\begin{aligned} \left(\sum_{i=1}^{k} |x_i(n') - x_i(n'')|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^{\infty} |x_i(n') - x_i(n'')|^p\right)^{\frac{1}{p}} \\ &< \frac{1}{2}r \end{aligned}$$

and $f_k\colon \mathbb{R}^k \to \mathbb{R}$ such that $f_k(z) = \left(\sum_{i=1}^{k} |x_i(n') - z_i|^p\right)^{\frac{1}{p}}$ is continuous and, for all $i \in \mathbb{N}$, $\lim x_i = a_i$, then

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty} |x_i(n') - a_i|^p\right)^{\frac{1}{p}} = \\
&\lim_{k\to\infty}\left(\sum_{i=1}^{k} |x_i(n') - a_i|^p\right)^{\frac{1}{p}} = \lim_{k\to\infty}\left(\lim_{n''\to\infty}\left(\sum_{i=1}^{k} |x_i(n') - x_i(n'')|^p\right)^{\frac{1}{p}}\right) \\
&\qquad \le \frac{1}{2}r \\
&\qquad < r
\end{aligned}$$

and hence $a = \lim x$. Moreover, since, for all $n' \in \mathbb{N}$,

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} |a_i|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{\infty} |a_i - x_i(n') + x_i(n')|^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{i=1}^{\infty} |a_i - x_i(n')|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |x_i(n')|^p\right)^{\frac{1}{p}} \\
&\le r + \left(\sum_{i=1}^{\infty} |x_i(n')|^p\right)^{\frac{1}{p}}
\end{aligned}$$

then $a \in \ell_p$, i.e. $\lim x \in \ell_p$. Therefore, since for every Cauchy sequence $x$ in $\ell_p$, $x$ is convergent and $\lim x \in \ell_p$, then $\ell_p$ is complete.

(4) Let $x \in (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ be a Cauchy sequence in $\ell_\infty$ and $r > 0$. Since $x$ is Cauchy, there exists $n \in \mathbb{N}$ such that, for all $n', n'' > n$, $d_\infty(x(n'), x(n'')) < \frac{1}{2}r$, i.e.

$$\sup_{i\in\mathbb{N}} |x_i(n') - x_i(n'')| < \frac{1}{2}r,$$

where the supremum exists because $x(n'), x(n'') \in \ell_\infty$, and hence both $\sup_{i\in\mathbb{N}} |x_i(n')|$ and $\sup_{i\in\mathbb{N}} |x_i(n'')|$ exist, and because for all $i \in \mathbb{N}$,

$$\begin{aligned}
|x_i(n') - x_i(n'')| &\le |x_i(n')| + |x_i(n'')| \\
&\le \sup_{i\in\mathbb{N}} |x_i(n')| + \sup_{i\in\mathbb{N}} |x_i(n'')|.
\end{aligned}$$

Since, for all $i \in \mathbb{N}$ and all $n', n'' > n$

$$\begin{aligned}
|x_i(n') - x_i(n'')| &\le \sup_{i\in\mathbb{N}} |x_i(n') - x_i(n'')| \\
&< \frac{1}{2}r \\
&< r,
\end{aligned}$$

then, for all $i \in \mathbb{N}$, $x_i \in \mathbb{R}^{\mathbb{N}}$ is Cauchy in $\ell_1^1$. Since $\ell_1^1$ is complete, then, for all $i \in \mathbb{N}$, $x_i$ is convergent to a limit $a_i$.

Since, for all $k \in \mathbb{N}$ and all $n', n'' > n$,

$$\begin{aligned}\sup_{i\in\{1,\ldots,k\}} |x_i(n') - x_i(n'')| &\leq \sup_{i\in\mathbb{N}} |x_i(n') - x_i(n'')| \\ &< \frac{1}{2}r\end{aligned}$$

and $f_k \colon \mathbb{R}^k \to \mathbb{R}$ such that $f_k(z) = \sup_{i\in\{1,\ldots,k\}} |x_i(n') - z_i|$ is continuous and, for all $i \in \mathbb{N}$, $\lim x_i = a_i$, then

$$\begin{aligned}&\sup_{i\in\mathbb{N}} |x_i(n') - a_i| = \\ &\lim_{k\to\infty} \sup_{i\in\{1,\ldots,k\}} |x_i(n') - a_i| = \lim_{k\to\infty} \bigl(\lim_{n''\to\infty} \sup_{i\in\{1,\ldots,k\}} |x_i(n') - x_i(n'')|\bigr) \\ &\qquad\qquad \leq \frac{1}{2}r \\ &\qquad\qquad < r\end{aligned}$$

and hence $a = \lim x$. Moreover, since, for all $n' \in \ltimes$,

$$\begin{aligned}\sup_{i\in\mathbb{N}} |a_i| &= \sup_{i\in\mathbb{N}} |a_i - x_i(n') + x_i(n')| \\ &\leq \sup_{i\in\mathbb{N}} |a_i - x_i(n')| + \sup_{i\in\mathbb{N}} |x_i(n')| \\ &\leq r + \sup_{i\in\mathbb{N}} |x_i(n')|\end{aligned}$$

then $a \in \ell_\infty$, i.e. $\lim x \in \ell_\infty$. Therefore, since for every Cauchy sequence $x$ in $\ell_\infty$, $x$ is convergent and $\lim x \in \ell_\infty$, then $\ell_\infty$ is complete.

(5) Exercise.
(6) Exercise.
(7) Exercise.
(8) Exercise.

Q.E.D.

The completeness of $\ell_\infty$ is also a consequence of the completeness, with respect to the sup metric, of the space of bounded functions taking values in a complete metric space.

The completeness of $B[0,1]$, the space of bounded real-valued functions on the interval $[0,1]$, with $d_\infty$, as well as that of $\ell_\infty$, is a direct consequence of the completeness with respect to $d_\infty$ of, on the one hand, the space of bounded functions from a metric space to a complete metric space and, on the other hand, the completeness of the real line with the usual metric. The completeness of $C[0,1]$ with $d_\infty$ is a consequence of $C[0,1]$ being a closed subset of $B[0,1]$ and the fact that every closed set in a complete metric space is itself complete.

From the lack of completeness of $C^1[0,1]$ one can deduce that it is not a closed subset of either $B[0,1]$ or of $C[0,1]$.

## THEOREMS

*On metrics.*

The following property allows to characterize a metric as a function which satisfies two conditions, namely, (1) it must assign a value of zero as the distance from any point to itself and (2) it must satisfy the triangular inequality.

**S1. Complete characterization of metrics.** *If $d$ is a metric in $X$, then*

(1) *for all $x, x' \in X$, $d(x', x'') \in \mathbb{R}_+$, and*
(2) *for all $x, x' \in X$, $d(x, x') = d(x', x)$*

*Proof.* Exercise.

**S2. Metric induced by an injective function into a metric space on its domain.** *If $(Y, d_Y)$ is a metric space, $f \in Y^X$ is injective and such that $f^{-1}(Y) = X$, and $d_X$ is a real valued function with domain $X \times X$ such that $d_X(x, x') = d_Y(f(x), f(x'))$, then $d_X$ is a metric in $X$.*

*Proof.* Exercise.

*Metrics induced in the cartesian product of finitely many metric spaces.*

**S3.** *If $(X_i, d_{X_i})$, for all $i \in \{1, \dots, n\}$, are metric spaces, then the real valued function $d_{\times_{i=1}^n X_i, p}$ with domain $(\times_{i=1}^n X_i) \times (\times_{i=1}^n X_i)$ and such that for all $x, x' \in \times_{i=1}^n X_i$,*

$$d_{\times_{i=1}^n X_i, p}(x, x') = \Big(\sum_{i=1}^n d_{X_i}(x_i, x_i')^p\Big)^{\frac{1}{p}}$$

*is a metric in $\times_{i=1}^n X_i$, for all $p \geq 1$.*

*Proof.* Exercise.

**S4.** *If $(X_i, d_{X_i})$, for all $i \in \{1, \dots, n\}$, are metric spaces, then the real valued function $d_{\times_{i=1}^n X_i, p}$ with domain $(\times_{i=1}^n X_i) \times (\times_{i=1}^n X_i)$ and such that for all $x, x' \in \times_{i=1}^n X_i$,*

$$d_{\times_{i=1}^n X_i, \infty}(x, x') = \sup_{i \in \{1, \dots, n\}} \big\{d_{X_i}(x_i, x_i')\big\}$$

*is a metric in $\times_{i=1}^n X_i$.*

*Proof.* Exercise.

*Subspaces of metric spaces.*

If we restrict a space to any of its subsets and restrict the metric of the original space to this subset, the result is a metric space. Such a metric space created from a subset and the original metric are said to be a subspace of the original space.

**S5. Restriction of a metric to a subspace.** *If $(X, d)$ is a metric space and $X' \subset X$, then $(X', d')$, where $d'$ is the restriction of $d$ to $X' \times X'$,[13] is a metric space.*

*Proof.* Exercise

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[13] i.e. $d' = d \cap ((X' \times X') \times \mathbb{R})$.

*On closure, limit, interior points.*

The notion of metric generalizes our intuition about distances in physical space to more abstract spaces and allows us to define some different ways in which a point can relate to sets in an arbitrary space. In particular, a closure point for a given set is a point for which some element of the set (which may include the point itself) is within an arbitrarily short distance, while an

accumulation point is characterized by having a distinct (i.e. excluding the point itself) element of the set within an arbitrarily small distance.

An immediate consequence of this last definition is that, for every accumulation point of a given set, there are infinitely many elements of the set that are within any specified distance. The following proposition establishes this fact.

**S6. Complete characterization of accumulation points.** *If $(X, d)$ is a metric space, $A \subset X$ and $a \in A$, then $a$ is an accumulation point of $A$ if, and only if, for all $r > 0$, $B_r(a) \cap A$ is not finite.*

*Proof.* Assume $a$ is an accumulation point of $A$ and let $r > 0$. Then there exists $a_1 \in B_r(a) \cap A$ such that $B_r(a) \setminus \{a_1\} \cap A \neq \phi$. Thus let $a_1' \in B_r(a) \setminus \{a_1\} \cap A$. Either $d(a, a_1) > 0$ or $d(a, a_1') > 0$, or both.

If $d(a, a_1) > 0$ and $d(a, a_1') = 0$, let $x_1 = a_1$. If $d(a, a_1) = 0$ and $d(a, a_1') > 0$, let $x_1 = a_1'$. If $d(a, a_1) > 0$ and $d(a, a_1') > 0$, let $x_1 = \text{argmin}\{d(a, x) | x \in \{a_1, a_1'\}\}$.

Let $r_2 = \frac{1}{2}d(a, x_1)$. Since $a$ is an accumulation point of $A$, there exists $a_2 \in B_{r_2}(a) \cap A$ such that $B_{r_2}(a) \setminus \{a_2\} \cap A \neq \phi$. Thus let $a_2' \in B_r(a) \setminus \{a_2\} \cap A$. Either $d(a, a_2) > 0$ or $d(a, a_2') > 0$, or both.

If $d(a, a_2) > 0$ and $d(a, a_2') = 0$, let $x_2 = a_2$. If $d(a, a_2) = 0$ and $d(a, a_2') > 0$, let $x_2 = a_2'$. If $d(a, a_2) > 0$ and $d(a, a_2') > 0$, let $x_2 = \text{argmin}\{d(a, x) | x \in \{a_2, a_2'\}\}$. Since $d(a, x_2) < \frac{1}{2}d(a, x_1)$ and $d(a, x_2) > 0$, $d(a, x_1) > 0$, then $x_2 \neq x_1$. Proceeding this way, $B_r(a) \cap A$ contains the image of an injective sequence and therefore $B_r(a) \cap A$ is not finite.

Conversely, assume that, for any $r > 0$, $B_r(a) \cap A$ is not finite and $a$ is not an accumulation point of $A$. Then there exists $r > 0$ such that, for all $a_1 \in B_r(a) \cap A$, $B_r(a) \setminus \{a_1\} \cap A = \phi$ holds or, equivalently, there exists $r > 0$ such that, for all $a_1 \in B_r(a) \cap A$, $B_r(a) \cap A = \{a_1\}$ holds, i.e. there exists $r > 0$ such that, $B_r(a) \cap A$ is finite. Q.E.D.

Below we establish the equivalence of two characterizations of accumulation points: one that requires them to have within any positive distance, two distinct points of the set and another that requires them to have within any positive distance, infinitely many distinct points of the set. An immediate consequence of this equivalence is that finite sets cannot have accumulation points.

**S7. Every ball containing a point $x$ contains two distinct points of a set as well iff every ball containing $x$ contains infinitely many points of the set.** *If $(X, d)$ is a metric space, $A \subset X$ and $x \in X$, then, for all $r > 0$, $B_r(x) \cap A \neq \phi$ and is not a singleton if, and only if, for all $r > 0$, $B_r(x) \cap A$ is not finite.*

*Proof.* Exercise.

**S8. The derived set of every finite set is empty.** *If $(X, d)$ is a metric space and $A \subset X$ is finite, then the derived set of $A$ is empty.*

*Proof.* Exercise.

*On the closure and interior of a set.*

**S9. Every set is in its closure and contains its interior.** *If* $(X, d)$ *is a metric space and* $A \subset X$*, then*

(1) $A \subset \mathrm{Cl}A$*, and*
(2) $\mathrm{Int}A \subset A$*.*

*Proof.* Exercise.

**S10. The boundary of a set is what remains of its closure after removing its interior.** *If* $(X, d)$ *is a metric space and* $A \subset X$*, then* $\partial A = \mathrm{Cl}A \setminus \mathrm{Int}A$*.*

*Proof.* Exercise.

The closure of a set is obtained by adding to it every accumulation point of the set that is not already in it, i.e. by "closing" it with the points not in the set but immediately close to it. Thus the closure of a set must contain the closure of anything that was already in the set. Similarly, the interior of a set is obtained by removing from it any point that is not completely surrounded by points of the set. Thus the interior of a set must contain the interior of anything that was already in the set. That is what the next proposition establishes.

**S11.** *If* $(X, d)$ *is a metric space,* $A, B \subset X$ *and* $A \subset B$*, then*

(1) $\mathrm{Cl}A \subset \mathrm{Cl}B$ *and*
(2) $\mathrm{Int}A \subset \mathrm{Int}B$*.*

*Proof.* Exercise

**S12. Any closure is closed and any interior is open.** *If* $(X, d)$ *is a metric space and* $A \subset X$ *then,*

(1) $\mathrm{Cl}(\mathrm{Cl}A) = \mathrm{Cl}A$ *and hence the closure of every set is a closed set of the metric space, and*
(2) $\mathrm{Int}(\mathrm{Int}A) = \mathrm{Int}A$ *and hence the interior of every set is an open set of the metric space.*

*Proof.*

(1) Let $x \in \mathrm{Cl}(\mathrm{Cl}A)$ and let $r > 0$. Since $x$ is a closure point of $\mathrm{Cl}A$, then $B_r(x) \cap \mathrm{Cl}A \neq \phi$. Let $x' \in B_r(x) \cap \mathrm{Cl}A$ and let $s > 0$ be such that $s < r - d(x, x')$. Since $x' \in \mathrm{Cl}A$, then $B_s(x') \cap A \neq \phi$, and since $s < r - d(x, x')$, then $B_s(x') \subset B_r(x)$ (in effect, for any $x'' \in B_s(x')$, $d(x'', x') < s < r - d(x, x')$ holds, and hence $d(x'', x) \leq d(x'', x') + d(x, x') < r$, i.e. $x'' \in B_r(x)$). Therefore $B_r(x) \cap A \neq \phi$, for any $r > 0$, i.e. $x$ is a closure point of $A$

    Conversely, since every set of points of a metric space is contained in its closure $\mathrm{Cl}(\mathrm{Cl}A) \supset \mathrm{Cl}A$.

(2) In effect, since every set of points of a metric space contains its interior, then $\mathrm{Int}(\mathrm{Int}A) \subset \mathrm{Int}A$.

    Conversely, let $x \in \mathrm{Int}A$. Then there is $r > 0$ such that $B_r(x) \subset A$. Let $x' \in B_r(x)$ and $s > 0$ be such that $s < r - d(x, x')$. Then $B_s(x') \subset B_r(x)$ (in effect, for any $x'' \in B_s(x')$, $d(x'', x') < s < r - d(x, x')$ holds, and hence

$d(x'', x) \leq d(x'', x') + d(x, x') < r$, i.e. $x'' \in B_r(x)$), and hence $B_s(x') \subset A$, i.e. $x' \in \mathrm{Int}A$. Therefore $B_r(x) \subset \mathrm{Int}A$, and hence $x \in \mathrm{Int}(\mathrm{Int}A)$.

Q.E.D.

**S13. Distributivity of closure with respect to unions and of interior with respect to intersections.** *If* $(X, d)$ *is a metric space and* $A, B \subset X$*, then*

(1) $\mathrm{Cl}(A \cup B) = \mathrm{Cl}A \cup \mathrm{Cl}B$ *and*
(2) $\mathrm{Int}(A \cap B) = \mathrm{Int}A \cap \mathrm{Int}B$.

*Proof.*

(1) Let $x \notin \mathrm{Cl}A \cup \mathrm{Cl}B$. Since $x \notin \mathrm{Cl}A$, then there exists $r > 0$ such that $B_r(x) \cap A = \phi$. Since $x \notin \mathrm{Cl}B$, then there exists $s > 0$ such that $B_s(x) \cap B = \phi$. Let $t = \min\{r, s\}$. Since $B_t(x) \subset B_r(x)$ and $B_t(x) \subset B_s(x)$, then $B_t(x) \cap A = \phi$ and $B_t(x) \cap B = \phi$, and hence $B_t(x) \cap A \cup B = \phi$. Therefore $x \notin \mathrm{Cl}A \cup \mathrm{Cl}B$.

Conversely, since $A \subset A \cup B$ and $B \subset A \cup B$, then $\mathrm{Cl}A \subset \mathrm{Cl}(A \cup B)$ and $\mathrm{Cl}B \subset \mathrm{Cl}(A \cup B)$. Therefore $\mathrm{Cl}(A \cup B) \supset \mathrm{Cl}A \cup \mathrm{Cl}B$.

(2) Let $x \in \mathrm{Int}(A \cap B)$. Then there is $r > 0$ such that $B_r(x) \subset A \cap B$, i.e. $B_r(x) \subset A$ and $B_r(x) \subset B$. Therefore $x \in \mathrm{Int}A$ and $x \in \mathrm{Int}B$, i.e. $x \in \mathrm{Int}A \cap \mathrm{Int}B$.

Conversely, let $x \in \mathrm{Int}A \cap \mathrm{Int}B$. Then $x \in \mathrm{Int}A$, and hence there exists $r > 0$ such that $B_r(x) \subset A$, and $x \in \mathrm{Int}B$, and hence there exists $s > 0$ such that $B_s(x) \subset B$. Let $t = \min\{r, s\}$. Then $B_t(x) \subset A$ and $B_t(x) \subset B$, i.e. $B_t(x) \subset A \cap B$. Therefore $x \in \mathrm{Int}(A \cap B)$.

Q.E.D.

In contrast to the results of S13, the closure of an intersection of sets need not contain the intersection of set closures, nor is the interior of a union of sets necessarily contained in the union of the sets' interiors. Notwithstanding this contrast, the converse statements stated in S13 remain true.

**S14. Behavior of closure with respect to intersections and of interior with respect to unions.** *If* $(X, d)$ *is a metric space and* $A, B \subset X$*, then*

(1) $\mathrm{Cl}(A \cap B) \subset \mathrm{Cl}A \cap \mathrm{Cl}B$ *and*
(2) $\mathrm{Int}(A \cup B) \supset \mathrm{Int}A \cup \mathrm{Int}B$.

*Proof.* Exercise. Provide counterexamples of the reversed inclusions.

**S15. The empty set and the whole space are both open and closed.** *If* $(X, d)$ *is a metric space, then*

(1) $\mathrm{Int}\phi = \phi = \mathrm{Cl}\phi$ *so* $\phi$ *is both an open and a closed set of* $(X, d)$*, and*
(2) $\mathrm{Int}X = X = \mathrm{Cl}X$ *so* $X$ *is both an open and a closed set of* $(X, d)$.

*Proof.* Exercise.

*On open sets and closed sets.*

**S16. Every ball is open.** *If $(X,d)$ is a metric space, $x \in X$ and $r > 0$, then $B_r(x)$ is an open set of $(X,d)$.*

*Proof.* Let $(X,d)$ be a metric space, $x \in X$, $r > 0$ and $x' \in B_r(x)$. If $s < r - d(x,x')$, then $B_s(x') \subset B_r(x)$, since for any $x'' \in B_s(x')$, $d(x'',x') < s < r - d(x,x')$ and hence $d(x'',x) \leq d(x'',x') + d(x,x') < r$, i.e. $x'' \in B_r(x)$. Therefore, $B_r(x)$ is an open set of $(X,d)$. Q.E.D.

**S17. The interior of any given set is the biggest open within that set.** *If $(X,d)$ is a metric space and $A \subset X$, then* Int$A$ *is the maximum, by the partial order $\subset$, of the subset of open sets of $(X,d)$ that are included in $A$.*

*Proof.* Exercise.

**S18. The closure of a given set is the smallest closed set containing that set.** *If $(X,d)$ is a metric space and $A \subset X$, then* Cl$A$ *is the minimum, by the partial order $\subset$, of the subset of closed sets of $(X,d)$ that include $A$.*

*Proof.* Exercise.

**S19. A set is open iff its complement is closed.** *If $(X,d)$ is a metric space and $A \subset X$, then $A$ is an open set of $(X,d)$ if, and only if, $A^C$ is a closed set of $(X,d)$.*

*Proof.* Let $A$ be an open set in a metric space $(X,d)$ and let $x \in A$. Then there exists $r > 0$ such that $B_r(x) \subset A$, i.e. $B_r(x) \cap A^C = \phi$. Hence if $x \in A$, then $x$ is not a closure point of $A^C$, i.e. $A^C$ contains all its closure points, that is to say $\mathrm{Cl}A^C \subset A^C$. Therefore $\mathrm{Cl}A^C = A^C$, i.e. $A^C$ is closed.

Conversely, let $A$ not be open. Then there exists $x \in A$ such that for all $r > 0$, $B_r(x) \not\subset A$ holds, i.e. $B_r(x) \cap A^C \neq \phi$. Thus $x \in \mathrm{Cl}A^C$ and hence $\mathrm{Cl}A^C \not\subset A^C$. Therefore $A^C$ is not closed. Q.E.D.

An alternative to the paradigm that measures the closeness of objects in abstract spaces by defining a metric, is one that relies instead on providing a description of what constitutes an open set in such spaces. Since points contained in an appropriately specified open set (e.g. a "neighborhood") of a given point can in a sense be considered to be "close" to that point, we might guess that the manner in which the open sets of a space are defined is in some way related to the concept of spatial proximity. It turns out that this is indeed the case, and it is this fact that makes this alternative paradigm (a topological approach) valid. When the open sets of a space are defined in a way that satisfies certain properties (described below), the resulting collection of open sets is called a **topology** for the space. It follows that a **topological space** consists of a set and one of its (potentially many) topologies. In order to give a sense of the relationship between the metrics and topologies, we note that while every metric induces a topology, not every topology is induced by a metric, i.e. not every topology is metrizable. In what follows, we will stick to metrics rather than general topologies.

The next two propositions show that the empty set ($\phi$) and the entire space ($X$) are simultaneously open and closed. If these two sets are the only sets that are both open and closed, then the space is a connected metric space.

**S20. Topological properties of open sets.** *If $(X,d)$ is a metric space, then*

(1) *both $\phi$* **and** *$X$* **are open** *sets of $(X,d)$,*

(2) **any union of open sets is an open set**, *i.e. if* $\mathcal{A} \subset \mathcal{P}(X)$ *is such that for all* $A \in \mathcal{A}$*,* $A$ *is an open set of* $(X,d)$*, then* $\cup_{\mathcal{A}} A$ *is an open set of* $(X,d)$*,*

(3) **any finite intersection of open sets is an open set** *i.e. if* $\mathcal{A} \subset \mathcal{P}(X)$ *is such that for all* $A \in \mathcal{A}$*,* $A$ *is an open set of* $(X,d)$ *and* $\mathcal{A}$ *is finite, then* $\cap_{\mathcal{A}} A$ *is an open set of* $(X,d)$*.*

*Proof.*

(1) If $x \in X$ and $r > 0$, then $B_r(x) \subset X$ holds trivially. Therefore $X$ is open.

If $x \in \phi$ and $r > 0$, then $B_r(x) \subset \phi$, since anything can claimed from a false proposition.[14] Therefore $\phi$ is open.

(2) Let $\mathcal{A}$ be a set of open sets in $(X,d)$ and let $x \in \cup_{A\in\mathcal{A}} A$. Then there exists $A_0$ such that $x \in A_0$. Since $A_0$ is open, then there exists $r > 0$ such that $B_r(x) \subset A_0$. Since $A_0 \subset \cup_{A\in\mathcal{A}} A$, then $B_r(x) \subset \cup_{A\in\mathcal{A}} A$.

(3) Let $\mathcal{A}$ be a finite set of open sets in $(X,d)$ and let $x \in \cap_{A\in\mathcal{A}} A$. Then, for each $A \in \mathcal{A}$, since it is open, there exists $r_A > 0$ such that $B_{r_A}(x) \subset A$. Since $\mathcal{A}$ is finite, then $\{r_A\}_{A\in\mathcal{A}}$ is finite. Thus $\inf_{A\in\mathcal{A}} r_A \in \{r_A\}_{A\in\mathcal{A}}$ and hence it is positive. Let $r = \inf_{A\in\mathcal{A}} r_A$. For all $A \in \mathcal{A}$, since $r \leq r_A$, then $B_r(x) \subset B_{r_A}(x)$, and since $B_{r_A}(x) \subset A$, then $B_r(x) \subset A$. Therefore, $B_r(x) \subset \cap_{A\in\mathcal{A}} A$, hence $\cap_{A\in\mathcal{A}} A$ is open.

Q.E.D.

Since the properties of being closed versus open are complementary concepts, the properties below (relating to closed rather than open sets) could be used in place of a topology (i.e. defined in terms of open sets) to define a given notion of distance for a space. Although the proofs in the remainder of this section can be completed based entirely on concepts that we've presented in the notes up to this point, they can also be readily obtained using De Morgan's laws of unions and intersection.[15]

**S21. Topological properties of closed sets.** *If* $(X,d)$ *is a metric space, then*

(1) *both* $\phi$ **and** $X$ **are closed** *sets of* $(X,d)$*,*

(2) **any intersection of closed sets is a closed set**, *i.e. if* $\mathcal{A} \subset \mathcal{P}(X)$ *is such that for all* $A \in \mathcal{A}$*,* $A$ *is a closed set of* $(X,d)$*, then* $\cup_{\mathcal{A}} A$ *is a closed set of* $(X,d)$*,*

(3) **any finite union of closed sets is a closed set** *i.e. if* $\mathcal{A} \subset \mathcal{P}(X)$ *is such that for all* $A \in \mathcal{A}$*,* $A$ *is a closed set of* $(X,d)$ *and* $\mathcal{A}$ *is finite, then* $\cap_{\mathcal{A}} A$ *is a closed set of* $(X,d)$*.*

*Proof.* Exercise.

**S22. Every finite set is closed.** *If* $(X,d)$ *is a metric space and* $A \subset X$ *is finite, then* $A$ *is a closed set of* $(X,d)$*.*

*Proof.* Exercise.

Next we characterize closed sets as those that contain all their accumulation points, i.e. those sets that coincide with their closures.

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[14] The implication "if $p$, then $q$" takes the value *false* only if $q$ takes the value *false*. In particular it takes the value *true* whenever $p$ takes the value *false*, no matter what the value of $q$ is. In order to see it, compute the truth table of the statement "if $p$, then $q$" noticing that it stands for "either $q$ or not $p$".

[15] See any reference in set theory or real analysis.

**S23. A set is closed iff it contains all its accumulation points.**

1. *If $(X,d)$ is a metric space, $A \subset X$ is closed, and $x \in X$ is an accumulation point of $A$, then $x \in A$.*
2. *If $(X,d)$ is a metric space, $A \subset X$ and, for all $x \in X$ such that $x$ is an accumulation point of $A$, $x \in A$, then $A$ is a closed set of $(X,d)$.*

*Proof.* Let $(X,d)$ be a metric space and $A \subset X$ be closed. Let $x$ be an accumulation point of $A$. Then $x$ is a closure point of $A$ as well. Since $A$ is closed, then $x \in A$.

Conversely, let $(X,d)$ be a metric space and let $A \subset X$ contain all its accumulation points, and let $x \in X$ be a closure point of $A$. Then either $x$ is an accumulation point of $A$, and hence $x \in A$, or $x$ is an isolated point of $A$, and hence $x \in A$ as well. Therefore $A$ contains all its closure points and is thus closed. Q.E.D.

**S24. Complete characterization of the open and closed sets of a metric subspace.** *If $(X,d)$ is a metric space, $X' \subset X$ and $d'$ is the restriction of $d$ to $X' \times X'$, then $A' \subset X'$ is an open (resp. closed) set of $(X',d')$ if, and only if, there is an open (resp. closed) set $A$ of $(X,d)$ such that $A' = A \cap X'$.*

*Proof.* Exercise.

*On connected metric spaces.*

**S25. Complete characterization of connected spaces.** *If $(X,d)$ is a metric space, then it is connected if, and only if, if $A \subset X$ is both an open and a closed set of $(X,d)$, then either $A = X$ or $A = \phi$.*

*Proof.* Assume $(X,d)$ is connected and let $A \subset X$ be open and closed. Assume $A \neq \phi$ and $A \neq X$. Then $A^C$ is open and closed and $A^C \neq \phi$ and $A^C \neq X$ as well.

Since $A, A^C$ are nonempty, open and disjoint and $(X,d)$ is connected. then $(A \cup A^C)^C \neq \phi$, but $A \cup A^C = X$ and $X^C = \phi$! Therefore either $A = X$ or $A = \phi$.

Conversely, assume $(X,d)$ is not connected. Then there exist $A, A' \subset X$ nonempty, disjoint and open such that $(A \cup A')^C = \phi$, i.e. $A \cup A' = X$. Then, since $A$ and $A'$ are disjoint, $A = {A'}^C$ and since $A'$ is open, then $A$ is closed. Therefore $A$ is both open and closed but $A \neq \phi$ and $A \neq X$, since $A^C = A' \neq \phi$. Q.E.D.

*On convergent sequences and their limits.*

The following result is at the foundation of the concepts of convergence of a sequence and of limits of convergent sequences. It states that convergence can be characterized by the property that only finitely many terms of a sequence fall outside any neighborhood of some point. This point is said to be the limit of the convergent sequence, and it is unique. This means that if for a given sequence in a metric space there are two points for which at most finitely many terms of the sequence fall outside any ball centered at these points, then the points are actually the same point.

**S26. The limit of a convergent sequence is unique.** *If $(X,d)$ is a metric space, $s \in X^{\mathbb{N}}$ is a sequence in $X$, and $x, x' \in X$ are such that, for all $r > 0$, $s^{-1}(B_r(x)^C)$ is finite and $x \neq x'$, then there exists $r > 0$ such that $s^{-1}(B_r(x')^C)$ is not finite.*

*Proof.* Let $(X,d)$ be a metric space, $s \in X^{\mathbb{N}}$, and $x, x' \in X$ be such that, for all $r > 0$, $s^{-1}(B_r(x)^C)$ is finite and $x \neq x'$.

Letting $r = \frac{1}{2}d(x,x') > 0$, then $B_r(x) \cap B_r(x') = \phi$ (since should there be $x'' \in B_r(x) \cap B_r(x')$, then $d(x'',x) < \frac{1}{2}d(x,x')$ and $d(x'',x') < \frac{1}{2}d(x,x')$ would hold, and hence $d(x'',x) + d(x'',x') < d(x,x')$!), i.e. $B_r(x') \subset B_r(x)^C$, and hence $s^{-1}(B_r(x')) \subset s^{-1}(B_r(x)^C)$. Since $s^{-1}(B_r(x)^C)$ is finite, then $s^{-1}(B_r(x'))$ is necessarily finite too. Therefore, since

$$\begin{gathered}\mathbb{N} = \\ s^{-1}(X) = \\ s^{-1}(B_r(x') \cup B_r(x')^C) = s^{-1}(B_r(x')) \cup s^{-1}(B_r(x')^C)\end{gathered}$$

is not finite, $s^{-1}(B_r(x')^C)$ is not finite. Q.E.D.

We can think of the set of terms of a sequence in a metric space as the range of a function that represents the sequence. Then properties linking the limit of a convergent sequence with the range of the sequence appear naturally. For example, the limit of a convergent sequence is a closure point of the range of the sequence since the limit is either reached eventually (in which case it is an element of the range and therefore also a closure point) or approached but never reached (in which case it is an accumulation point and therefore also a closure point).

**S27. The limit of a convergent sequence is a closure point of its range.** *If $(X,d)$ is a metric space and $s \in X^{\mathbb{N}}$ is convergent to $x \in X$, then $x \in \mathrm{Cl}s(\mathbb{N})$.*

*Proof.* Let $(X,d)$ be a metric space, $s \in X^{\mathbb{N}}$ be convergent to $x \in X$, and $r > 0$.

Since $s \in X^{\mathbb{N}}$ is convergent to $x \in X$, then $s^{-1}(B_r(x)^C)$ is finite. Hence, since

$$\begin{gathered}\mathbb{N} = \\ s^{-1}(X) = \\ s^{-1}(B_r(x) \cup B_r(x)^C) = s^{-1}(B_r(x)) \cup s^{-1}(B_r(x)^C)\end{gathered}$$

is not finite, then $s^{-1}(B_r(x))$ is not finite, Thus, $s^{-1}(B_r(x)) \neq \phi$, and hence $s(\mathbb{N}) \cap B_r(x) \neq \phi$.

Therefore, for all $r > 0$, $s(\mathbb{N}) \cap B_r(x) \neq \phi$, i.e. $x \in \mathrm{Cl}s(\mathbb{N})$ Q.E.D.

**S28. The limit of a convergent sequence with infinitely many distinct terms is an accumulation point of its range.** *If $(X,d)$ is a metric space and $s \in X^{\mathbb{N}}$ is convergent to $x \in X$ and such that $s(\mathbb{N})$ is not finite, then $x$ is an accumulation point of $s(\mathbb{N})$.*

*Proof.* Let $(X,d)$ be a metric space and $s \in X^{\mathbb{N}}$ be convergent to $x \in X$ and such that $s(\mathbb{N})$ is not finite.

Let $r > 0$. Since $x = \lim s$, then $s^{-1}(B_r(x)^C)$ is finite. Since $s^{-1}(B_r(x)^C)$ is finite and $s^{-1}(s(\mathbb{N}) \cap B_r(x)^C) = s^{-1}(B_r(x)^C)$, then $s^{-1}(s(\mathbb{N}) \cap B_r(x)^C)$ is finite. Since $s^{-1}(s(\mathbb{N}) \cap B_r(x)^C)$ is finite, then $s(\mathbb{N}) \cap B_r(x)^C$ is finite. Since $s(\mathbb{N}) \cap B_r(x)^C$ is finite and

$$\begin{aligned}s(\mathbb{N}) &= s(\mathbb{N}) \cap X \\ &= s(\mathbb{N}) \cap (B_r(x) \cup B_r(x)^C) \\ &= (s(\mathbb{N}) \cap (B_r(x)) \cup (s(\mathbb{N}) \cap B_r(x)^C)\end{aligned}$$

is not finite, then $s(\mathbb{N}) \cap B_r(x)$ is not finite. Since, for all $r > 0$, $s(\mathbb{N}) \cap B_r(x)$ is not finite, then $x$ is an accumulation point of $s(\mathbb{N})$.

Q.E.D.

**S29. Every convergent sequence is bounded.** *If $(X, d)$ is a metric space and $s \in X^{\mathbb{N}}$ is convergent, then $s(\mathbb{N})$ is a bounded set of $(X, d)$.*

*Proof.* Let $(X, d)$ be a metric space, $s \in X^{\mathbb{N}}$ be convergent to $x \in X$ and $r > 0$. Then, either $s(\mathbb{N}) \subset B_r(x)$ or $s(\mathbb{N}) \not\subset B_r(x)$.

(1) If $s(\mathbb{N}) \subset B_r(x)$, then $s(\mathbb{N})$ is a bounded set of $(X, d)$.
(2) If $s(\mathbb{N}) \not\subset B_r(x)$, since $x$ is the limit of $s$, then $s^{-1}(B_r(x)^C) = s^{-1}(s(\mathbb{N}) \cap B_r(x)^C)$ is finite, and hence $s(\mathbb{N}) \cap B_r(x)^C$ is finite. Hence there exists $\max_{x' \in s(\mathbb{N}) \cap B_r(x)^C} d(x', x)$. Let it be $r' - 1$, which is necessarily such that $r \leq r' - 1$, i.e. $r \leq r + 1 < r'$ and hence $B_r(x) \subset B_{r'}(x)$. Thus, for all $x \in S(\mathbb{N})$, if $x' \in S(\mathbb{N}) \cap B_r(x)$, then $x' \in B_{r'}(x)$, since $B_r(x) \subset B_{r'}(x)$; and if $x' \in S(\mathbb{N}) \cap B_r(x)^C$, then $d(x', x) \leq r' - 1 < r'$, i.e. $x' \in B_{r'}(x)$. Therefore, $s(\mathbb{N}) \subset B_{r'}(x)$, i.e. $s(\mathbb{N})$ is a bounded set of $(X, d)$.

Q.E.D.

**S30. The limit of a convergent sequence is a cluster point.** *If $(X, d)$ is a metric space and $s \in X^{\mathbb{N}}$ converges to $x \in X$, then, for all $r > 0$, $s^{-1}(B_r(x))$ is not finite.*

*Proof.* Let $(X, d)$ be a metric space and $s \in X^{\mathbb{N}}$ converge to $x \in X$.

Since $s \in X^{\mathbb{N}}$ converges to $x \in X$, for all $r > 0$, $s^{-1}(B_r(x)^C)$ is finite. Since

$$\begin{gathered}\mathbb{N} = \\ s^{-1}(X) = \\ s^{-1}(B_r(x) \cup B_r(x)^C) = s^{-1}(B_r(x)) \cup s^{-1}(B_r(x)^C)\end{gathered}$$

is not finite, then, for all $r > 0$, $s^{-1}(B_r(x))$ is not finite. Q.E.D.

Within any positive distance of a cluster point of a sequence one can find an infinite number of points of the sequence. By requiring this distance to approach (but never reach) zero, selecting any term of the sequence, and then repeatedly taking terms of the sequence that (1) fall within this shrinking distance of the cluster point and (2) are increasing in their index values, one can construct a subsequence converging to the cluster point

**S31. Every sequence with a cluster point has a subsequence converging to it.** *If $(X, d)$ is a metric space, $s \in X^{\mathbb{N}}$, and $x \in X$ is such that, for all $r > 0$, $s^{-1}(B_r(x))$ is not finite, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and such that, for all $r > 0$, $(s \circ h)^{-1}(B_r(x)^C)$ is finite.*

*Proof.* Let $(X, d)$ be a metric space, $s \in X^{\mathbb{N}}$, and $x \in X$ be such that, for all $r > 0$, $s^{-1}(B_r(x))$ is not finite.

Since for all $r > 0$, $s^{-1}(B_r(x))$ is not finite, then, for all $r > 0$, $s^{-1}(B_r(x)) \neq \phi$. Therefore,

(1) $s^{-1}(B_1(x)) \neq \phi$ and hence there exists $n_1 \in s^{-1}(B_1(x))$,
(2) since $s^{-1}(B_{\frac{1}{2}}(x))$ is not finite, then $s^{-1}(B_{\frac{1}{2}}(x)) \setminus \{1, \ldots, n_1\} \neq \phi$ and hence there exists $n_2 \in s^{-1}(B_{\frac{1}{2}}(x)) \setminus \{1, \ldots, n_1\}$ such that $n_2 > n_1$,

(3) since $s^{-1}(B_{\frac{1}{3}}(x))$ is not finite, then $s^{-1}(B_{\frac{1}{3}}(x)) \setminus \{1, \dots, n_2\} \neq \phi$ and hence there exists $n_3 \in s^{-1}(B_{\frac{1}{3}}(x)) \setminus \{1, \dots, n_2\}$ such that $n_3 > n_2$, and so on...

Let $h \in \mathbb{N}^{\mathbb{N}}$ be such that, for all $i \in \mathbb{N}$, $h(i) = n_i$. Then, for all $i, i' \in \mathbb{N}$ such that $i < i'$, $h(i) = n_i < n_{i'} = h(i')$, and $s(n_i) = s(h(i)) = (s \circ h)(i)$. Thus $s \circ h$ is a subsequence on $s$ and for all $r > 0$, $(s \circ h)(i) = s(h(i)) = s(n_i) \in B_r(x)$ if, and only if, $i > \frac{1}{r}$. Therefore, for all $r > 0$, $(s \circ h)^{-1}(B_r(x)^C)$ is finite. Q.E.D.

*Accumulation points and limits.*

The following proposition allows us to characterize the accumulation points of a set as the limits of convergent sequences in the set. Thus a point is an accumulation point of a set if, and only if, it can be approached arbitrarily close from within the set.

**S32. Every accumulation point of a set is the limit of a convergent sequence in the set without the point, and conversely.** *If $(X, d)$ is a metric space, $A \subset X$, and $x \in X$ then $x$ is an accumulation point of $A$ if, and only if, there exists $s \in (A \setminus \{x\})^{\mathbb{N}}$ convergent to $x$.*

*Proof.* Let $(X, d)$ be a metric space, $A \subset X$, and $x \in X$

Assume that $x \in X$ is an accumulation point of $A$ and let $r_1 > 0$.

(1) Since $x$ is an accumulation point of $A$, then $A \cap B_{r_1}(x) \setminus \{x\} \neq \phi$ and hence there exists $x_1 \in A \cap B_{r_1}(x) \setminus \{x\}$ and $r_2 = \frac{1}{2}d(x_1, x)$.

(2) Since $x$ is an accumulation point of $A$, then $A \cap B_{r_2}(x) \setminus \{x\} \neq \phi$ and hence there exists $x_2 \in A \cap B_{r_2}(x) \setminus \{x\}$ and $r_3 = \frac{1}{2}d(x_2, x)$.

(3) Since $x$ is an accumulation point of $A$, then $A \cap B_{r_3}(x) \setminus \{x\} \neq \phi$ and hence there exists $x_3 \in A \cap B_{r_3}(x) \setminus \{x\}$ and $r_4 = \frac{1}{2}d(x_3, x)$, and so on.

Thus there exists $s \in (A \setminus \{x\})^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $s(n) = x_n$ and, for all $r' > 0$, there exists $n(r') \in \mathbb{N}$ such that $r_{n(r')} < r$ and hence $B_{r_{n(r')}}(x) \subset B_r(x)$ and $s^{-1}(B_{r_{n(r')}}(x)^C)$ has at most $n(r') - 1$ elements, i.e. since $B_r(x)^C \subset B_{r_n}(x)^C$, $s^{-1}(B_r(x)^C)$ has at most $n(r') - 1$ elements. Thus there exists $s \in (A \setminus \{x\})^{\mathbb{N}}$ convergent to $x$.

Conversely, assume that $x$ is not an accumulation point of $A$ and let $s \in (A \setminus \{x\})^{\mathbb{N}}$, Since $x$ is not an accumulation point of $A$, then there exists $r > 0$ such that $A \cap B_r(x)$ is finite. Since $A \setminus \{x\} \subset A$ and $A \cap B_r(x)$ is finite, then $(A \setminus \{x\}) \cap B_r(x)$ is finite. Similarly, since $s(\mathbb{N}) \subset A \setminus \{x\}$, then $s(\mathbb{N}) \cap B_r(x) \subset (A \setminus \{x\}) \cap B_r(x)$, and since $(A \setminus \{x\}) \cap B_r(x)$ is finite, then $s(\mathbb{N}) \cap B_r(x)$ is finite. Then there exists $r' = \min_{x' \in s(\mathbb{N}) \cap B_r(x)} d(x', x)$. Since $s(\mathbb{N}) \subset A \setminus \{x\}$, then $r' = \min_{x' \in s(\mathbb{N}) \cap B_r(x)} d(x', x) > 0$. Thus $s(\mathbb{N}) \cap B_{r'}(x) = \phi$, and hence $s(\mathbb{N}) \subset B_{r'}(x)^C$. Since $s(\mathbb{N}) \subset s(\mathbb{N})$, then $s^{-1}(s(\mathbb{N})) = \mathbb{N}$, and since $s(\mathbb{N}) \subset B_{r'}(x)^C$, then $s^{-1}(s(\mathbb{N})) \subset s^{-1}(B_{r'}(x)^C)$, i.e. $\mathbb{N} \subset s^{-1}(B_{r'}(x)^C)$. Thus there exists $r' > 0$ such that $s^{-1}(B_{r'}(x)^C)$ is not finite and hence $x$ is not the limit of $s$. Q.E.D.

The previous property and the fact that a set is closed if, and only if, it contains all its accumulation points, allow us to characterize the closed sets of a metric space as those containing the limit of every convergent sequence within them.

**S33. A set is closed iff it contains the limit of every convergent sequence contained in it.** *If $(X, d)$ is a metric space and $A \subset X$, then $A$ is a closed set of $(X, d)$ if, and only if, for all $s \in A^{\mathbb{N}}$ convergent, $\lim s \in A$.*

*Proof.* Let $(X, d)$ be a metric space and $A \subset X$.

Assume that $A$ is closed and let $s \in A^{\mathbb{N}}$ be convergent to $x \in X$. Either $x \in s(\mathbb{N})$ or $x \notin s(\mathbb{N})$.

(1) If $x \in s(\mathbb{N})$, since $s(\mathbb{N}) \subset A$, then $x \in A$.
(2) If $x \notin s(\mathbb{N})$, then $s(\mathbb{N}) \subset A \setminus \{x\}$. Since $x$ is the limit of $s \in (A \setminus \{x\})^{\mathbb{N}}$, then $x$ is an accumulation point of $A$. Since $A$ is closed, then $A$ contains all its accumulation points and hence $x \in A$.

Conversely, assume that $A$ contains the limit of every convergent sequence contained in $A$ and let $x$ be an accumulation point of $A$. Since $x$ is an accumulation point of $A$, then there exists $s \in (A\setminus\{x\})^{\mathbb{N}}$ convergent to $x$. Since $A\setminus\{x\} \subset A$, then there exists $s \in A^{\mathbb{N}}$ convergent to $x$. Since $A$ contains the limit of every convergent sequence contained in $A$, then $x \in A$. Therefore, $A$ contains all its accumulation points and hence $A$ is closed. Q.E.D.

*On totally bounded sets in metric spaces.*

A totally bounded set can be assimilated to a finite set with a precision as high as it may be wished. No matter how fine we require a "grid" of points of the space to be, a finite number of points of this grid is always sufficient to guarantee that every point of a totally bounded set is within a distance smaller than the separation of the points of the grid. Thus should we take this finite set for the actual set, we would be incurring a (metric) error that is arbitrarily small. Every totally bounded set is bounded. We will use this property in the proof of the boundedness of every compact set below.

**S34. Every totally bounded set is bounded.** *If $(X, d)$ is a metric space and $A$ is a totally bounded set of $(X, d)$, then $A$ is a bounded set of $(X, d)$.*

*Proof.* Let $(X, d)$ be a metric space, $A$ be a totally bounded set of $(X, d)$, and let $\varepsilon > 0$. Since $A$ is a totally bounded set of $(X, d)$, then there exists $N \subset X$ finite and such that $A \subset \cup_{x\in N} B_\varepsilon(x)$. Since, for all $x \in N$, $B_\varepsilon(x)$ is a bounded set of $(X, d)$, then $\cup_{x\in N} B_\varepsilon(x)$ is a bounded set of $(X, d)$, and hence $A$ is a bounded set of $(X, d)$ too. Q.E.D.

*On compact sets in metric spaces.*

Compactness is an extremely useful property for sets associated with real-valued functions, since it is associated with the existence of maxima and minima. The following theorem establishes an equivalence between sets whose sequences (i.e. those that are entirely contained in the set) all have subsequences converging to limits within the set, and sets for which every non-finite subset has an accumulation point within the set. This equivalence allows us to characterize the compact sets of a metric space in terms of either of the two sets of conditions. In particular, the characterization of compact sets as those for which every non-finite subset has an accumulation point in the set, is used below to prove the boundedness of every compact set.

**S35. Every sequence in a set has a subsequence convergent in the set iff every non-finite subset has an accumulation point in the set.** *If $(X, d)$ is a metric space and $A \subset X$, then, for all $s \in A^{\mathbb{N}}$, there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and*

*$x \in A$ such that $s \circ h$ converges to $x$, if, and only if, for all $A' \subset A$ not finite, there exists $x \in A$ such that $x$ is an accumulation point of $A'$.*

*Proof.* Let $(X, d)$ be a metric space and $A \subset X$.

Assume that for all $s \in A^{\mathbb{N}}$, there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and $x \in A$ such that, for all $r > 0$, $(s \circ h)^{-1}(B_r(x)^C)$ is finite, i.e. $x$ is the limit of $s \circ h$, and let $A' \subset A$ be not finite. Since $A'$ is not finite, there exists $s \in {A'}^{\mathbb{N}}$ injective. Since $A' \subset A$, then $s \in A^{\mathbb{N}}$, and hence there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and $x \in A$ such that, for all $r > 0$, $(s \circ h)^{-1}(B_r(x)^C)$ is finite, i.e. $x$ is the limit of $s \circ h$. Since $h$ is increasing, then $h$ is injective. Since $s$ is injective and $h$ is injective, then $s \circ h$ is injective. Since $s \circ h$ is injective, then $(s \circ h)(\mathbb{N})$ is not finite. Since $x$ is the limit of $s \circ h$ and $(s \circ h)(\mathbb{N})$ is not finite, then, $x$ is an accumulation point of $(s \circ h)(\mathbb{N})$. Since $(s \circ h)(\mathbb{N}) \subset s(\mathbb{N})$ and $s(\mathbb{N}) \subset A'$, then $(s \circ h)(\mathbb{N}) \subset A'$. Since $x$ is an accumulation point of $(s \circ h)(\mathbb{N})$ and $(s \circ h)(\mathbb{N}) \subset A'$, then $x$ is an accumulation point of $A'$. Therefore, for all $A' \subset A$ not finite, there exists $x \in A$ such that $x$ is an accumulation point of $A'$.

Conversely, assume that for all $A' \subset A$ not finite, there exists $x \in A$ such that, $x$ is an accumulation point of $A'$, and let $s \in A^{\mathbb{N}}$. Either $s(\mathbb{N})$ is finite or $s(\mathbb{N})$ is not finite.

1. If $s(\mathbb{N})$ is finite, then since $\mathbb{N} = s^{-1}(s(\mathbb{N})) = \cup_{x \in s(\mathbb{N})} s^{-1}(x)$ is not finite, there exists $x \in s(\mathbb{N})$ such that $s^{-1}(x)$ is not finite. Let $h \in \mathbb{N}^{\mathbb{N}}$ be such that for all $n \in \mathbb{N}$, $h(n)$ is the n-th smallest integer in $s^{-1}(x)$, hence $h$ is increasing. Since, for all $n \in \mathbb{N}$, $h(n) \in s^{-1}(x)$, then for all $n \in \mathbb{N}$, $s(h(n)) = x$. Hence the subsequence $s \circ h$ of $s$ is constant and, hence, convergent to $x$. Since $x \in s(\mathbb{N})$ and $s(\mathbb{N}) \subset A$, then $x \in A$.
2. If $s(\mathbb{N})$ is not finite, then there exists $x \in A$ such that $x$ is an accumulation point of $s(\mathbb{N})$. Since $x$ is an accumulation point of $s(\mathbb{N})$, then for all $r > 0$, $s(\mathbb{N}) \cap B_r(x)$ is not finite, and hence $s^{-1}(s(\mathbb{N}) \cap B_r(x)) = s^{-1}(B_r(x))$ is not finite. Since for all $r > 0$, $s^{-1}(B_r(x))$ is not finite, i.e. $x$ is a cluster point of $s$, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and such that, for all $r > 0$, $(s \circ h)^{-1}(B_r(x)^C)$ is finite, i.e. $s \circ h$ converges to $x$.

Therefore, for all $s \in A^{\mathbb{N}}$, there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and $x \in A$ such that $s \circ h$ converges to $x$. Q.E.D.

Compact sets will coincide with closed and bounded sets in $\mathbb{R}^n$ with any metric, but in general the implication goes only one way: compact sets are closed and bounded, as the next two propositions establish.

**S36. Every compact set is bounded.** *If $(X, d)$ is a metric space and $A$ is a compact set of $(X, d)$, then $A$ is a bounded set of $(X, d)$.*

*Proof.* Let $(X, d)$ be a metric space and $A$ be a non bounded set of $(X, d)$. Since $A$ is not a bounded set of $(X, d)$, then $A$ is not totally bounded, i.e. there exists $\varepsilon > 0$ such that, for all $N \subset X$ finite, $A \not\subset \cup_{x \in N} B_\varepsilon(x)$.

Let $x_1 \in A$. Then

1. since $N_1 = \{x_1\} \subset A$ is finite, $A \not\subset \cup_{x \in N_1} B_\varepsilon(x)$, i.e. there exists $x_2 \in A \cap (\cup_{x \in N_1} B_\varepsilon(x))^C$ such that $x_2 \notin \{x_1\}$, since $\{x_1\} \subset \cup_{x \in N_1} B_\varepsilon(x)$,
2. since $N_2 = \{x_1, x_2\} \subset A$ is finite, $A \not\subset \cup_{x \in N_2} B_\varepsilon(x)$, i.e. there exists $x_3 \in A \cap (\cup_{x \in N_2} B_\varepsilon(x))^C$ such that $x_3 \notin \{x_1, x_2\}$, since $\{x_1, x_2\} \subset \cup_{x \in N_2} B_\varepsilon(x)$,

(3) since $N_3 = \{x_1, x_2, x_3\} \subset A$ is finite, $A \not\subset \cup_{x \in N_3} B_\varepsilon(x)$, i.e. there exists $x_4 \in A \cap (\cup_{x \in N_3} B_\varepsilon(x))^C$, such that $x_4 \notin \{x_1, x_2, x_3\}$, since $\{x_1, x_2, x_3\} \subset \cup_{x \in N_3} B_\varepsilon(x)$, and so on.

Therefore, there exists $\{x_1, x_2, \ldots,\} \subset A$ not finite such that for all $n \in \mathbb{N}$, $B_\varepsilon(x_n) = \{x_n\}$.

Is there an accumulation point of $\{x_1, x_2, \ldots,\}$ in $A$? Let $x \in A$. Either $\{x_1, x_2, \ldots,\} \cap B_{\frac{1}{2}r}(x) = \phi$ or $\{x_1, x_2, \ldots,\} \cap B_{\frac{1}{2}r}(x) \neq \phi$. If $\{x_1, x_2, \ldots,\} \cap B_{\frac{1}{2}r}(x) = \phi$, then $x$ is not an accumulation point of $\{x_1, x_2, \ldots,\}$. If $\{x_1, x_2, \ldots,\} \cap B_{\frac{1}{2}r}(x) \neq \phi$, let $x_m, x_n \in \{x_1, x_2, \ldots,\} \cap B_{\frac{1}{2}r}(x)$. Then

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x_n, x) \\ &< \frac{1}{2}r + \frac{1}{2}r \\ &= r \end{aligned}$$

i.e. $x_m \in B_r(x_n) = \{x_n\}$. Therefore, $x_m = x_n$. Hence $\{x_1, x_2, \ldots,\} \cap B_{\frac{1}{2}r}(x)$ is finite and thus $x$ is not an accumulation point of $\{x_1, x_2, \ldots,\}$. Therefore, no point of $A$ is an accumulation point of $\{x_1, x_2, \ldots,\}$. i.e. $\{x_1, x_2, \ldots,\}$ is a non finite subset of $A$ with no accumulation point in $A$, which contradicts that $A$ is sequentially compact. Q.E.D.

Compactness imposes stronger conditions on the behavior of sequences within a set than does closedness: while closedness requires that all the convergent sequences of a set have limits in the set, compactness requires that every sequence of a set have a subsequence with a limit in the set. Clearly then compactness implies closedness, since (1) every convergent sequence in a compact set must have a subsequence with a limit in the set and (2) the limit of the subsequence cannot be distinct from the limit of the sequence. This implies that the sequence itself has a limit that is in the set. Now, since every convergent sequence in a compact set converges to a point in the set, we know that a compact set is necessarily closed. A proof of this fact from scratch, i.e. showing that every compact set must contain all its accumulation points and hence be closed, follows.

**S37. Every compact set is closed.** *If $(X, d)$ is a metric space and $A$ is a compact set of $(X, d)$, then $A$ is a closed set of $(X, d)$.*

*Proof.* Let $(X, d)$ be a metric space, $A$ be a compact set of $(X, d)$, and $x \in X$ be an accumulation point of $A$. Since $x$ is an accumulation point of $A$, then there exists $s \in (A \setminus \{x\})^{\mathbb{N}}$ convergent to $x$. Since $x$ is the limit of $s$, then, for all $r > 0$, $s^{-1}(B_r(x)^C)$ is finite. Since $\mathbb{N} \subset s^{-1}(A \setminus \{x\})$, then

$$\begin{aligned} \mathbb{N} &= \\ s^{-1}(s(\mathbb{N})) &= s^{-1}(s(\mathbb{N}) \cap (B_r(x) \cup B_r(x)^C)) \\ &= s^{-1}((s(\mathbb{N}) \cap (B_r(x)) \cup (s(\mathbb{N}) \cap B_r(x)^C))) \\ &= s^{-1}(s(\mathbb{N}) \cap B_r(x)) \cup s^{-1}(s(\mathbb{N}) \cap B_r(x)^C) \end{aligned}$$

is not finite, and therefore, since $s^{-1}(B_r(x)^C) = s^{-1}(s(\mathbb{N}) \cap B_r(x)^C)$ is finite, $s^{-1}(s(\mathbb{N}) \cap B_r(x))$ is not finite. Since $s(\mathbb{N}) \cap B_r(x) \subset s(\mathbb{N})$, then $s(s^{-1}(s(\mathbb{N}) \cap B_r(x))) = s(\mathbb{N}) \cap B_r(x)$, and since $s^{-1}(s(\mathbb{N}) \cap B_r(x))$ is not finite, then $s(\mathbb{N}) \cap B_r(x)$

is not finite. Therefore, since $s(\mathbb{N}) \cap B_r(x) \subset s(\mathbb{N})$, then $s(\mathbb{N})$ is not finite. Since $A$ is compact and $s(\mathbb{N}) \subset A$ is not finite, then there exists an accumulation point $x' \in A$ of $s(\mathbb{N})$. If $x \neq x'$, then $d(x, x') > 0$. If $\varepsilon = \frac{1}{2}d(x, x')$, then $B_\varepsilon(x) \cap B_\varepsilon(x') = \phi$. Therefore $B_\varepsilon(x) \subset B_\varepsilon(x')^C$, i.e. $B_\varepsilon(x') \subset B_\varepsilon(x)^C$. Since $x'$ is an accumulation point of $s(\mathbb{N})$, then $s(\mathbb{N}) \cap B_\varepsilon(x')$ is not finite. Thus, since $s(\mathbb{N}) \cap B_\varepsilon(x') \subset s(\mathbb{N}) \cap B_\varepsilon(x)^C$, then $s(\mathbb{N}) \cap B_\varepsilon(x)^C$ is not finite, i.e. $x$ would not be the limit of $s$. Thus $x = x'$, and since $x' \in A$, then $x \in A$. Therefore, $A$ contains all its accumulation points, i.e. $A$ is a closed set of $(X, d)$. Q.E.D.

**S38. Every closed set in a compact metric space is compact.** *If $(X, d)$ is a compact metric space and $A$ is a closed set of $(X, d)$, then $A$ is a compact set of $(X, d)$.*

*Proof.* Let $(X, d)$ be a compact metric space, $A$ be a closed set of $(X, d)$, and $s \in A^{\mathbb{N}}$. Since $(X, d)$ is a compact metric space, then there exists a subsequence of $s$ convergent to $x \in X$. Since $x$ is the limit of a sequence in $A$, then $x$ is an accumulation point of $A$. Thus, since $A$ is closed, it contains all its accumulation points, and hence $x \in A$. Therefore, every sequence in $A$ has a convergent subsequence whose limit is in $A$, i.e. $A$ is compact. Q.E.D.

**S39. A complete characterization of compacts sets in $(\mathbb{R}^n, d_2)$.** *For any integer $n$, a set in $(\mathbb{R}^n, d_2)$ is compact iff it is closed and bounded.*

*Proof.* (Converse statement) Let $A$ be a closed and bounded set of $(\mathbb{R}^n, d_2)$.

Since $A$ is bounded, there exists $B_\varepsilon(x)$ such that $A \subset B_\varepsilon(x)$. Hence

$$\forall a \in A, \sqrt{\sum_{i=1}^{n}(a^i - x^i)^2} < \varepsilon.$$

Let $\{a_k\}$ be a sequence in $A$. Then

$$\forall k \in \mathbb{N}, \sqrt{\sum_{i=1}^{n}(a_k^i - x^i)^2} < \varepsilon$$

and hence

$$\forall i \in \{1, \ldots, n\}, \forall k \in \mathbb{N}, |a_k^i - x^i| < \sqrt{\sum_{i=1}^{n}(a_k^i - x^i)^2} < \varepsilon.$$

Thus every coordinate sequence of $\{a_k\}$ is a bounded sequence and hence with a convergent subsequence in $(\mathbb{R}, d_2)$. Therefore, there is an increasing function $h^1$ from $\mathbb{N}$ to itself such that $\{a^1_{h^1(k)}\}$ converges to some $a^1$.

Then $\{a^2_{h^1(k)}\}$ is a bounded sequence and hence with a convergent subsequence in $(\mathbb{R}, d_2)$. Therefore, there is an increasing function $h^2$ from $\mathbb{N}$ to itself such that $\{a^2_{h^1(h^2(k))}\}$ converges to some $a^2$ while $\{a^1_{h^1(h^2(k))}\}$ is a subsequence of $\{a^1_{h^1(k)}\}$ and still converges to $a^1$. And so on.

Thus there exists a convergent subsequence $\{a_{h(k)}\}$ of $\{a_k\}$ whose limit is the point $(a^1, a^2, \ldots, a^n) = a$.

Either the range of $\{a_{h(k)}\}$ is finite or not.

If the range of $\{a_{h(k)}\}$ is finite then necessarily $a$ is in this range and hence in $A$.

If the range of $\{a_{h(k)}\}$ is not finite then $a$ is an accumulation point of this range and hence of $A$. But $A$ is closed and contains all its accumulation points. In particular it contains $a$.

Therefore, for any sequence in $A$ there is a convergent subsequence whose limit is in $A$, i.e. $A$ is compact. Q.E.D.

*Compactness and sequential compactness.*

**S40. Every sequentially compact set is totally bounded.** *If $(X, d)$ is a metric space and $A$ is a sequentially compact set of $(X, d)$, then, for all $r > 0$, there exist $C \subset A$ finite and such that $A \subset \cup_{x \in C} B_r(x)$.*

*Proof.* Let $(X, d)$ be a metric space and $A$ be a sequentially compact set of $(X, d)$.

Assume that there exists $r > 0$ such that, for all $C \subset A$ finite, $A \not\subset \cup_{x \in N} B_r(x)$.

Let $x_1 \in A$. Then

(1) since $C_1 = \{x_1\} \subset A$ is finite, $A \not\subset \cup_{x \in C_1} B_r(x)$, i.e. there exists $x_2 \in A \cap (\cup_{x \in C_1} B_r(x))^C$ such that $x_2 \notin \{x_1\}$, since $\{x_1\} \subset \cup_{x \in C_1} B_r(x)$,
(2) since $C_2 = \{x_1, x_2\} \subset A$ is finite, $A \not\subset \cup_{x \in C_2} B_r(x)$, i.e. there exists $x_3 \in A \cap (\cup_{x \in C_2} B_r(x))^C$ such that $x_3 \notin \{x_1, x_2\}$, since $\{x_1, x_2\} \subset \cup_{x \in C_2} B_r(x)$,
(3) since $C_3 = \{x_1, x_2, x_3\} \subset A$ is finite, $A \not\subset \cup_{x \in C_3} B_r(x)$, i.e. there exists $x_4 \in A \cap (\cup_{x \in C_3} B_r(x))^C$, such that $x_4 \notin \{x_1, x_2, x_3\}$, since $\{x_1, x_2, x_3\} \subset \cup_{x \in C_3} B_r(x)$, and so on.

Let $s \in A^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s(n) = x_n$. Then $s$ is injective and, for all $n \in \mathbb{N}$, $B_r(s(n)) = \{s(n)\}$. Since $A$ is a sequentially compact set of $(X, d)$, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and $x \in A$ such that $\lim s \circ h = x$. Since $\lim s \circ h = x$, then $(s \circ h)^{-1}(B_{\frac{r}{2}}(x)^C)$ is finite. For all $n, n' > \max(s \circ h)^{-1}(B_{\frac{r}{2}}(x)^C)$ distinct, since $s$ and $h$ are injective and hence $s \circ h$ is injective, $(s \circ h)(n) \neq (s \circ h)(n')$, and $d((s \circ h)(n), x) < \frac{r}{2}$ and $d((s \circ h)(n'), x) < \frac{r}{2}$. Hence

$$\begin{aligned} d((s \circ h)(n), (s \circ h)(n')) &\leq d((s \circ h)(n), x) + d((s \circ h)(n'), x) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r, \end{aligned}$$

i.e. $B_r((s \circ h)(n)) \not\subset \{(s \circ h)(n)\}$. Therefore, there exists $h(n) \in \mathbb{N}$ such that $B_r(s(h(n))) \not\subset \{s(h(n))\}$! Q.E.D.

**S41.** *If $(X, d)$ is a metric space, $A$ is a sequentially compact set of $(X, d)$, and $\mathcal{O}$ is an open cover of $A$, then there exists $r > 0$ such that, for all $x \in A$, there exists $O_x \in \mathcal{O}$ such that $B_r(x) \subset O_x$.*

*Proof.* Let $(X, d)$ be a metric space, $A$ be a sequentially compact set of $(X, d)$, and $\mathcal{O}$ be an open cover of $A$.

Assume that for all $r > 0$, there exists $x \in A$ such that, for all $O \in \mathcal{O}$, $B_r(x) \not\subset O$. Then for all $n \in \mathbb{N}$, there exists $x_n \in A$ such that, for all $O \in \mathcal{O}$, $B_{\frac{1}{n}}(x_n) \not\subset O$. Let $s \in A^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $s(n) = x_n$. Since $A$ is sequentially compact, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and such that $s \circ h$ is convergent to $x \in A$. Since $A \subset \cup_{O \in \mathcal{O}} O$ and $x \in A$, then there exists $O \in \mathcal{O}$ such that $x \in O$. Since $O$

is open, then there exists $r > 0$ such that $B_r(x) \subset O$. Since $x$ is the limit of $s \circ h$, then $(s \circ h)^{-1}(B_{\frac{r}{2}}(x)^C)$ is finite. For all $n \in \mathbb{N}$ such that $n > \max(s \circ h)^{-1}(B_{\frac{r}{2}}(x)^C)$ and $n > \frac{2}{r}$,

(1) since $n > \max(s \circ h)^{-1}(B_{\frac{r}{2}}(x)^C)$, then $(s \circ h)(n) \in B_{\frac{r}{2}}(x)$, that is to say $d((s \circ h)(n), x) < \frac{r}{2}$,
(2) since $n > \frac{2}{r}$, $\frac{1}{n} + \frac{r}{2} < r$.

Therefore, for all $n \in \mathbb{N}$ such that $n > \max(s \circ h)^{-1}(B_{\frac{r}{2}}(x)^C)$ and $n > \frac{2}{r}$, and all $x' \in B_{\frac{1}{n}}((s \circ h)(n))$,

$$\begin{aligned} d(x', x) &\leq d((s \circ h)(n), x') + d((s \circ h)(n), x) \\ &< \frac{1}{n} + \frac{r}{2} \\ &< r \end{aligned}$$

i.e. $x' \in B_r(x)$. Hence, for all $n \in \mathbb{N}$ such that $n > \max(s \circ h)^{-1}(B_{\frac{r}{2}}(x)^C)$ and $n > \frac{2}{r}$, $B_{\frac{1}{n}}(x_n) \subset B_r(x)$, and since $B_r(x) \subset O$, then $B_{\frac{1}{n}}(x_n) \subset O$!

Q.E.D.

**S42. In a metric space a set is compact iff it is sequentially compact.** *If $(X, d)$ is a metric space and $A \subset X$, then $A$ is a compact set of $(X, d)$ if, and only if, $A$ is a sequentially compact set of $(X, d)$.*

*Proof.* Let $(X, d)$ be a metric space and $A \subset X$.

Assume that $A$ is a compact set of $(X, d)$. Let $s \in A^{\mathbb{N}}$ be such that $s$ does not have a convergent subsequence whose limit is in $A$. Then there is no cluster point of $s$ in $A$, i.e. for all $x \in A$, there exists $r_x > 0$ such that, $s^{-1}(B_{r_x}(x))$ is finite, and since $s(\mathbb{N}) \cap B_{r_x}(x) \subset s(s^{-1}(B_{r_x}(x)))$, then $s(\mathbb{N}) \cap B_{r_x}(x)$ is finite. Since $A \subset \cup_{x \in A} B_{r_x}(x)$, for all $x \in A$, $B_{r_x}(x)$ is open, and $A$ is compact, then there exists $C \subset A$ finite such that $A \subset \cup_{x \in C} B_{r_x}(x)$. Since, for all $x \in C$, $s(\mathbb{N}) \cap B_{r_x}(x)$ is finite, and $C$ is finite, then

$$\cup_{x \in C}(s(\mathbb{N}) \cap B_{r_x}(x)) = s(\mathbb{N}) \cap (\cup_{x \in C} B_{r_x}(x))$$

is finite. Since $s(\mathbb{N}) \subset A$ and $A \subset \cup_{x \in C} B_{r_x}(x)$, then $s(\mathbb{N}) \subset \cup_{x \in C} B_{r_x}(x)$ and hence $s(\mathbb{N}) \cap (\cup_{x \in C} B_{r_x}(x)) = s(\mathbb{N})$, i.e. $s(\mathbb{N})$ is finite. Therefore, since

$$\begin{aligned} \mathbb{N} &= s^{-1}(s(\mathbb{N})) \\ &= \cup_{x \in s(\mathbb{N})} s^{-1}(x) \end{aligned}$$

is not finite, then there exists $x' \in s(\mathbb{N})$ such that $s^{-1}(x')$ is not finite. Let $h \in \mathbb{N}^{\mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, $h(n)$ is the $n$-th smallest element of $s^{-1}(x')$. Then $h$ is increasing and $s \circ h$ is a subsequence of $s$ such that, for all $n \in \mathbb{N}$, $(s \circ h)(n) = x'$. Thus, for all $r > 0$, $(s \circ h)(B_r(x')^C)$ is finite, i.e. $s \circ h$ is convergent to $x'$. Since $x' \in s(\mathbb{N})$ and $s(\mathbb{N}) \subset A$, then $x' \in A$, i.e. $s$ has a convergent subsequence whose limit is in $A$!

Conversely, assume that $A$ is a sequentially compact set of $(X, d)$. Let $\mathcal{O}$ be an open cover of $A$. Then,

(1) there exists $r > 0$ such that, for all $x \in A$, there exists $O_x \in \mathcal{O}$ such that $B_r(x) \subset O_x$,
(2) there exists $C \subset A$ finite and such that $A \subset \cup_{x \in C} B_r(x)$.

Therefore, for all $x \in C$, there exists $O_x \in \mathcal{O}$ such that $B_r(x) \subset O_x$, and hence

$$\begin{aligned} A &\subset \cup_{x \in C} B_r(x) \\ &\subset \cup_{x \in C} O_x, \end{aligned}$$

i.e. $\{O_x\}_{x \in C}$ is a finite subcover of $A$. Q.E.D.

*On Cauchy sequences.*

The next property provides a necessary condition for a sequence to be convergent, namely that the sequence be a Cauchy sequence. A sequence that is not Cauchy cannot be convergent.

**S43. Every convergent sequence is a Cauchy sequence.** *If $(X, d)$ is a metric space and $s \in X^{\mathbb{N}}$ is convergent then $s$ is a Cauchy sequence.*

*Proof.* Let $(X, d)$ be a metric space, $s \in X^{\mathbb{N}}$ be convergent to $x$, and $r > 0$. Then there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $d(s(n'), x) < \frac{1}{2}r$. Let $n', n'' > n$. Then both $d(s(n'), x) < \frac{1}{2}r$ and $d(s(n''), x) < \frac{1}{2}r$ hold and hence

$$\begin{aligned} d(s(n'), s(n'')) &\leq d(s(n'), x) + d(x, s(n'')) \\ &< \frac{1}{2}r + \frac{1}{2}r \\ &= r, \end{aligned}$$

i.e. $s$ is a Cauchy sequence. Q.E.D.

The converse is not true in general. For instance, let $s \in \mathbb{Q}^{\mathbb{N}}$ be a sequence of rational numbers, with the usual distance, such that $s(n)$ is the exact decimal expansion of $\sqrt{2}$ up to the $n$-th decimal. This is a Cauchy sequence since the difference between any two terms $s(n), s(n')$ with $n < n'$ is smaller than $10^{-n}$, and hence for any $r > 0$, letting $n$ be such that $10^{-n} < r$, then $d(s(n'), s(n'')) < 10^{-n} < r$ for any $n', n'' > n$. Notwithstanding this there is no rational number to which this sequence converges.

The following property, together with the previous one, provides another necessary condition for a sequence to be convergent, namely boundedness. In effect, a sequence that is not bounded cannot be a Cauchy sequence, so it cannot be a convergent sequence either.

**S44. Every Cauchy sequence is bounded.** *If $(X, d)$ is a metric space and $x \in S^{\mathbb{N}}$ is a Cauchy sequence, then $x$ is a bounded sequence.*

*Proof.* Let $x \in X^{\mathbb{N}}$ be a Cauchy sequence, and let $s > 0$. Then there exists $n \in \mathbb{N}$ such that, for all $n', n'' > n$, $d(x_{n'}, x_{n''}) < s$. Hence for any given $n' > n$ and for all $n'' > n$, $x_{n''} \in B_s(x_{n'})$, i.e. at most the $n$ first terms of the sequence are not in $B_s(x_{n'})$. Let $r = \max\{s, d(x_1, x_{n'}), \ldots, d(x_n, x_{n'})\} + 1$. Therefore since for all $n'' \in \mathbb{N}$, $d(x_{n''}, x_{n'}) \leq \max\{s, d(x_1, x_{n'}), \ldots, d(x_n, x_{n'})\} < r$, then, for all $n'' \in \mathbb{N}$, $x_{n''} \in B_r(x_{n'})$. Q.E.D.

**S45. Every Cauchy sequence converges to the limit of any convergent subsequence.** *If $(X, d)$ is a metric space, $s \in X^{\mathbb{N}}$ is Cauchy and such that there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and $x \in X$ such that $x = \lim s \circ h$, then $x = \lim s$.*

*Proof.* Let $(X, d)$ be a metric space, $s \in X^{\mathbb{N}}$ be Cauchy and such that there exists $h \in \mathbb{N}^{\mathbb{N}}$ increasing and $x \in X$ such that $x = \lim s \circ h$, and let $r > 0$.

Since $x = \lim s \circ h$, then there exists $n' \in \mathbb{N}$ such that, for all $n > n'$, $d((s \circ h)(n), x) < \frac{1}{2}r$. Since $s$ is Cauchy, there exists $n'' \in \mathbb{N}$ such that, for all $m, n > n''$, $d(s(m), s(n)) < \frac{1}{2}r$. Then, for all $m, n > \max\{n', n''\}$, since $h(n) > n$,

$$\begin{aligned} &d((s \circ h)(n), s(m)) = \\ &\quad d(s(h(n)), s(m)) < \frac{1}{2}r \end{aligned}$$

and $d((s \circ h)(n), x) < \frac{1}{2}r$, and hence

$$\begin{aligned} d(s(m), x) &\leq d((s \circ h)(n), s(m)) + d((s \circ h)(n), x) \\ &< \frac{1}{2}r + \frac{1}{2}r \\ &= r. \end{aligned}$$

Therefore, for all $r > 0$, there exists $n \in \mathbb{N}$ such that, for all $m > n$, $d(s(m), x) < r$, i.e. $x = \lim s$. Q.E.D.

The following proposition establishes that every compact metric space is also complete. This fact makes compact metric spaces an extremely useful framework in which to (1) characterize points as maxima or minima of continuous, real valued functions (e.g. their existence is guaranteed by the compactness of the domain) and (2) characterize points as limits of Cauchy sequences (i.e. sequences whose convergence is guaranteed by the completeness implied by the compactness).

**S46. Every compact metric space is complete.** *If $(X, d)$ is a compact metric space, then $(X, d)$ is a complete metric space.*

*Proof.* Let $(X, d)$ be a compact metric space and $s \in X^{\mathbb{N}}$ be Cauchy. Since $(X, d)$ is a compact metric space, then there exists $h \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$ such that $x = \lim s \circ h$. Since $s \circ h$ is convergent and $s$ is Cauchy, then $\lim s \circ h = \lim s$, i.e. $x = \lim s$. Therefore, since, for all $s \in X^{\mathbb{N}}$ Cauchy, there exists $x \in X$ such that $x = \lim s$, then $X$ is a complete set of $(X, d)$, i.e. $(X, d)$ is a complete metric space. Q.E.D.

The next property, the continuity of the distance from a given point to any other point for any metric, will be needed in the following proposition.

**S47. The distance from a given point to any other point is continuous for any metric.** *If $(X, d)$ is a metric space, $x' \in X$ and $f \in \mathbb{R}^X$ is such that $f(x) = d(x, x')$ for all $x \in X$, then $f$ is continuous.*

*Proof.* Let $x \in X$, $r > 0$ and $x'' \in X$ be such that $d(x'', x) < r$.

Since $d(x'', x') \leq d(x'', x) + d(x', x)$, then

$$\begin{aligned} d(x'', x') - d(x, x') &\leq d(x'', x) \\ &< r \end{aligned}$$

and since $d(x, x') \leq d(x'', x) + d(x'', x')$, then

$$\begin{aligned} d(x, x') - d(x'', x') &\leq d(x'', x) \\ &< r, \end{aligned}$$

i.e. $-r < d(x'', x') - d(x, x') < r$ or, equivalently, $|f(x'') - f(x)| < r$.

Therefore, for all $x \in X$ and all $r > 0$, there exists $s > 0$ (namely $s = r$) such that if $d(x'', x) < s$, then $|f(x'') - f(x)| < r$, i.e. $f$ is continuous. Q.E.D.

**S48. The space of bounded functions to a complete metric space is itself complete with respect to the sup metric.** *If $X$ is a set and $(Y, d_Y)$ is a complete metric space, then $(\mathrm{B}(Y^X), d_\infty)$ is complete.*

*Proof.* Let $X$ be a set and $(Y, d_Y)$ be a complete metric space.

In $(\mathrm{B}(Y^X), d_\infty)$, let $f \in (\mathrm{B}(Y^X))^{\mathbb{N}}$ be Cauchy, and let $r > 0$. Since $f$ is Cauchy, then there exists $n \in \mathbb{N}$ such that, for all $n', n'' > n$, $d_\infty(f_{n'}, f_{n''}) < \frac{1}{2}r$. Since, for all $n', n'' > n$, $d_\infty(f_{n'}, f_{n''}) < \frac{1}{2}r$, then, for all $x \in X$ and all $n', n'' > n$,

$$\begin{aligned} &d_Y(f_{n'}(x), f_{n''}(x)) \leq \\ &\sup_{x \in X} d_Y(f_{n'}(x), f_{n''}(x)) = d_\infty(f_{n'}, f_{n''}) \\ &\qquad < \frac{1}{2}r. \end{aligned}$$

Since, for all $r > 0$, there exists $n \in \mathbb{N}$ such that, for all $x \in X$ and all $n', n'' > n$, $d_Y(f_{n'}(x), f_{n''}(x)) < \frac{1}{2}r$, then, for all $x \in X$, $f(x) \in Y^{\mathbb{N}}$ is Cauchy. Since, for all $x \in X$, $f(x)$ is Cauchy and $(Y, d_Y)$ is complete, then, for all $x \in X$, $f(x)$ is convergent. Let $\phi \in Y^X$ be such that, for all $x \in X$, $(x, \lim f(x)) \in \phi$. Since, and all $r > 0$, there exists $n \in \mathbb{N}$ such that, for all $x \in X$ and all $n', n'' > n$, $d_Y(f_{n'}(x), f_{n''}(x)) < \frac{1}{2}r$, and $d_Y$ is continuous, then, for all $r > 0$, there exists $n \in \mathbb{N}$ such that, for all $x \in X$ and all $n' > n$,

$$\begin{aligned} &d_Y(f_{n'}(x), \phi(x)) = \\ &d_Y(f_{n'}(x), \lim_{n'' \to \infty} f_{n''}(x)) = \lim_{n'' \to \infty} d_Y(f_{n'}(x), f_{n''}(x)) \\ &\qquad \leq \frac{1}{2}r. \end{aligned}$$

Since, for all $r > 0$, there exists $n \in \mathbb{N}$ such that, for all $x \in X$ and all $n' > n$, $d_Y(f_{n'}(x), \phi(x)) \leq \frac{1}{2}r$, then, for all $r > 0$, there exists $n \in \mathbb{N}$ such that, for all $n' > n$,

$$\begin{aligned} &d_\infty(f_{n'}, \phi) = \\ &\sup_{x \in X} d_Y(f_{n'}(x), \phi(x)) \leq \frac{1}{2}r \\ &\qquad < r \end{aligned}$$

Since, for all $r > 0$, there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $d_\infty(f_{n'}, \phi) < r$, then $\phi = \lim f$.

Let $r > 0$. Since $\phi = \lim f$, then there exists $n \in \mathbb{N}$ such that, for all $n' > n$, $d_\infty(f_{n'}, \phi) < r$. Since, for all $n' > n$, $d_\infty(f_{n'}, \phi) < r$, then, for all $x \in X$ and all

$n' > n$,

$$\begin{aligned} & d_Y(f_{n'}(x), \phi(x)) \leq \\ & \sup_{x \in X} d_Y(f_{n'}(x), \phi(x)) = d_\infty(f_{n'}, \phi) \\ & \qquad < r. \end{aligned}$$

Since, for all $n \in \mathbb{N}$, $f_n \in \mathrm{B}(Y^X)$, then, for all $n \in \mathbb{N}$, there exist $y \in Y$ and $r' > 0$ such that, for all $x \in X$, $d_Y(f_n(x), y) < r'$. Let $n' > n$. Since $n' > n$, then, for all $x \in X$, $d_Y(f_{n'}(x), \phi(x)) < r$ and there exist $y \in Y$ and $r' > 0$ such that, for all $x \in X$, $d_Y(f_{n'}(x), y) < r'$. Since, for all $x \in X$, $d_Y(f_{n'}(x), \phi(x)) < r$ and $d_Y(f_{n'}(x), y) < r'$, then, for all $x \in X$,

$$\begin{aligned} d_Y(\phi(x), y) &\leq d_Y(f_{n'}(x), \phi(x)) + d_Y(f_{n'}(x), y) \\ &< r + r'. \end{aligned}$$

Since, for all $r > 0$, there exist $y \in Y$ and $r' > 0$ such that, for all $x \in X$, $d_Y(\phi(x), y) < r + r'$, then $\phi = \lim f \in \mathrm{B}(Y^X)$.

Therefore, in $(\mathrm{B}(Y^X), d_\infty)$, since, for all $f \in (\mathrm{B}(Y^X))^{\mathbb{N}}$ Cauchy, $f$ is convergent and $\lim f \in \mathrm{B}(Y^X)$, then $(\mathrm{B}(Y^X), d_\infty)$ is complete. Q.E.D.

As a consequence of the previous proposition both $\ell_\infty^n$ and $\ell_\infty$ are complete, as is the space of continuous, bounded functions, taking values in a complete metric space with respect to the sup metric. The reason is that it is a closed set of the previous metric space, and any closed set of a complete metric space is complete as well. As a consequence any metric subspace that is a closed set of an ambient metric space which is complete, is complete as well. Conversely, if a metric subspace is complete it must be closed with respect to the ambient metric space.

**S49. Every closed set of a complete metric space is complete.** *If $(X, d)$ is a complete metric space and $A$ is a closed set of $(X, d)$, then $A$ is a complete set of $(X, d)$.*

*Proof.* Let $(X, d)$ be a complete metric space, $A$ is a closed set of $(X, d)$, and $s \in A^{\mathbb{N}}$ be Cauchy.

Since $s(\mathbb{N}) \subset A$ and $A \subset X$, then $s(\mathbb{N}) \subset X$, i.e. $s \in X^{\mathbb{N}}$. Since $s$ is Cauchy and $X$ is a complete set of $(X, d)$, then there exists $x \in X$ such that $x = \lim s$. Since $x = \lim s$, then $x \in \mathrm{Cl}s(\mathbb{N})$, i.e. for all $r > 0$, $s(\mathbb{N}) \cap B_r(x) \neq \phi$. Since $s(\mathbb{N}) \subset A$, then, for all $r > 0$, $s(\mathbb{N}) \cap B_r(x) \subset A \cap B_r(x)$. Since, for all $r > 0$, $s(\mathbb{N}) \cap B_r(x) \subset A \cap B_r(x)$ and $s(\mathbb{N}) \cap B_r(x) \neq \phi$, then $A \cap B_r(x) \neq \phi$, i.e. $x \in \mathrm{Cl}A$. Since $x \in \mathrm{Cl}A$ and $\mathrm{Cl}A \subset A$, then $x \in A$, i.e. $\lim s \in A$. Therefore, since, for all $s \in A^{\mathbb{N}}$ Cauchy, $s$ is convergent and $\lim s \in A$ then $A$ is a complete set of $(X, d)$. Q.E.D.

**S50. Every complete metric subspace is closed in the ambient space.** *If $(X, d)$ is a metric space, $X' \subset X$ and $(X', d|_{X'})$ is complete, then $X'$ is a closed set of $(X, d)$.*

*Proof.* Let $(X, d)$ be a metric space, $X' \subset X$, $(X', d|_{X'})$ be complete, and $x \in \mathrm{Cl}X'$ with respect to $(X, d)$.

Since $x$ is a closure point of $X'$ with respect to $(X, d)$, then either $x$ is an isolated point of $X'$ or $x$ is an accumulation point of $X'$.

(1) If $x$ is an isolated point of $X'$, then $x \in X'$.

(2) If $x$ is an accumulation point of $X'$, then there exists $s \in (X' \setminus \{x\})^{\mathbb{N}}$ such that $\lim s = x$. Since $s$ is convergent, then $s$ is Cauchy. Since $s$ is Cauchy and $(X', d|_{X'})$ is complete, then $\lim s \in X'$. Since $\lim s \in X'$ and $x = \lim s$, then $x \in X'$.

Therefore, $\mathrm{Cl}X' \subset X'$, i.e. $X'$ is a closed set of $(X, d)$.
Q.E.D.

**S51. The space of continuous, bounded functions from a metric space to a complete metric space is itself complete with respect to the sup metric.** *If* $(X, d_X)$ *and* $(Y, d_Y)$ *are metric spaces and* $(Y, d_Y)$ *is complete, then* $\mathrm{CB}(Y^X)$ *with* $d_\infty$ *is a complete metric space.*

*Proof.* Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, $(Y, d_Y)$ be complete, $\{f_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $\mathrm{CB}(Y^X)$ with respect to $d_\infty$, $f \in \mathrm{B}(Y^X)$ be its limit, $x \in X$ and $r > 0$.

Since $f$ is the limit of $\{f_n\}_{n\in\mathbb{N}}$, there exists $N \in \mathbb{N}$ such that, for all $n > N$,

$$d_\infty(f_n, f) =$$

$$\sup_{x\in X} d_Y(f_n(x), f(x)) < \frac{r}{3}$$

i.e.

$$d_Y(f_n(x), f(x)) < \frac{r}{3}$$

for all $x \in X$.

Let $n > N$. Since $f_n$ is continuous, there exists $s > 0$ such that

$$f_n(B_s(x)) \subset B_{\frac{r}{3}}(f_n(x)).$$

Let $x' \in B_s(x)$ Then

$$\begin{aligned} d_Y(f(x'), f(x)) &\le d_Y(f(x'), f_n(x')) + d_Y(f_n(x'), f_n(x)) + d_Y(f_n(x), f(x)) \\ &< \frac{r}{3} + \frac{r}{3} + \frac{r}{3} \\ &= r \end{aligned}$$

i.e. $f(B_s(x)) \subset B_r(f(x))$ and hence $f$ is continuous at any $x \in X$. Therefore $f \in \mathrm{CB}(Y^X)$ as well and $\mathrm{CB}(Y^X)$ is thus closed.

Finally, since $\mathrm{B}(Y^X)$ with $d_\infty$ is complete, because $(Y, d_Y)$ is complete, and $\mathrm{CB}(Y^X) \subset \mathrm{B}(Y^X)$ is closed, then $\mathrm{CB}(Y^X)$ with $d_\infty$ is a complete metric space.
Q.E.D.

**S52.** *If* $(X, d_X)$ *and* $(Y, d_Y)$ *are metric spaces* $(Y, d_Y)$ *is complete,* $\{f_n\}_{n\in\mathbb{N}}$ *is a convergent sequence in* $\mathrm{CB}(Y^X)$ *with respect to* $d_\infty$*, and* $x \in X^{\mathbb{N}}$ *is convergent, then*

$$\lim_{n'\to\infty}\big((\lim_{n\to\infty} f_n)(x_{n'})\big) = \big(\lim_{n\to\infty} f_n\big)\big(\lim_{n\to\infty} x_{n'}\big).$$

*Proof.* Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces $(Y, d_Y)$ be complete, $\{f_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $\mathrm{CB}(Y^X)$ with respect to $d_\infty$ and $x \in X^{\mathbb{N}}$ be convergent.

Since $\{f_n\}_{n\in\mathbb{N}}$ is convergent, then $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy and $\mathrm{CB}(Y^X)$ with $d_\infty$ is complete, then $\lim_{n'\to\infty} f'_n \in \mathrm{CB}(Y^X)$, i.e. $\lim_{n'\to\infty} f'_n$ is continuous. Since $\lim_{n'\to\infty} f'_n$ is continuous and $x$ is convergent, then

$$\lim_{n\to\infty}\big((\lim_{n'\to\infty} f_{n'})(x_n)\big) = \big(\lim_{n'\to\infty} f_{n'}\big)(\lim_{n'\to\infty} x_n).$$

Q.E.D.