Lecture Notes for Econ 898

CONSTRAINED MAXIMA

Julio Dávila

Department of Economics
University of Pennsylvania

In this section we consider the conditions that characterize the maxima of real-valued functions in a constrained domain. It may be surprising how little is needed to characterize sufficiently a maximizer x^* of some real-valued function f among the x's that make non-negative some finite set of real-valued functions g_i, $i = 1, \dots, m$, i.e. a solution x^* to

$$\begin{aligned} \max_x \ & f(x) \\ 0 \leq & g(x). \end{aligned} \tag{1}$$

In effect, it suffices that there exist some set of non-negative reals (**multipliers**) $\lambda^* \geq 0$ such that x^* maximizes the auxiliary (**Lagrangian**) function $f(x) + \lambda^* g(x)$ while λ^* minimizes $f(x^*) + \lambda g(x^*)$, for all $\lambda \geq 0$, i.e.

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*) \leq f(x^*) + \lambda \cdot g(x^*). \tag{2}$$

As a matter of fact, that is all that is needed. No condition on any derivative is needed, actually the functions do not even need to be differentiable. What is more, the space X on which the functions are defined needs not be a vector space (needless to say, finite-dimensionality cannot and does not play any role here) or even have any kind of structure on it. It needs just be a plain set.

Quite on the contrary, for a similar necessary characterization of a solution x^* to (1) above (namely **Slater's Theorem**) a lot of structure has to be added, namely X has to be an separable, complete, inner product space[1] whose unit ball has a compact boundary. Moreover, the constraints have to satisfy Slater's condition: there must exist some $\tilde{x}$ such that $g(\tilde{x}) \in \mathbb{R}^m_{++}$. Under these conditions for every solution x^* to (1) above there exists some $\lambda^* \geq 0$ such that, for all admissible x and all $\lambda \geq 0$, conditions (2) above are satisfied.

The need for so much structure on the ambient space X (and for Slater's condition as well) is a consequence of the fact that what drives this sufficient characterization is the **separating hyperplane theorem**, according to which two disjoint, nonempty, convex sets in a complete, separable, inner product space whose unit ball has a compact boundary, are "separated" by some hyperplane, i.e. there exists

[1] That is to say, a separable Hilbert space.

some hyperplane that splits the space in two half-spaces that contain only one of the convex sets each.

When the space in which the functions are defined is a vector space and the functions are differentiable at the solution x^* (or candidate to solution) of the problem (1) above, the idea that x^* has to maximize the Lagrangian function for some λ^* that, in its turn, minimizes the same Lagrangian given x^* still holds true.

Once again, the sufficient characterization is very little demanding and requires only (i) that the graph of the Lagrangian is entirely below its linearization at x^* given λ^* (this would be guaranteed by the concavity of this function, but this is an unnecessarily strong requirement), and (ii) that

(1) the differential at x^* of the Lagrangian with respect to x takes non-positive values within the constrained domain,
(2) it vanishes at x^*, and
(3) the vector of multipliers λ^* and the vector of constraints values $g(x^*)$ are orthogonal.

When the space is $\mathbb{R}^n$ these conditions are

$$\begin{aligned} Df(x^*) + \lambda^* Dg(x^*) &\leq 0 \\ \big(Df(x^*) + \lambda^* Dg(x^*)\big) \cdot x^* &= 0 \\ \lambda^* \cdot g(x^*) &= 0. \end{aligned}$$

It is straightforward to see that the sufficient conditions for x^* to be a solution top (1) are equivalent to have x^* maximizing the Lagrangian $\mathcal{L}$ for some λ^* that minimizes $\mathcal{L}$ given x^* (the **Kuhn-Tucker conditions**).

For a necessary characterization using the Kuhn-Tucker conditions another set of conditions is needed. In particular, the binding constraints at a solution x^* to (1) must satisfy the **constraints qualification condition** by which there must exist a differentiable, feasible path starting from x^* whose derivative coincides with any vector with non-negative inner products with the gradients of the constraints that are binding at x^*. Whenever that is the case, we can use **Farkas lemma** to prove that the Kuhn-Tucker conditions are necessarily satisfied by a solution to (1). (Farkas lemma states that in a separable, complete, inner product space[2] with compact closed unit ball any vector b with non-negative inner product with every vector having non-negative inner products with vectors a^i of a finite set, has to be in the positive cone of the latter. It is a consequence of the strict separation from the null vector of any nonempty, convex, closed set in that space that does not contain the null vector.)

Finally, still another entirely equivalent way to state the Kuhn-Tucker conditions is as complementary slackness conditions.

Precise statements and their proofs follow.

Sufficient Conditions for Constrained Maxima

Sufficient condition for a constrained maximum. *If X is a set, $f \in \mathbb{R}^X$, $g \in (\mathbb{R}^m)^X$, and $x^* \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$, $\lambda^* \in \mathbb{R}^m_+$ are such that, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ and all $\lambda \in \mathbb{R}^m_+$,*

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*) \leq f(x^*) + \lambda \cdot g(x^*)$$

[2] That is to say, in a separable Hilbert space.

then

$$x^* \in \arg\max_x f(x)$$
$$g(x) \geq 0.$$

Proof. Let X be a set, $f \in \mathbb{R}^X$, $g \in (\mathbb{R}^m)^X$, and $x^* \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$, $\lambda^* \in \mathbb{R}^m_+$ be such that, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ and all $\lambda \in \mathbb{R}^m_+$,

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*) \leq f(x^*) + \lambda \cdot g(x^*).$$

Since, for all $\lambda \in \mathbb{R}^m_+$, $f(x^*) + \lambda^* \cdot g(x^*) \leq f(x^*) + \lambda \cdot g(x^*)$, then, for all $\lambda \in \mathbb{R}^m_+$,

$$0 \leq (\lambda - \lambda^*) \cdot g(x^*).$$

Since, for all $\lambda \in \mathbb{R}^m_+$, $0 \leq (\lambda - \lambda^*) \cdot g(x^*)$, then

$$0 \leq g(x^*)$$

and

$$0 \leq -\lambda^* \cdot g(x^*).$$

Since $0 \leq g(x^*)$, $0 \leq -\lambda^* \cdot g(x^*)$, and $\lambda^* \in \mathbb{R}^m_+$, then

$$0 = \lambda^* \cdot g(x^*).$$

Since $\lambda^* \in \mathbb{R}^m_+$, $0 = \lambda^* \cdot g(x^*)$ and $f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*)$, then, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ such that $g(x) \geq 0$,

$$\begin{aligned} f(x) &\leq \\ f(x) + \lambda^* \cdot g(x) &\leq f(x^*) + \lambda^* \cdot g(x^*) \\ &= f(x^*). \end{aligned}$$

Q.E.D.

Necessary Conditions for Constrained Maxima

Preliminary Lemmas.

Lemma. *If X is a complete inner product space*[3] *and $C \subset X$ is nonempty, closed, and convex, then*

$$\arg\min_{x \in C} \|x\| \neq \phi.$$

Proof. Let X be a complete inner product space and $C \subset X$ be nonempty, closed, and convex.

Since $C \neq \phi$ and, for all $x \in C$, $0 \leq \|x\|$, then there exist $\inf_{x \in C} \|x\|$ and $x \in C^{\mathbb{N}}$ such that

$$\lim_{n \to \infty} \|x_n\| = \inf_{x \in C} \|x\|.$$

[3] That is to say, in a Hilbert space.

Since, for all $n, n' \in \mathbb{N}$, $x_n, x_{n'} \in C$ and C is convex, then

$$\frac{1}{2}(x_n + x_{n'}) \in C.$$

Since, for all $n, n' \in \mathbb{N}$, $\frac{1}{2}(x_n + x_{n'}) \in C$, then, for all $n, n' \in \mathbb{N}$,

$$\inf_{x \in C} \|x\| \leq \frac{1}{2}\|x_n + x_{n'}\|.$$

Since, for all $n, n' \in \mathbb{N}$, $\inf_{x \in C} \|x\| \leq \frac{1}{2}\|x_n + x_{n'}\|$ and

$$\|x_n - x_{n'}\|^2 + \|x_n + x_{n'}\|^2 = 2\|x_n\|^2 + 2\|x_{n'}\|^2,$$

then, for all $n, n' \in \mathbb{N}$,

$$\|x_n - x_{n'}\|^2 + (2 \inf_{x \in C} \|x\|)^2 \leq 2\|x_n\|^2 + 2\|x_{n'}\|^2.$$

Since, for all $n, n' \in \mathbb{N}$, $\|x_n - x_{n'}\|^2 + (2 \inf_{x \in C} \|x\|)^2 \leq 2\|x_n\|^2 + 2\|x_{n'}\|^2$ and $\lim_{n \to \infty} \|x_n\| = \inf_{x \in C} \|x\|$, then,

$$\limsup_{n,n' \in \mathbb{N}} \|x_n - x_{n'}\|^2 + (2 \inf_{x \in C} \|x\|)^2 \leq 2(\inf_{x \in C} \|x\|)^2 + 2(\inf_{x \in C} \|x\|)^2.$$

i.e.

$$\limsup_{n,n' \in \mathbb{N}} \|x_n - x_{n'}\|^2 \leq 0$$

Since $\limsup_{n,n' \in \mathbb{N}} \|x_n - x_{n'}\|^2 \leq 0$ and

$$0 \leq \liminf_{n,n' \in \mathbb{N}} \|x_n - x_{n'}\|^2 \leq \limsup_{n,n' \in \mathbb{N}} \|x_n - x_{n'}\|^2 \leq 0,$$

then

$$0 = \liminf_{n,n' \in \mathbb{N}} \|x_n - x_{n'}\|^2 = \lim_{n,n' \to \infty} \|x_n - x_{n'}\|^2 = \limsup_{n,n' \in \mathbb{N}} \|x_n - x_{n'}\|^2 = 0.$$

Since $0 = \lim_{n,n' \to \infty} \|x_n - x_{n'}\|^2$ and X is complete, then there exists $\lim_{n \to \infty} x_n$. Since $x \in C^{\mathbb{N}}$, there exists $\lim_{n \to \infty} x_n$, and C is closed, then

$$\lim_{n \to \infty} x_n \in C.$$

Since $\| \lim_{n \to \infty} x_n\| = \lim_{n \to \infty} \|x_n\| = \inf_{x \in C} \|x\|$, and $\lim_{n \to \infty} x_n \in C$, then

$$\lim_{n \to \infty} x_n \in \arg\min_{x \in C} \|x\|.$$

Since there exists $x \in C^{\mathbb{N}}$ convergent such that $\lim_{n \to \infty} x_n \in \arg\min_{x \in C} \|x\|$, then

$$\arg\min_{x \in C} \|x\| \neq \phi.$$

Q.E.D.

Lemma. *If X is an inner product space, $C \subset X$ is convex, and $p \in \arg\min_{x \in C} \|x\|$, then, for all $x \in C$,*

$$0 \leq p \cdot (x - p).$$

Proof. Let X be an inner product space, $C \subset X$ be convex, and $p \in \arg\min_{x \in C} \|x\|$.

Let $x \in C$. Since $p \in \arg\min_{x \in C} \|x\|$, then $p \in C$. Since $x, p \in C$ and C is convex, then, for all $\alpha \in (0, 1)$,

$$\alpha x + (1 - \alpha)p \in C.$$

Since, for all $\alpha \in (0, 1)$, $\alpha x + (1 - \alpha)p \in C$ and $p \in \arg\min_{x \in C} \|x\|$, then, for all $\alpha \in (0, 1)$,

$$\|p\| \leq \|\alpha x + (1 - \alpha)p\|$$

and hence

$$\begin{aligned} \|p\|^2 &\leq \|p + \alpha(x - p)\|^2 \\ &= \|p\|^2 + 2\alpha(x - p) \cdot p + \alpha^2 \|x - p\|^2 \end{aligned}$$

i.e.

$$0 \leq p \cdot (x - p) + \frac{1}{2}\alpha \|x - p\|^2.$$

Since, for all $\alpha \in (0, 1)$, $0 \leq p \cdot (x - p) + \frac{1}{2}\alpha \|x - p\|^2$, then

$$\begin{aligned} p \cdot (x - p) = \\ p \cdot (x - p) + \lim_{\alpha \to 0} \frac{1}{2}\alpha \|x - p\|^2 &= \lim_{\alpha \to 0} (p \cdot (x - p) + \frac{1}{2}\alpha \|x - p\|^2) \\ &\geq 0. \end{aligned}$$

Q.E.D.

Separating hyperplane 1. *If X is a complete inner product space*[4] *and $C \subset X$ is nonempty, closed, and convex, then there exists $p \in C$ such that, for all $x \in C$,*

$$0 \leq p \cdot (x - p).$$

Proof. This is a dtraightforward corollary of the two previous lemmas.

Separating hyperplane 2. *If X is a separable, complete, inner product space*[5] *such that $\partial B_1(0)$ is compact, and $C \subset X$ is nonempty, convex, and such that $0 \notin C$, then there exist $p \in X \setminus \{0\}$ and $q \in X$ such that, for all $x \in C$,*

$$0 \leq p \cdot (x - q).$$

Proof. Let X be a separable, complete, inner product space such that $\partial B_1(0)$ is compact, and $C \subset X$ be nonempty, convex, and such that $0 \notin C$.

[4] That is to say, a Hilbert space.

[5] That is to say, a separable Hilbert space.

Since X is separable and $C \subset X$, then there exists $x \in C^{\mathbb{N}}$ such that

$$C \subset \mathrm{Cl}x(\mathbb{N}).$$

Let, for all $N \in \mathbb{N}$,

$$C_N = \mathrm{Co}\{x_n\}_{n=1}^N.$$

Since $C_N = \mathrm{Co}\{x_n\}_{n=1}^N$, then, for all $N, N' \in \mathbb{N}$ such that $N \leq N'$,

$$C_N \subset C_{N'} \subset C.$$

Since X is a complete inner product space and, for all $N \in \mathbb{N}$, C_N is nonempty, closed, and convex, then, for all $N \in \mathbb{N}$, there exist $p_N \in \arg\min_{x \in C_N} \|x\|$ such that, for all $x \in C_N$,

$$0 \leq p_N \cdot (x - p_N).$$

Since, for all $N \in \mathbb{N}$, $C_N \subset C$ and $0 \notin C$, then, for all $N \in \mathbb{N}$, $0 \notin C_N$. Since for all $N \in \mathbb{N}$, $0 \notin C_N$ and $p_N \in C_N$, then for all $N \in \mathbb{N}$,

$$p_N \neq 0.$$

Since, for all $N \in \mathbb{N}$, $p_N \neq 0$, $\|p_N\|^{-1} p_N \in \partial B_1(0)$ and $\partial B_1(0)$ is compact, then there exists a convergent subsequence $\{\|p_{h(N)}\|^{-1} p_{h(N)}\}_{N \in \mathbb{N}}$. Let

$$\begin{aligned} p = {} & \lim_{N \to \infty} \|p_{h(N)}\|^{-1} p_{h(N)} \\ & \in \partial B_1(0). \end{aligned}$$

Since $p \in \partial B_1(0)$ and $0 \notin \partial B_1(0)$, then

$$p \neq 0.$$

Since, for all $N, N' \in \mathbb{N}$ such that $N \leq N'$, $C_{h(N)} \subset C_{h(N')}$ and, for all $N \in \mathbb{N}$, $p_{h(N)} \in \arg\min_{x \in C_{h(N)}} \|x\|$ then, for all $N, N' \in \mathbb{N}$ such that $N \leq N'$,

$$\begin{aligned} & \|p_{h(N')}\| = \\ & \min_{x \in C_{h(N')}} \|x\| \leq \min_{x \in C_{h(N)}} \|x\| \\ & \qquad = \|p_{h(N)}\|. \end{aligned}$$

Since, for all $N, N' \in \mathbb{N}$ such that $N \leq N'$, $\|p_{h(N')}\| \leq \|p_{h(N)}\|$ and, for all $N \in \mathbb{N}$, $0 \leq \|p_N\|$, then $\{\|p_{h(N)}\|\}_{N \in \mathbb{N}}$ is convergent and

$$0 \leq \lim_{N \to \infty} \|p_{h(N)}\|.$$

Let $x_0 \in C$. Since $x_0 \in C$ and $C \subset \mathrm{Cl}x(\mathbb{N})$, then there exists $x' \in C^{\mathbb{N}}$ such that, for all $N \in \mathbb{N}$, $x'_N \in C_N$ and

$$\lim_{N \to \infty} x'_N = x_0.$$

Since $\lim_{N \to \infty} x'_N = x_0$ then

$$\lim_{N \to \infty} x'_{h(N)} = x_0.$$

Since, for all $N \in \mathbb{N}$, $x'_N \in C_N$ and, for all $x \in C_N$, $0 \leq p_N \cdot (x - p_N)$ then, for all $N \in \mathbb{N}$,

$$0 \leq p_N \cdot (x'_N - p_N)$$

i.e.

$$\|p_N\| \leq \|p_N\|^{-1} p_N \cdot x'_N.$$

Since, for all $N \in \mathbb{N}$, $\|p_{h(N)}\| \leq \|p_{h(N)}\|^{-1} p_{h(N)} \cdot x_{h(N)}$, $\{\|p_{h(N)}\|\}_{N \in \mathbb{N}}$ is convergent, $\lim_{N\to\infty} \|p_{h(N)}\|^{-1} p_{h(N)} = p$, and $\lim_{N\to\infty} x'_{h(N)} = x_0$, then

$$\begin{aligned} \lim_{N\to\infty} \|p_{h(N)}\| &\leq \big(\lim_{N\to\infty} \|p_{h(N)}\|^{-1} p_{h(N)}\big) \cdot \big(\lim_{N\to\infty} x'_{h(N)}\big) \\ &\leq p \cdot x. \end{aligned}$$

Since $p \neq 0$, then $p(X) = \mathbb{R}$. Since $p(X) = \mathbb{R}$ and $\lim_{N\to\infty} \|p_{h(N)}\| \in \mathbb{R}$, then there exists $q \in X$ such that $p \cdot q = \lim_{N\to\infty} \|p_{h(N)}\| \geq 0$. Since $p \cdot q = \lim_{N\to\infty} \|p_{h(N)}\|$ and $\lim_{N\to\infty} \|p_{h(N)}\| \leq p \cdot x$, then

$$0 \leq p \cdot (x - q).$$

Q.E.D.

Separating Hyperplane Theorem. *If X is a separable, complete, inner product space*[6] *such that $\partial B_1(0)$ is compact, and $A, B \subset X$ are nonempty, disjoint, and convex, then there exist $p \in X \setminus \{0\}$ such that, for all $a \in A$ and all $b \in B$,*

$$p \cdot a \leq p \cdot b.$$

Proof. Let X be a separable, complete inner product space such that $\partial B_1(0)$ is compact, and $A, B \subset X$ be nonempty, disjoint, and convex.

Since A, B are nonempty and disjoint, then $B - A$ is nonempty. Since A, B are disjoint, then

$$0 \notin B - A.$$

Since A, B are convex, then $B - A$ is convex. Since $0 \notin B - A$ and $B - A$ is convex, then there exist $p \in X \setminus \{0\}$ and $q \in X$ such that, $0 \leq p \cdot q$ and, for all $c \in B - A$,

$$0 \leq p \cdot (c - q).$$

Let $a \in A$ and $b \in B$. Since $a \in A$ and $b \in B$, then

$$b - a \in B - A.$$

Since $b - a \in B - A$ and, for all $c \in B - A$, $0 \leq p \cdot (c - q)$, then

$$0 \leq p \cdot ((b - a) - q)$$

Since $0 \leq p \cdot q$, then

$$p \cdot a \leq p \cdot a + p \cdot q.$$

Since $p \cdot q \leq p \cdot (b - a)$, then

$$p \cdot a + p \cdot q \leq p \cdot b.$$

Since $p \cdot a \leq p \cdot a + p \cdot q$ and $p \cdot a + p \cdot q \leq p \cdot b$, then

$$p \cdot a \leq p \cdot a + p \cdot q \leq p \cdot b.$$

Q.E.D.

[6] That is to say, a separable Hilbert space.

Necessary condition for a constrained maximum (Slater's Theorem). *If X is a separable, complete, inner product space*[7] *such that $\partial B_1(0)$ is compact, $f \in \mathbb{R}^X$ and $g \in (\mathbb{R}^m)^X$ are concave,*[8] *there exists $\tilde{x} \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ such that*

$$g(\tilde{x}) \in \mathbb{R}^m_{++}, \qquad \text{(Slater's cond.)}$$

and

$$\begin{aligned} x^* \in \arg\max_x & f(x) \\ & g(x) \geq 0, \end{aligned}$$

then there exists $\lambda^* \in \mathbb{R}^m_+$ such that, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ and all $\lambda \in \mathbb{R}^m_+$,

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*) \leq f(x^*) + \lambda \cdot g(x^*).$$

Proof. Let X be a separable, complete, inner product space such that $\partial B_1(0)$ is compact, $f \in \mathbb{R}^X$ and $g \in (\mathbb{R}^m)^X$ be concave,[9] there exist $\tilde{x} \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ such that

$$g(\tilde{x}) \in \mathbb{R}^m_{++},$$

and

$$\begin{aligned} x^* \in \arg\max_x & f(x) \\ & g(x) \geq 0. \end{aligned}$$

Let $Y \subset \mathbb{R}^{1+m}$ be such that $(y_0, y) \in Y$ if, and only if, there exists $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ such that

$$\begin{aligned} y_0 &\leq f(x) \\ y &\leq g(x), \end{aligned}$$

and $Z \subset \mathbb{R}^{1+m}$ be such that $(z_0, z) \in Z$ if, and only if,

$$\begin{aligned} f(x^*) &< z_0 \\ 0 &< z. \end{aligned}$$

(1) $Y \cap Z = \phi$:

Let $(y_0, y) \in Y$ and $(z_0, z) \in Z$ be such that $(y_0, y) = (z_0, z)$. Since $(y_0, y) \in Y$, then there exists $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ such that

$$\begin{aligned} f(x^*) < z_0 = y_0 &\leq f(x) \\ 0 < z = y &\leq g(x) \end{aligned}$$

and therefore

$$\begin{aligned} x^* \notin \arg\max_x & f(x) \\ & g(x) \geq 0. \end{aligned}$$

[7] That is to say, a separable Hilbert space.

[8] And hence $f^{-1}(\mathbb{R})$ and $g^{-1}(\mathbb{R}^m)$ are convex. In the case of g, the function is coordinate-wise concave.

[9] See previous footnote.

(2) Y is convex :

Let $(y_0, y), (y_0', y') \in Y$. Since $(y_0, y), (y_0', y') \in Y$, then there exist $x, x' \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ such that

$$\begin{aligned} y_0 &\leq f(x) & y_0' &\leq f(x') \\ y &\leq g(x) & y' &\leq g(x') \end{aligned}$$

Since f, g are concave, then, for all $\alpha \in (0,1)$,

$$\begin{aligned} \alpha y_0 + (1-\alpha) y_0' &\leq \alpha f(x) + (1-\alpha) f(x') \leq f(\alpha x + (1-\alpha)x') \\ \alpha y + (1-\alpha) y' &\leq \alpha g(x) + (1-\alpha) g(x') \leq g(\alpha x + (1-\alpha)x') \end{aligned}$$

Since there exists $\alpha x + (1-\alpha)x' \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ such that

$$\begin{aligned} \alpha y_0 + (1-\alpha) y_0' &\leq f(\alpha x + (1-\alpha)x') \\ \alpha y + (1-\alpha) y' &\leq g(\alpha x + (1-\alpha)x') \end{aligned}$$

then $\alpha(y_0, y) + (1-\alpha)(y_0', y') \in Y$.

(3) Z is convex :

Since Z is a cartesian product of half-spaces and every half space is convex, then Z is convex

Since Y, Z are disjoint and convex, then[10] there exists $(p_0, p) \in \mathbb{R}^{1+m} \setminus \{0\}$ such that, for all $(y_0, y) \in Y$ and all $(z_0, z) \in Z$,

$$p_0 y_0 + p \cdot y \leq p_0 z_0 + p \cdot z$$

(1) $(p_0, p) \geq 0$:

In effect, should not $(p_0, p) \geq 0$ hold, then $p_0 z_0 + p \cdot z$ would be unbounded below in Z!

(2) $p_0 \neq 0$:

Let $\varepsilon > 0$ and

$$\begin{aligned} (\tilde{y}_0, \tilde{y}) &= (f(\tilde{x}), g(\tilde{x})) \\ (\tilde{z}_0, \tilde{z}) &= (f(x^*) + \varepsilon, \varepsilon \mathbf{1}). \end{aligned}$$

Then $(\tilde{y}_0, \tilde{y}) \in Y$ and $(\tilde{z}_0, \tilde{z}) \in Z$. Therefore

$$\begin{aligned} p_0 f(\tilde{x}) + p \cdot g(\tilde{x}) &= \\ p_0 \tilde{y}_0 + p \cdot \tilde{y} &\leq p_0 \tilde{z}_0 + p \cdot \tilde{z} \\ &= p_0 (f(x^*) + \varepsilon) + p \cdot \varepsilon \mathbf{1} \end{aligned}$$

Since, for all $\varepsilon > 0$,

$$p_0 f(\tilde{x}) + p \cdot g(\tilde{x}) \leq p_0 (f(x^*) + \varepsilon) + p \cdot \varepsilon \mathbf{1}$$

[10] By the separating hyperplane theorem.

then

$$\begin{aligned} p_0 f(\tilde{x}) + p \cdot g(\tilde{x}) &\leq \lim_{\varepsilon \to 0} \left(p_0(f(x^*) + \varepsilon) + p \cdot \varepsilon \mathbf{1} \right) \\ &= p_0 f(x^*) \end{aligned}$$

Assume that $p_0 = 0$. Since $p_0 = 0$ and $(p_0, p) \neq 0$, then

$$p \neq 0.$$

Since $p_0 = 0$ and $p_0 f(\tilde{x}) + p \cdot g(\tilde{x}) \leq p_0 f(x^*)$, then

$$p \cdot g(\tilde{x}) \leq 0$$

But since $p \geq 0$, $p \neq 0$, and $g(\tilde{x}) > 0$, then

$$p \cdot g(\tilde{x}) > 0!$$

Therefore

$$p_0 \neq 0.$$

Let $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$, and

$$\begin{aligned} (y_0, y) &= (f(x), g(x)) \\ (z_0, z) &= (f(x^*) + \varepsilon, \varepsilon \mathbf{1}) \end{aligned}$$

Then $(y_0, y) \in Y$ and $(z_0, z) \in Z$. Therefore

$$\begin{aligned} & p_0 f(x) + p \cdot g(x) = \\ & \qquad p_0 y_0 + p \cdot y \leq p_0 z_0 + p \cdot z \\ & \qquad\qquad = p_0(f(x^*) + \varepsilon) + p \cdot \varepsilon \mathbf{1}. \end{aligned}$$

Since, for all $\varepsilon > 0$,

$$p_0 f(x) + p \cdot g(x) \leq p_0(f(x^*) + \varepsilon) + p \cdot \varepsilon \mathbf{1},$$

then

$$\begin{aligned} p_0 f(x) + p \cdot g(x) &\leq \lim_{\varepsilon \to 0} \left(p_0(f(x^*) + \varepsilon) + p \cdot \varepsilon \mathbf{1} \right) \\ &= p_0 f(x^*). \end{aligned}$$

Let $\lambda^* = p_0^{-1} p$. Since $p_0 f(x) + p \cdot g(x) \leq p_0 f(x^*)$ and $\lambda^* = p_0^{-1} p$, then

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*).$$

Since

$$\begin{aligned} x^* \in \arg \max_x & f(x) \\ & g(x) \geq 0, \end{aligned}$$

then

$$\begin{aligned} x^* &\in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m) \\ g(x^*) &\geq 0. \end{aligned}$$

Since $x^* \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ and, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$, $f(x) + \lambda^* \cdot g(x) \leq f(x^*)$, then

$$f(x^*) + \lambda^* \cdot g(x^*) \leq f(x^*)$$

i.e.

$$\lambda^* \cdot g(x^*) \leq 0.$$

Since $g(x^*) \geq 0$, $\lambda^* \geq 0$, and $\lambda^* \cdot g(x^*) \leq 0$, then

$$\lambda^* \cdot g(x^*) = 0.$$

Since $f(x) + \lambda^* \cdot g(x) \leq f(x^*)$ and $\lambda^* \cdot g(x^*) = 0$, then

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*).$$

Since $\lambda^* \cdot g(x^*) = 0$ and $g(x^*) \geq 0$ then, for all $\lambda \geq 0$,

$$f(x^*) + \lambda^* \cdot g(x^*) \leq f(x^*) + \lambda \cdot g(x^*)$$

Therefore, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ and all $\lambda \in \mathbb{R}^m_+$,

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*) \leq f(x^*) + \lambda \cdot g(x^*).$$

Q.E.D.

Conditions for constrained maxima with differentiability

Sufficient conditions for a constrained maximum. *If $f \in \mathbb{R}^{\mathbb{R}^n}$ and $g \in (\mathbb{R}^m)^{\mathbb{R}^n}$ are differentiable at $x^* \in \mathbb{R}^n_+$ such that $g(x^*) \geq 0$, $\lambda^* \in \mathbb{R}^m_+$ is such that, for all $x \geq 0$,*

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*) + (x - x^*) \cdot \big(Df(x^*) + \lambda^* Dg(x^*)\big)$$

and

$$\begin{aligned} Df(x^*) + \lambda^* Dg(x^*) &\leq 0 \\ \big(Df(x^*) + \lambda^* Dg(x^*)\big) \cdot x^* &= 0 \\ \lambda^* \cdot g(x^*) &= 0 \end{aligned}$$

then

$$\begin{aligned} x^* \in \arg\max_x & f(x) \\ g(x) &\geq 0 \\ x &\geq 0. \end{aligned}$$

Proof. Let $f \in \mathbb{R}^{\mathbb{R}^n}$ and $g \in (\mathbb{R}^m)^{\mathbb{R}^n}$ be differentiable at $x^* \in \mathbb{R}^n_+$ such that $g(x^*) \geq 0$, $\lambda^* \in \mathbb{R}^m_+$ be such that, for all $x \geq 0$,

$$f(x) + \lambda^* g(x) \leq f(x^*) + \lambda^* g(x^*) + (x - x^*) \cdot \big(Df(x^*) + \lambda^* Dg(x^*)\big)$$

and

$$\begin{aligned} Df(x^*) + \lambda^* Dg(x^*) &\leq 0 \\ \big(Df(x^*) + \lambda^* Dg(x^*)\big) \cdot x^* &= 0 \\ \lambda^* \cdot g(x^*) &= 0. \end{aligned}$$

Let $x \geq 0$ be such that

$$g(x) \geq 0.$$

Since $\lambda^* \geq 0$, $g(x) \geq 0$, $\lambda^* \cdot g(x^*) = 0$, $\big(Df(x^*) + \lambda^* Dg(x^*)\big) \cdot x^* = 0$, $x \geq 0$, and $Df(x^*) + \lambda^* Dg(x^*) \leq 0$, then

$$\begin{aligned} f(x) &\leq \\ f(x) + \lambda^* g(x) &\leq f(x^*) + \lambda^* \cdot g(x^*) + (x - x^*) \cdot \big(Df(x^*) + \lambda^* Dg(x^*)\big) \\ &= f(x^*) + x \cdot \big(Df(x^*) + \lambda^* Dg(x^*)\big) \\ &\leq f(x^*). \end{aligned}$$

Q.E.D.

Sufficient Kuhn-Tucker conditions for a constrained maximum. *If $f \in \mathbb{R}^{\mathbb{R}^n}$ and $g \in (\mathbb{R}^m)^{\mathbb{R}^n}$ are differentiable at $x^* \in \mathbb{R}^n_+$, $\mathcal{L}(x, \lambda) = f(x) + \lambda \cdot g(x)$, $\lambda^* \in \mathbb{R}^m_+$ is such that, for all $x \geq 0$,*

$$\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) + (x - x^*) \cdot D_x \mathcal{L}(x^*, \lambda^*)$$

and

$$\begin{aligned} D_x \mathcal{L}(x^*, \lambda^*) &\leq 0 \\ x^* \cdot D_x \mathcal{L}(x^*, \lambda^*) &= 0 \\ x^* &\geq 0 \\ D_\lambda \mathcal{L}(x^*, \lambda^*) &\geq 0 \\ \lambda^* \cdot D_\lambda \mathcal{L}(x^*, \lambda^*) &= 0 \\ \lambda^* &\geq 0 \end{aligned}$$

then

$$\begin{aligned} x^* \in \arg\max_x & f(x) \\ g(x) &\geq 0 \\ x &\geq 0 \end{aligned}$$

Proof. This is a straightforward corollary of the previous sufficient conditions.

S2. Sufficient conditions for a constrained maximum. *If X is a real vector space, $f \in \mathbb{R}^X$ and $g \in (\mathbb{R}^m)^X$ are differentiable at $x^* \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m_+)$, $\lambda^* \in \mathbb{R}^m_+$ is such that, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$,*

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*) + \big(df_{x^*} + \lambda^* dg_{x^*}\big)(x - x^*)$$

and

$$\begin{aligned} \big(df_{x^*} + \lambda^* dg_{x^*}\big)(x) &\leq 0 \\ \big(df_{x^*} + \lambda^* dg_{x^*}\big)(x^*) &= 0 \\ \lambda^* \cdot g(x^*) &= 0 \end{aligned}$$

then

$$\begin{aligned} x^* \in \arg\max_x & f(x) \\ g(x) &\geq 0. \end{aligned}$$

Proof. Let X be a real vector space, $f \in \mathbb{R}^X$ and $g \in (\mathbb{R}^m)^X$ be differentiable at $x^* \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m_+)$, $\lambda^* \in \mathbb{R}^m_+$ be such that, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$,

$$f(x) + \lambda^* \cdot g(x) \leq f(x^*) + \lambda^* \cdot g(x^*) + \big(df_{x^*} + \lambda^* dg_{x^*}\big)(x - x^*)$$

and

$$\begin{aligned}\big(df_{x^*} + \lambda^* dg_{x^*}\big)(x) &\leq 0\\ \big(df_{x^*} + \lambda^* dg_{x^*}\big)(x^*) &= 0\\ \lambda^* \cdot g(x^*) &= 0.\end{aligned}$$

Let $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ be such that

$$g(x) \geq 0.$$

Since $g(x) \geq 0$ and $\lambda^* \geq 0$, then

$$\begin{aligned} f(x) &\leq \\ f(x) + \lambda^* g(x) &\leq f(x^*) + \lambda^* g(x^*) + \big(df_{x^*} + \lambda^* dg_{x^*}\big)(x - x^*)\\ &= f(x^*) + \big(df_{x^*} + \lambda^* dg_{x^*}\big)(x)\\ &\leq f(x^*).\end{aligned}$$

Since $x^* \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m_+)$ and, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m_+)$, $f(x) \leq f(x^*)$, then

$$\begin{aligned} x^* &\in \arg\max f(x)\\ & g(x) \geq 0.\end{aligned}$$

Q.E.D.

Corollary (sufficient Kuhn-Tucker conditions for a constrained maximum). *If X is a real vector space, $f \in \mathbb{R}^X$ and $g \in (\mathbb{R}^m)^X$ are differentiable at $x^* \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$, $\mathcal{L} \in \mathbb{R}^{X \times \mathbb{R}^m}$ is such that, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$ and all $\lambda \in \mathbb{R}^m$,*

$$\mathcal{L}(x, \lambda) = f(x) + \lambda \cdot g(x),$$

$\lambda^* \in \mathbb{R}^m_+$ is such that, for all $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}^m)$,

$$\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) + (x - x^*) \cdot D_x \mathcal{L}(x^*, \lambda^*),$$

and

$$\begin{aligned} d_1\mathcal{L}_{(x^*,\lambda^*)}(x) &\leq 0\\ d_1\mathcal{L}_{(x^*,\lambda^*)}(x^*) &= 0\\ d_2\mathcal{L}_{(x^*,\lambda^*)}(\lambda) &\geq 0\\ d_2\mathcal{L}_{(x^*,\lambda^*)}(\lambda^*) &= 0\end{aligned}$$

then

$$\begin{aligned} x^* &\in \arg\max_x f(x)\\ & g(x) \geq 0.\end{aligned}$$

Proof. This is a straightforward corollary of S2.

Preliminary lemmas for the necessary conditions.

Strictly separating hyperplane. *If X is a separable, complete, inner product space*[11] *such that $\partial B_1(0)$ is compact, and $C \subset X$ is nonempty, closed, convex, and such that $0 \notin C$, then there exist $p \in X \setminus \{0\}$ such that, for all $x \in C$,*

$$0 < p \cdot x.$$

Proof. Exercise.

Farkas lemma. *If X is a separable, complete, inner product space*[12] *such that $\partial B_1(0)$ is compact, I is finite, $a \in X^I$, $b \in X$ and, for all $x \in X$ such that, for all $i \in I$, $0 \leq x \cdot a^i$,*

$$0 \leq x \cdot b,$$

then there exists $\lambda \in (\mathbb{R}_+)^I$ such that

$$b = \sum_{i \in I} \lambda^i a^i.$$

Proof. Let X be a separable, complete, inner product space such that $\partial B_1(0)$ is compact, I be finite, $a \in X^I$, and $b \in X$.

Assume that, for all $\lambda \in (\mathbb{R}_+)^I$,

$$b \neq \sum_{i \in I} \lambda^i a^i.$$

Since, for all $\lambda \in (\mathbb{R}_+)^I$, $b \neq \sum_{i \in I} \lambda^i a^i$, then

$$b \notin \langle a^i \rangle_{i \in I}^+.$$

Since $b \notin \langle a^i \rangle_{i \in I}^+$ and $b \in \{b\}$, then

$$0 \notin \langle a^i \rangle_{i \in I}^+ - \{b\}.$$

Since $0 \notin \langle a^i \rangle_{i \in I}^+ - \{b\}$ and $\langle a^i \rangle_{i \in I}^+ - \{b\}$ is nonempty, closed, and convex, then there exists $x \in X \setminus \{0\}$ such that, for all $x' \in \langle a^i \rangle_{i \in I}^+ - \{b\}$,

$$0 < x \cdot x'.$$

Let $\lambda \in (\mathbb{R}_+)^I$ and $\alpha > 0$. Since $\lambda \in (\mathbb{R}_+)^I$ and $\alpha > 0$, then

$$\sum_{i \in I} a^i \alpha \lambda^i \in \langle a^i \rangle_{i \in I}^+.$$

[11] That is to say, a separable Hilbert space.

[12] That is to say, a separable Hilbert space.

Since $\sum_{i\in I} a^i\alpha\lambda^i \in \langle a^i\rangle^+_{i\in I}$ and $b \in \{b\}$, then

$$\sum_{i\in I} a^i\alpha\lambda^i - b \in \langle a^i\rangle^+_{i\in I} - \{b\}.$$

Since $\sum_{i\in I} a^i\alpha\lambda^i - b \in \langle a^i\rangle^+_{i\in I} - \{b\}$ and, for all $x' \in \langle a^i\rangle^+_{i\in I} - \{b\}$, $0 < x \cdot x'$, then

$$0 < x \cdot \big(\sum_{i\in I} a^i\alpha\lambda^i - b\big)$$

i.e.

$$\frac{1}{\alpha}\, x \cdot b < x \cdot \sum_{i\in I} a^i\lambda^i$$

Since, for all $\lambda \in (\mathbb{R}_+)^I$ and all $\alpha > 0$, $\frac{1}{\alpha}\, x \cdot b < x \cdot \sum_{i\in I} a^i\lambda^i$, then, for all $\lambda \in (\mathbb{R}_+)^I$,

$$0 \leq \sum_{i\in I} x \cdot a^i\lambda^i.$$

Since, for all $\lambda \in (\mathbb{R}_+)^I$, $0 \leq \sum_{i\in I} x \cdot a^i\lambda^i$, then, for all $i \in I$,

$$0 \leq x \cdot a^i.$$

Since $0 \in \langle a^i\rangle^+_{i\in I}$ and $b \in \{b\}$, then

$$-b \in \langle a^i\rangle^+_{i\in I} - \{b\}.$$

Since $-b \in \langle a^i\rangle^+_{i\in I} - \{b\}$ and, for all $x' \in \langle a^i\rangle^+_{i\in I} - \{b\}$, $0 < x \cdot x'$, then

$$0 < x \cdot (-b)$$

i.e.

$$x \cdot b < 0.$$

Therefore, if, for all $\lambda \in (\mathbb{R}_+)^I$, $b \neq \sum_{i\in I} \lambda^i a^i$, then there exists $x \in X$ such that, for all $i \in I$, $0 \leq x \cdot a^i$ and

$$x \cdot b < 0.$$

Q.E.D.

Necessary conditions for a constrained maximum. *If $f \in \mathbb{R}^{\mathbb{R}^n}$ and $g \in (\mathbb{R}^m)^{\mathbb{R}^n}$ are differentiable at*

$$\begin{aligned} x^* \in \arg\max_x & f(x) \\ g(x) &\geq 0 \\ x &\geq 0 \end{aligned}$$

and,[13] *for all $x' \in \mathbb{R}^n$ such that,*

(1) *for all $i \in I_{x^*} = \{i \mid x^*_i = 0\}$,*

$$x' \cdot e^i \geq 0,$$

(2) *for all $j \in J_{x^*} = \{j \mid g_j(x^*) = 0\}$,*

$$x' \cdot Dg_j(x^*) \geq 0,$$

[13] Constraints qualification condition.

there exist $\varepsilon > 0$ and $x \in (\mathbb{R}^n)^{[0,\varepsilon)}$ such that

(1) $x(0) = x^*$,
(2) *for all* $t \in [0, \varepsilon)$,

$$\begin{aligned} g(x(t)) &\geq 0 \\ x(t) &\geq 0, \end{aligned}$$

(3) *and* $Dx(0) = x'$,

then there exists $\lambda^* \in \mathbb{R}^m_+$ *such that*

$$\begin{aligned} Df(x^*) + \lambda^* Dg(x^*) &\leq 0 \\ \big(Df(x^*) + \lambda^* Dg(x^*)\big) \cdot x^* &= 0 \\ \lambda^* \cdot g(x^*) &= 0 \end{aligned}$$

Proof. Let $f \in \mathbb{R}^{\mathbb{R}^n}$ and $g \in (\mathbb{R}^m)^{\mathbb{R}^n}$ be differentiable at

$$\begin{aligned} x^* \in \arg\max_x f(x) \\ g(x) \geq 0 \\ x \geq 0 \end{aligned}$$

and let, for all $x' \in \mathbb{R}^n$ such that,

(1) for all $i \in I_{x^*}$, $x' \cdot e^i \geq 0$,
(2) for all $j \in J_{x^*}$, $x' \cdot Dg_j(x^*) \geq 0$,

there exist $\varepsilon > 0$ and $x \in (\mathbb{R}^n)^{[0,\varepsilon)}$ such that

(1) $x(0) = x^*$,
(2) for all $t \in [0, \varepsilon)$, $g(x(t)) \geq 0$ and $x(t) \geq 0$,
(3) and $Dx(0) = x'$.

Let $x' \in \mathbb{R}^n$ be such that,

(1) for all $i \in I_{x^*}$, $x' \cdot e^i \geq 0$,
(2) for all $j \in J_{x^*}$, $x' \cdot Dg_j(x^*) \geq 0$.

Then there exist $\varepsilon > 0$ and $x \in (\mathbb{R}^n)^{[0,\varepsilon)}$ such that

(1) $x(0) = x^*$,
(2) for all $t \in [0, \varepsilon)$, $g(x(t)) \geq 0$ and $x(t) \geq 0$
(3) and $Dx(0) = x'$,

Since

$$\begin{aligned} x^* \in \arg\max_x f(x) \\ g(x) \geq 0 \\ x \geq 0 \end{aligned}$$

and, for all $t \in [0, \varepsilon)$, $g(x(t)) \geq 0$ and $x(t) \geq 0$, then, for all $t \in [0, \varepsilon)$,

$$f(x(t)) \leq f(x^*)$$

$$f(x(0)) + Df(x(0))\, Dx(0) \cdot (t - 0) + \cdots \leq f(x^*)$$

or equivalently

$$Df(x^*) \cdot x' t \leq 0.$$

Since, for all $t \in [0, \varepsilon)$, $Df(x^*) \cdot x' t \leq 0$, then

$$Df(x^*) \cdot x' \leq 0.$$

Since, for all x' such that

(1) for all $i \in I_{x^*}$, $x' \cdot e^i \geq 0$, and
(2) for all $j \in J_{x^*}$, $x' \cdot Dg_j(x^*) \geq 0$,

$$x' \cdot (-Df(x^*)) \geq 0,$$

then[14] there exist, for all $i \in I_{x^*}$, $\rho_i \geq 0$, and all $j \in J_{x^*}$, $\sigma_j \geq 0$, such that

$$-Df(x^*) = \sum_{i \in I_{x^*}} \rho_i e^i + \sum_{j \in J_{x^*}} \sigma_j Dg_j(x^*),$$

or letting

$$\lambda_j^* = \begin{cases} \sigma_j & \text{if } j \in J_{x^*} \\ 0 & \text{if } j \notin J_{x^*}, \end{cases}$$

equivalently,

$$\begin{aligned} Df(x^*) + \lambda^* Dg(x^*) &= \\ Df(x^*) + \sum_{j \in J_{x^*}} \sigma_j Dg_j(x^*) &= -\sum_{i \in I_{x^*}} \rho_i e^i \\ &\leq 0. \end{aligned}$$

Moreover, since, for all $i \in I_{x^*}$, $x_i^* = 0$, then

$$\begin{aligned} \big(Df(x^*) + \lambda^* Dg(x^*)\big) \cdot x^* &= \\ \big(-\sum_{i \in I_{x^*}} \rho_i e^i\big) \cdot x^* &= -\sum_{i \in I_{x^*}} \rho_i e^i \cdot x_i^* \\ &= 0. \end{aligned}$$

Finally, since, for all $j \in J_{x^*}$, $g_j(x^*) = 0$, then

$$\begin{aligned} \lambda^* \cdot g(x^*) &= \\ \sum_{j \in J_{x^*}} \sigma_j g_j(x^*) &= 0. \end{aligned}$$

Q.E.D.

Necessary Kuhn-Tucker conditions for a maximum. *If $f \in \mathbb{R}^{\mathbb{R}^n}$ and $g \in (\mathbb{R}^m)^{\mathbb{R}^n}$ are differentiable at*

$$\begin{aligned} x^* \in \arg\max_x f(x) \\ g(x) \geq 0 \\ x \geq 0 \end{aligned}$$

[14] By Farkas Lemma.

x^* satisfies the constraint qualification condition, and

$$\mathcal{L}(x, \lambda) = f(x) + \lambda g(x)$$

then there exists $\lambda^* \in \mathbb{R}^m_+$ such that

$$\begin{aligned}
D_x \mathcal{L}(x^*, \lambda^*) &\leq 0 \\
x^* \cdot D_x \mathcal{L}(x^*, \lambda^*) &= 0 \\
x^* &\geq 0 \\
D_\lambda \mathcal{L}(x^*, \lambda^*) &\geq 0 \\
\lambda^* \cdot D_\lambda \mathcal{L}(x^*, \lambda^*) &= 0 \\
\lambda^* &\geq 0
\end{aligned}$$

Proof. This is a straightforward corollary of the previous necessary conditions.

Still another avatar of the same conditions

Necessary complementary slackness conditions for a maximum. *If $f \in \mathbb{R}^{\mathbb{R}^n}$ and $g \in (\mathbb{R}^m)^{\mathbb{R}^n}$ are differentiable at*

$$\begin{aligned}
x^* \in \arg\max_x & f(x) \\
g(x) &\geq 0 \\
x &\geq 0
\end{aligned}$$

x^* satisfies the constraint qualification condition, then there exists $\lambda^* \in \mathbb{R}^m_+$ such that, for all $i = 1, \dots, n$,

$$\begin{aligned}
D_{x_i} f(x^*) + \lambda^* \cdot D_{x_i} g(x^*) &\leq 0, \ (= 0 \textit{ if } x_i^* > 0) \\
x_i^* &\geq 0, \ (= 0 \textit{ if } D_{x_i} f(x^*) + \lambda^* \cdot D_{x_i} g(x^*) < 0)
\end{aligned}$$

and for all $j = 1, \dots, m$,

$$\begin{aligned}
g_j(x^*) &\geq 0, \ (= 0 \textit{ if } \lambda_j^* > 0) \\
\lambda_j^* &\geq 0, \ (= 0 \textit{ if } g_j(x^*) > 0).
\end{aligned}$$

Sufficient complementary slackness conditions for a maximum. *If $f \in \mathbb{R}^{\mathbb{R}^n}$ and $g \in (\mathbb{R}^m)^{\mathbb{R}^n}$ are differentiable at $x^* \in \mathbb{R}^n_+$, $\lambda^* \in \mathbb{R}^m_+$ is such that, for all $x \geq 0$,*

$$f(x) + \lambda^* g(x) \leq f(x^*) + \lambda^* g(x^*) + (x - x^*) \cdot \big(Df(x^*) + \lambda^* Dg(x^*)\big)$$

and, for all $i = 1, \dots, n$,

$$\begin{aligned}
D_{x_i} f(x^*) + \lambda^* \cdot D_{x_i} g(x^*) &\leq 0, \ (= 0 \textit{ if } x_i^* > 0) \\
x_i^* &\geq 0, \ (= 0 \textit{ if } D_{x_i} f(x^*) + \lambda^* \cdot D_{x_i} g(x^*) < 0)
\end{aligned}$$

and all $j = 1, \dots, m$,

$$\begin{aligned}
g_j(x^*) &\geq 0, \ (= 0 \textit{ if } \lambda_j^* > 0) \\
\lambda_j^* &\geq 0, \ (= 0 \textit{ if } g_j(x^*) > 0),
\end{aligned}$$

then

$$\begin{aligned}
x^* \in \arg\max_x & f(x) \\
g(x) &\geq 0 \\
x &\geq 0.
\end{aligned}$$