

Conditional Moments and Independence

Áureo de Paula *

University of Pennsylvania

March 7, 2008

ABSTRACT

Consider two random variables X and Y . In initial probability and statistics courses, a discussion of various concepts of dissociation between X and Y is customary. These concepts typically involve independence and uncorrelatedness. An example is shown where $E(Y^n|X) = E(Y^n)$ and $E(X^n|Y) = E(X^n)$ for $n = 1, 2, \dots$ and yet X and Y are not stochastically independent. The bivariate distribution is constructed using a well-known example in which the distribution of a random variable is not uniquely determined by its sequence of moments. Other similar families of distributions with identical moments can be used to display such a pair of random variables. It is interesting to note in class that even such a degree of dissociation between the moments of X and Y does not imply stochastic independence.

Key Words: indeterminate distributions, moments, independence.

*Áureo de Paula is Assistant Professor, Department of Economics, University of Pennsylvania, Philadelphia, PA 19104. (e-mail: aureo@sas.upenn.edu) This paper was written in response to a student question in the first graduate econometrics course at the University of Pennsylvania (ECON705). The author is grateful to the class for the thought provoking interactions. The author would also like to thank the Editor, the Associate Editor and two anonymous referees whose comments immensely helped improve the article.

In this article, I construct two random variables X and Y such that $E(Y^n|X) = E(Y^n)$ and $E(X^n|Y) = E(X^n)$ for all positive integer n ; consequently, $\text{Cov}(X^m, Y^n) = 0$ for all positive integers m and n . Yet X and Y are not stochastically independent. The is that even such a strong degree of dissociation is not enough to assure independence between two random variables. In constructing X and Y , I use a family of (distinct) distributions which are well known to possess the same sequence of moments. I also point out that similar families can be used to construct bivariate distributions with similar properties.

Here is my example. Take two random variables X and Y distributed according to the following probability density function:

$$f(x, y) = \frac{1}{2\pi} \frac{1}{xy} \exp \left\{ -[(\ln x)^2 + (\ln y)^2]/2 \right\} [1 + \sin(2\pi x) \sin(2\pi y)] \quad (1)$$

for all positive x and y , and zero otherwise. This positive function integrates to 1 and is thus a valid probability density function. The marginal probability density functions for X and Y are log-normal distributions. From this expression, the conditional pdf of Y given X is

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} \frac{1}{y} \exp \left[-(\ln y)^2/2 \right] [1 + \sin(2\pi \ln x) \sin(2\pi \ln y)] \quad (2)$$

for all positive x and y and the conditional pdf of X given Y is

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp \left[-(\ln x)^2/2 \right] [1 + \sin(2\pi \ln y) \sin(2\pi \ln x)]$$

for all positive x and y . These expressions can be used to show that

$$E(Y^n|X) = E(Y^n) \text{ and } E(X^n|Y) = E(X^n)$$

for all positive integer n . To see this, check that

$$\begin{aligned} E(Y^n|X = x) &= \int_{\mathbf{R}_+} y^n f_{Y|X}(y|x) dy \\ &= \int_{\mathbf{R}_+} y^n \frac{1}{\sqrt{2\pi}} \frac{1}{y} \exp(-(\ln y)^2/2) [1 + \sin(2\pi \ln x) \sin(2\pi \ln y)] dy \\ &= \int_{\mathbf{R}_+} y^n \frac{1}{\sqrt{2\pi}} \frac{1}{y} \exp[-(\ln y)^2/2] dy \\ &= E(Y^n) \end{aligned}$$

which does not depend on X . Here \mathbf{R}_+ denotes the positive half line. The above calculation uses the surprising fact that, for any positive integer n

$$\int_0^\infty y^n \frac{1}{\sqrt{2\pi}} \frac{1}{y} \exp[-(\ln y)^2/2] \sin(2\pi \ln y) dy = \frac{1}{\sqrt{2\pi}} e^{n^2/2} \int_{-\infty}^\infty e^{-s^2/2} \sin(2\pi s) ds = 0$$

where the first equality follows by a change of variables argument and the second holds since the integrand is an odd function. Similar computations deliver conditional moments independence of X given Y .

Yet X and Y are dependent because the conditional pdf of Y given X clearly depends on X , because X determines the weight on $\sin(2\pi \ln y)$ in the formula (2) for the conditional pdf. A similar conclusion follows from the analysis of the pdf of X given Y . In other words, in general

$$f_{Y|X}(y|x) \neq f_Y(y) \quad \text{and} \quad f_{X|Y}(x|y) \neq f_X(x).$$

To obtain this example, I used the following class of pdf's:

$$f(x; \lambda) = \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp(-(\ln x)^2/2) [1 + \lambda \sin(2\pi \ln x)], & x > 0 \\ 0, & x \leq 0 \end{cases}$$

where $|\lambda| \leq 1$. When $\lambda = 0$ this corresponds to the pdf of a log-normal random variable. For different values of λ , these densities are easily seen to differ. This class of pdf's is nonetheless known to possess the same set of moments (see Billingsley (1995), Section 30, or Casella and Berger (2001), Example 2.3.10).

My example uses the above class of pdf's to generate the conditional pdf's for two variables X and Y . Since I use λ as the function of the conditioning random variable and the moments of this class of distributions do not depend on this parameter, the conditional moments are in the end equal to the unconditional ones. Finally, these two conditional distributions are consistent with a well defined bivariate joint distribution (i.e. they are compatible). In fact, they can be obtained from the joint pdf displayed in (1).

The above construction makes use of a well-known class of distinct distributions possessing the same sequence of moments. Other such families of distributions exist and can similarly be used to generate bivariate distributions with the same properties. Examples are given in Berg (1988) and Stieltjes (1894/1895). These families are typically given by:

$$f(x) = h(x)[1 + \lambda g(x)], \quad x \in \mathcal{S}$$

where $h(x)$ is a pdf and $g(x)$ is a periodic function. A condition $|\lambda| < c$ is typically imposed to assure positivity of $f(x)$. A bivariate distribution $f(x, y)$ can then be obtained from the following conditional distributions

$$f_{Y|X}(y|x) = h(y)[1 + \lambda' g(x)g(y)]$$

and

$$f_{X|Y}(x|y) = h(x)[1 + \lambda' g(x)g(y)]$$

where a bound on $|\lambda'|$ may be necessary to guarantee that the joint distribution is positive and $(x, y) \in \mathcal{S}^2$. That these conditional distributions are compatible with a consistent bivariate distribution can be obtained from Theorem 2.2.1 in Arnold *et al.* (1992). As a matter of fact, the conditional distributions above are compatible with the following joint distribution:

$$f_{X,Y}(x, y) = h(x)h(y)[1 + \lambda'g(x)g(y)]$$

with support \mathcal{S}^2 . Furthermore, they demonstrate that X and Y are not stochastically independent and the conditional moments can be computed and shown to be equal to the unconditional ones.

A Simple Application

I now present a stylized illustration for the points discussed in this note. I start with the observation that, since stock prices are a function of the cash-flow generated by companies, one would expect that dissociation of earnings across firms would be accompanied by dissociation of stock prices across firms. Pindyck and Rotemberg (1993) noted that while firm earnings across sufficiently distinct sectors tend to be uncorrelated, stock prices for the same firms are not. The following very simple illustration shows that such a phenomenon can occur when profits are associated in the subtle manner depicted by the family of multivariate distributions introduced previously in this article. In fact a stronger version of earnings uncorrelatedness, uncorrelatedness for all positive powers of earnings, is still consistent with correlated stock prices in this very simple environment.¹

Consider an economic environment with two firms, A and B . A traditional model for firm profits in period $t + 1$, P_{t+1}^A and P_{t+1}^B , given profits in the period $t + 1$ postulates that:

$$\frac{P_{t+1}^A}{P_t^A} = R_P^A \epsilon_{t+1}^A \quad \frac{P_{t+1}^B}{P_t^B} = R_P^B \epsilon_{t+1}^B$$

where $\log(\epsilon_{t+1}^A)$ and $\log(\epsilon_{t+1}^B)$ are standard normal random variables and R_P^A and R_P^B are the per-period returns for the two firms.² In this simple financial environment, stock prices in period $t + 1$, S_{t+1}^A and S_{t+1}^B , equal the expected discounted sum of utilities provided by earnings in each period conditional on profits at that period. Assuming a logarithmic utility function this delivers:³

$$S_{t+1}^A = E \left[\sum_{i=1}^{\infty} \beta^{i-1} \log(P_{t+i}^A) | P_{t+1}^A \right]$$

¹This is, admittedly, a very simple environment. For a comprehensive analysis of this issue, see Pindyck and Rotemberg (1993) or Barberis *et al.* (2005).

²This follows if one models P_t as a geometric Brownian motion as in McDonald and Siegel (1985).

³More general utility functions could be assumed without changing the qualitative conclusions of my illustration.

where β is the discount factor. After some calculation, this gives

$$S_{t+1}^A = \frac{1}{1-\beta} \log P_{t+1}^A + \frac{\beta}{(1-\beta)^2} \log R_P^A$$

and an analogous expression holds for S_{t+1}^B . If we assume that the joint probability density function of ϵ_{t+1}^A and ϵ_{t+1}^B is given by (1), the logarithms of ϵ_{t+1}^A and ϵ_{t+1}^B are standard normals and firm earnings are uncorrelated. As a matter of fact, all positive powers of P_{t+1}^A and P_{t+1}^B are uncorrelated! Yet,

$$\text{Cov}(S_{t+1}^A, S_{t+1}^B) \neq 0$$

since the logarithms of ϵ_{t+1}^A and ϵ_{t+1}^B are correlated. Therefore, even if all positive powers of profits are uncorrelated, those may still be dependent in ways that would lead to correlated stock prices. As the example indicates, independence can be described in terms of covariance, but nonlinear test functions are needed. More specifically, X and Y are independent if and only $\text{Cov}(f(X), g(Y)) = 0$ for all bounded continuous functions f and g .

References

- [1] ARNOLD, B., E. CASTILLO AND J.-M. SARABIA (1992), *Conditionally Specified Distributions*, Springer-Verlag.
- [2] BARBERIS, N., A. SHLEIFER AND J. WURLER (2005), “Comovement”, *Journal of Financial Economics* **75**, pp.283-317.
- [3] BERG, C. (1988), “The Cube of a Normal Distribution is Indeterminate”, *The Annals of Probability*, Vol.16, No.2, pp.910-913.
- [4] BILLINGSLEY, P. (1995), *Probability and Measure*, 3rd Edition, New York: Wiley.
- [5] CASELLA, G. AND R. BERGER (2001), *Statistical Inference*, 2nd Edition, Pacific Grove: Duxbury.
- [6] McDONALD AND SIEGEL (1985), “Investment and the Valuation of Firms When There is an Option to Shut Down”, *International Economic Review*, **26**(2), pp.331-49.
- [7] PINDYCK, R. AND J. ROTEMBERG (1993), “The Comovement of Stock Prices”, *The Quarterly Journal of Economics*, **108**(4), pp.1073-1104.
- [8] STIELTJES, T.J. (1894/1895), “Recherches Sur Les Fractions Continues”, *Ann. Fac. Sci. Toulouse Math.* **8**(J), pp.1-122, **9**(A), pp.1-47 (in *Oeuvres Complètes*, Vol.2, (1918), pp.402-566, Noordhoff, Gronigen)