

*Syllabus for Econ 898*

**MATHEMATICS FOR ECONOMIC THEORY**

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PRACTICAL MATTERS

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- (3) Course's webpage: <http://www.ssc.upenn.edu/~davilaj/econ898.html>
- (4) Meetings: Monday and Wednesday, from 10:30 to 12:00 at 395 McNeil.
- (5) 1st exam: Monday, October 21, 2002, 10:30-12:00.
- (6) 2nd exam: Wednesday, December 11, 2002, 10:30-12:00.
- (7) Evaluation: 1st exam 40%, 2nd exam 40%, homeworks 20%.

REFERENCES

Handouts with the contents of the course will be posted at the course's website <http://www.ssc.upenn.edu/~davilaj/econ898.html>. They are permanently under (hopefully convergent) construction, and are intended to cover all what you will be required to know for the homeworks and the exams. Nevertheless, although they are to a reasonable extent self-contained, it may be extremely helpful to review the contents of the course in different sources in order to get a good understanding of the material that will be covered in it. Please find below a list of references useful for such a review.

Basic references in real analysis,

- (1) Michel and Herguet, *Applied Algebra and Functional Analysis*, Dover, ISBN 0-486-67598.
- (2) Haaser and Sullivan, *Real Analysis*, Ch. 1, 2, 3, 4, 5, 7, 11 Dover, ISBN 0-486-66509-7.
- (3) Kolmogorov and Fomin, *Introductory Real Analysis*, Ch. 1, 2, 4, 5, Dover, ISBN 0-486-61226-0.

References oriented to graduate students in Economics:

- (1) Moore, James C., *Mathematical Methods for Economic Theory*, Springer, ISBN 3-540-66235-9.
- (2) Efe A. Ok, *Real Analysis and Probability with Economic Applications*, Ch. 2, 3, 4, 5, unpublished lecture notes, May 2000.
- (3) Takayama, *Mathematical Economics*, Ch. 5, 8 Cambridge University Press, ISBN 0-52131-498-4.
- (4) Stokey and Lucas, *Recursive Methods in Economic Analysis*, Ch. 3, 4, 17, Harvard University Press, ISBN 0-674-75096-9.

## OVERVIEW OF THE COURSE

This course is intended to acquaint you with some of the basic higher mathematics that you need to proceed with the courses and research leading to your Ph.D. degree in Economics. Basically, we will review the mathematics needed to get two of the theorems behind many basic results in economics, namely **Bellman's principle of optimality** and **Kakutani's fixed point theorem**. These results allow, respectively, to solve dynamic programming problems arising for instance (but not only) in macroeconomics, and to obtain the existence of equilibria both in games and economies. Finally, basic theorems of differential calculus will allow for results on at least local uniqueness of equilibria and comparative statics whenever the framework allows it.

After an introduction intended to motivate the need for more advanced mathematics in order to deal with simple economic problems, we will nonetheless review in Section 1 basic results about constrained maxima in  $\mathbb{R}^n$ , namely the **Kuhn-Tucker conditions** and the lemmas necessary to derive them, i.e. **separating hyperplanes theorems** and **Farkas lemma**.

From there we will start rebuilding the mathematics that will allow us to reach our initial goal, i.e. Bellman's principle of optimality and Kakutani's fixed point theorem. Section 2 introduces the basic set-theoretic concept of **relation** and from it we introduce different concepts of **order** and the concept of **function** as special types of relations, as well as those of **correspondence**, **operation** and **sequence** as special types of functions. Sections 3, 4, 5, and 6 build on the introduction of a notion of distance between the elements of a set, leading to the concept of **metric space**. Section 4 presents increasingly strong notions of **continuity** of functions from a metric space to another metric space, ending up with the notion of **contraction**. Contractions have a very important property when they operate on metric spaces that are **complete**: in complete metric spaces contractions have a point of the space mapped into itself and this point is unique. This result is known as the Contraction Mapping Theorem or **Banach's Fixed Point Theorem**. The importance of such result is that it will allow to prove the existence of solutions to a functional equation like Bellman's equation (under the adequate assumptions), which is the bread and butter of current macroeconomics and many other fields in economics. Not only this result guarantees the existence of such solution, but it provides also a way to compute numerical approximations of the solution and other related functions.

Section 5 deals with correspondences and some notions of continuity available for them, paving the way towards a property, the **Theorem of the Maximum**, which establishes good properties for the dependence of the set of maximizers of a real-

valued function and the maximum of the function on the parameters determining its domain. This result is then used to show **Bellman's principle of optimality** that allows to reduce the standard dynamic optimization problem to **Bellman's equation**, whose solution we already know how to find (at least approximately and under the adequate conditions).

From here the course takes a different direction, the reason being that we need to introduce additional structure, other than endowing the space with a metric, in order to get more results. This structure is that of a **vector space**. The structure of vector space allows us to "operate" with the elements of a set (renamed *vectors* for that purpose) in a way that extends the way we operate with numbers. Based on these operations, the notions of **linear combination** of vectors, **convex set** of vectors, and **linearly dependent** and **independent** sets of vectors are introduced, as well as the idea of **basis**, i.e. of a "minimal" subset of points of the space with which we are able to generate the entire space by means of these operations, in such a way that having a basis amounts to having the space. Many of the properties of a vector space depend on whether it has a finite basis or not. The number of elements in a basis (every basis happens to have the same number of them) is the **dimension** of the vector space, and finite- versus infinite-dimensional vector spaces exhibit different properties. The interest in reintroducing all these familiar concepts in an abstract way without necessarily making reference to the special vector space  $\mathbb{R}^n$  with which we are used to and which will eventually be identified to any  $n$ -dimensional real vector space, is (1) to allow to disentangle what is peculiar to this particular space as opposed to any general vector space, and (2) to allow noticing that many other objects we will naturally encounter, e.g. function spaces, constitute vector spaces as well, and hence we can handle them in a similar way.

The interest of the vector space structure comes, on the one hand, from the notion of convexity, i.e. the property some sets have of containing any **convex linear combination** of their elements. Once we have the notion of convexity, we will be in a position to get another application of the Maximum's Theorem in the context of finite-dimensional real vector spaces: a version of the **Brouwer's Fixed Point Theorem** based on a particular version of the same theorem specialized to the unit ball. We will finish this part getting **Kakutani's Fixed Point Theorem** from Brouwer's, and then from Kakutani's we will obtain **Nash's Theorem** of the existence of an equilibrium for games of finitely many players with continuous pay-offs (quasiconcave in their own strategies) and compact, convex, finite-dimensional strategy sets. Kakutani's theorem is also at the heart of the existence of a Walrasian equilibrium of an economy of finitely many consumers and producers with convex preferences and technologies.

On the other hand, the interest of the vector space structure comes also from the existence of a particularly simple type of functions between vector spaces, namely the **linear functions** introduced in Section 12, that are useful to approximate locally other functions. Focusing on the idea of approximation, notice that it calls for the introduction of a metric in each vector space, and here is where the two strands of the course mix with each other.

The way to introduce a metric consistently with the already existing vector space structure is through the notion of **norm**, which gives a "length" to each of the elements of the vector space, to obtain a **normed vector space**, which both a vector space and a metric space. In effect, each norm induces a metric by means of setting the length of the difference between any two vector as the distance between

them. An even geometrically richer way to metrize a vector space is to endow it with an **inner product** to obtain an **inner product space**. An inner product is a way to operate vectors that not only induces a norm and hence a metric, but also allows to define "angles" between vectors, carrying with it all the geometric intuitions of the usual space into abstract vector spaces, an extremely useful fact that helps keeping track of some geometric intuition in spaces that are hardly intuitive. Section 12 deals then with the issue of continuity of linear functions in normed vector space, a matter that relies heavily on the dimension of the vector space on which they are defined and for which our intuitions received from  $\mathbb{R}^n$  will not always be honored.

The full benefits of the structure of normed vector spaces are obtained exploiting the idea of approximating functions locally by means of linear functions. The functions for which such approximation is possible are known as **differentiable functions**. These functions and their properties are presented in Section 13, among them the familiar results of multivariate differential calculus as well as the extremely useful **Implicit Function Theorem** in the most general set-up. The Implicit Function Theorem guarantees allows to see the inverse image of a (regular) value of a function (or the set of solutions to a system of equations, in the finite-dimensional case) as a function itself, which is at the foundations of any comparative statics-like argument. In order to establish this theorem we will make use again of the existence and uniqueness of a fixed point of a contraction. The **Inverse Function Theorem** is presented as a corollary of the Implicit Function Theorem. Finally, the fact that the Implicit Function Theorem allows to see as graphs of differentiable functions some subsets of vector spaces makes sense of the idea of a **tangent space** to a "surface" at a point in an abstract setup.

## LIST OF TOPICS

### (0) **Introduction**

#### (1) **Review of constrained maximization in $\mathbb{R}^n$**

Preliminary lemmas (separating hyperplanes, Farkas lemma), necessary and sufficient conditions for constrained maxima in  $\mathbb{R}^n$  with and without differentiability.

#### (2) **Relations, orders, functions, fields...**

Relation from a set to a set. Inverse of a relation. Image and inverse image of a set by a relation. Range and domain of a relation. Injective, surjective and bijective relations. Composition of relations. Extension and restriction of a relation.

Binary relations. Upper and lower bound, greatest and least element, supremum and infimum, maximum and minimum, maximal and minimal of a set by a binary relation. Reflexivity, antireflexivity, symmetry, asymmetry, antisymmetry, transitivity, negative transitivity, completeness, and totality of a binary relation. Equivalence relations.

Preorders. Partial orders. Complete orders. Linear orders.

Function from a set to a set. Set of functions from a set to a set. Extension of a function. Increasing functions between completely ordered sets. Power set of a set.

Operations. Sums and multiplications. Fields. The field of real numbers. The integers. The field of rationals. The irrationals.

Sequence in a set. Subsequence of a sequence. Finite sets.  
Correspondence from a set to a set. Graph of a correspondence.

(3) **Metric spaces**

Metrics. Metric spaces. *Discrete spaces.  $\ell_p^n$ ,  $\ell_\infty^n$ ,  $\ell_p$  and  $\ell_\infty$  spaces. The space of bounded real valued functions with the sup metric.* Subspaces.

Closure, limit and interior points. Closure and interior of a set. Closed sets and open sets.

Bounded sets.

Compact sets.

Connected, separable, compact, complete metric spaces.

Sequences in metric spaces. Convergence. Limit of a convergent sequence. Cauchy sequences. *Sequences in  $\mathbb{R}$ , in  $\mathbb{R}^n$ , in  $\ell_p$  and  $\ell_\infty$ .* Totally bounded sets.

Compactness and sequential compactness.

(4) **Functions between metric spaces**

Limit of a function at a closure point of its domain. Uniqueness of the limit. Characterization of the limit by sequences.

Continuity. Continuous functions and compact sets.

Uniform continuity.

Lipschitz continuity.

Contractions. Banach's fixed point theorem: existence and uniqueness of fixed points of contractions on complete metric spaces. Sufficient characterizations of contractions. Blackwell's sufficient condition.

Solutions to functional equations. *Bellman's functional equation.*

(5) **Correspondences between metric spaces**

Upper and lower hemicontinuity, and continuity of a correspondence between metric spaces.

Characterization of upper and lower hemicontinuous correspondences by opens, closed sets, and sequences.

Upper hemicontinuity of every correspondence taking values in a compact metric space and with closed graph. Closed graph of every closed-valued upper hemicontinuous correspondence.

Preservation of upper hemicontinuity by unions.

Preservation of upper hemicontinuity and compact-valuedness by intersections.

Preservation of compactness by an upper hemicontinuous compact-valued correspondence.

(6) **The theorem of the maximum**

(7) **The standard dynamic optimization problem**

Bellman's principle of optimality. (A version of) Brouwer's fixed point theorem.

(8) **Vector spaces**

Vector spaces. *The real vector space of  $n$ -tuples of real numbers. The real vector spaces of polynomials with real coefficients. The vector space of  $m \times n$  matrices of scalars.* Vector subspaces.

Linear combinations of vectors. Convex linear combinations. Convex sets. Linearly dependent and independent sets of vectors.

Bases. Unique decomposition with respect to a basis.

Finite dimensional vector spaces. Dimension. The vector space of  $n$ -tuples of scalars. *The finite dimensional real vector space  $\mathbb{R}^n$ .* Infinite dimensional vector spaces. *The infinite dimensional vector spaces of sequences of real numbers, of all the real polynomials, of real valued functions over a non finite domain.*

(9) **Brouwer's fixed point theorem**

Brouwer's fixed point theorem for the closed unit ball of a normed real vector space. Brouwer's fixed point theorem for compact, convex set of a normed real vector space.

(10) **Approximations and selections of correspondences.**

Approximation of correspondences. Von Neumann approximation lemma. Michael's selection theorem.

(11) **Kakutani's fixed point theorem and Nash's theorem.**

Kakutani's fixed point theorem.

Nash equilibrium existence theorem for games of finitely many players with compact, convex, finite-dimensional strategy sets and continuous pay-offs quasi-concave in their own strategies.

(12) **Linear functions**

Linear functions between real vector spaces. The vector subspaces formed by the range and the null space of a linear function. The complete characterization of linear functions by their values at any basis. Linearity of the composition of linear functions. Complete characterization of injective linear functions. Inverse of an injective linear function.

Isomorphisms. Inverse of an isomorphism. Equivalence by isomorphism of all the  $n$ -dimensional vector spaces over a given field. Dimension induced by an isomorphism.

Linearity of sums and multiplication by a scalar of linear functions. Vector spaces of linear functions. The identification of vector spaces of matrices of scalars and of linear functions. Matrix of the composition of linear functions between finite dimensional vector spaces Matrix of a linear function between finite dimensional spaces after a change of bases.

**Linear functions between normed vector spaces**

Norm in a vector space. Normed vector spaces. Metric induced by a norm. Norm of the Cartesian product of two normed real vector spaces. Equivalent norms. Equivalence relation between equivalent norms. Equivalence of all the norms in a finite dimensional space. Equivalence of convergence with respect to equivalent norms. Equivalence by homeomorphism of every  $n$ -dimensional real vector space.

Inner product in a vector space. Inner product Spaces. Norm induced by an inner product. Cauchy-Schwarz inequality in inner product spaces.

Linear functions between normed real vector spaces. Characterization of continuous linear functions by their continuity at the null vector. Characterization of the convergence of a sequence in a finite dimensional normed

real vector space by the convergence of coordinate sequences. Continuity of linear functions with finite dimensional domains. Sup norm in finite dimensional real vector spaces. Equivalence of continuity and bounded-linearity of linear functions. Stability of boundedness by continuous linear transformations.

The vector subspaces of bounded-linear or continuous functions. Norm of a continuous linear function.

(13) **Differentiable functions between normed real vector spaces**

Differential of a function at a point. Uniqueness of the differential.

Equivalence between continuity and continuity of the differential of differentiable functions. Rules of differentiation. *Differential of a constant function, of a linear function, of the sum of functions, of the multiplication by a scalar of function, of the composition of functions.* Properties of differentiable real valued functions on the real line. *Rolle's theorem. Generalized mean value theorem. Mean value theorem.*

Directional derivatives. Directional derivatives as values of differentials. Partial derivatives. Differentials in terms of partial derivatives. Differentiability at any point of an open implied by the continuity of partial derivatives in the open.

Equivalence of continuous differentiability and continuity of the partial derivatives in an open.

Jacobian matrix, gradient vector.

Mean value theorem for functions on the real line taking values in arbitrary normed real vector spaces. Mean value theorem for functions between arbitrary normed real vector spaces.

(14) **The implicit function theorem**

Partial functions. Differentiability of partial functions induced by the differentiability of a function on a cartesian product.

Partial differentials. Continuity of partial differentials induced by the continuity of the differential.

Implicit function theorem in arbitrary normed real vector spaces.

Inverse function theorem in arbitrary normed real vector spaces.

Tangent space to a function at a point where it is differentiable.