Lecture Notes for Econ 898

INTRODUCTION

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This introduction intends to motivate the need to take a course in higher mathematics like Econ 898 in order to pursue your PhD studies in Economics. To be specific, recall any of these two basic problems of microeconomics:

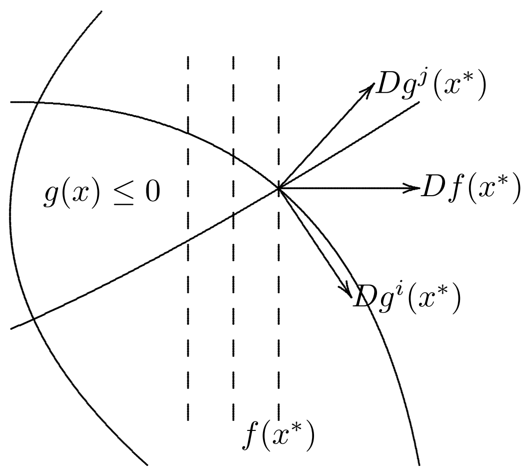
(1) the consumer's problem

$$\begin{aligned} \max \ & u(x_1, \ldots, x_n) \\ & p_1 x_1 + \cdots + p_n x_n \leq m \\ & 0 \leq x_1, \ldots, x_n, \end{aligned}$$

(2) the producer's problem

$$\begin{aligned} \max \ & py - (w_1 x_1 + \cdots + w_n x_n) \\ & y \leq f(x_1 \ldots, x_n) \\ & 0 \leq y, x_1, \ldots, x_n. \end{aligned}$$

Each of them consists of finding an extremum of a real-valued function defined in some $\mathbb{R}^n$ under inequality constraints. We know how to solve this type of problems: basically we move in the direction of the functions gradient at any point as far as possible within the region defined by the constraints. Eventually, when the value of the function cannot be increased any more and, hence, we have attained the maximum we were looking for, the gradient will have to be a convex linear combination of the gradients of the binding constraints. Thus the maximum is characterized by a system of equations and inequalities in a finite number of variables, known as the Kuhn-Tucker conditions.



Now consider the following basic problem in economics. Starting with an initial amount k_0 of a commodity, one has to decide how to split it at each subsequent date $t = 0, 1, 2, \dots$ between, on the one hand, a part c_t to be consumed, providing a utility $u(c_t)$ at date t which discounted at the initial date at the rate β is $\beta^t u(c_t)$ at date 0; and, on the other hand, another part k_{t+1} to be used to produce an amount $f(k_{t+1})$ of the commodity at the next period $t+1$. The goal is to choose a sequence of splits, i.e. of consumption-saving decisions, such that the total discounted utility is maximal, i.e.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$c_t + k_{t+1} = f(k_t)$$
$$0 \leq c_t, k_{t+1},\ t = 0, 1, 2, \dots$$
$$(k_0 \text{ given})$$

At first sight, this problem looks very much like the previous ones but for the notable differences that

(1) now there are infinitely many variables $c_0, c_1, \dots$ and $k_1, k_2, \dots$,
(2) there are infinitely many constraints as well.

That is to say, this problem amounts to finding an extremum of a real-valued function defined in some sort of space $\mathbb{R} \times \mathbb{R} \times \dots$ whose dimension seems to be... oh well, infinity! As a consequence, clearly the problem cannot be solved in the same way as before. This is an example of how a simple economic problem can lead us naturally to work with spaces more complicated than the usual $\mathbb{R}^n$, namely with *infinite-dimensional* spaces.

Alternatively, solving the consumption-saving problem leads us to deal with a completely different kind of equation. In effect, if we let V denote the maximum attained at the solution to the consumption-savings problem, clearly it is a function $V(k_0)$ of the given initial amount k_0 of the commodity. Moreover this function should satisfy

$$
\begin{aligned}
V(k_0) &= \max_{\substack{c_t+k_{t+1}=f(k_t)\\ 0\le c_t,k_{t+1}\\ t=0,1,2,\ldots\\ (k_0 \text{ given})}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&= \max_{\substack{c_0+k_1=f(k_0)\\ 0\le c_0,k_1\\ (k_0 \text{ given})}} \left\{ \max_{\substack{c_t+k_{t+1}=f(k_t)\\ 0\le c_t,k_{t+1}\\ t=1,2,\ldots\\ (c_0,k_1 \text{ given})}} \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \\
&= \max_{\substack{c_0+k_1=f(k_0)\\ 0\le c_0,k_1\\ (k_0 \text{ given})}} \left\{ u(c_0) + \max_{\substack{c_t+k_{t+1}=f(k_t)\\ 0\le c_t,k_{t+1}\\ t=1,2,\ldots\\ (k_1 \text{ given})}} \sum_{t=1}^{\infty} \beta^t u(c_t) \right\} \\
&= \max_{\substack{c_0+k_1=f(k_0)\\ 0\le c_0,k_1\\ (k_0 \text{ given})}} \left\{ u(c_0) + \beta \max_{\substack{c'_{t-1}+k'_t=f(k'_{t-1})\\ 0\le c'_{t-1},k'_t\\ t=1,2,\ldots\\ (k'_0 \text{ given})}} \sum_{t=1}^{\infty} \beta^{t-1} u(c'_{t-1}) \right\} \\
&= \max_{\substack{c_0+k_1=f(k_0)\\ 0\le c_0,k_1\\ (k_0 \text{ given})}} \left\{ u(c_0) + \beta \max_{\substack{c'_{t'}+k'_{t'+1}=f(k'_{t'})\\ 0\le c'_{t'},k'_{t'+1}\\ t'=0,1,2,\ldots\\ (k'_0 \text{ given})}} \sum_{t'=0}^{\infty} \beta^{t'} u(c'_{t'}) \right\} \\
&= \max_{\substack{c_0+k_1=f(k_0)\\ 0\le c_0,k_1\\ (k_0 \text{ given})}} \left\{ u(c_0) + \beta V(k'_0) \right\} \\
&= \max_{\substack{c_0+k_1=f(k_0)\\ 0\le c_0,k_1\\ (k_0 \text{ given})}} \left\{ u(c_0) + \beta V(k_1) \right\},
\end{aligned}
$$

where the following changes of variables have been used $c_t = c'_{t-1}$, $k_t = k'_{t-1}$, and $t-1 = t'$. That is to say, the *function* V must satisfy the equation

$$
V(k_0) = \max_{\substack{c_0+k_1=f(k_0)\\ 0\le c_0,k_1\\ (k_0 \text{ given})}} \left\{ u(c_0) + \beta V(k_1) \right\}
$$

for all k_0, or in general

$$
V(k_t) = \max_{\substack{c_t+k_{t+1}=f(k_t)\\ 0\le c_t,k_{t+1}\\ (k_t \text{ given})}} \left\{ u(c_t) + \beta V(k_{t+1}) \right\}.
$$

Note that the unknown in this equation, known as Bellman equation, is therefore not a number or a vector of numbers anymore, but a function. Thus solving the simple consumption-savings problem presented above amounts to solving a *functional equation*. If we are able to find the solution V to such an equation, then solving the consumption-savings problem becomes an easy task which requires only the Kuhn-Tucker tools that we already know. In effect, we just need, starting from k_0, to solve iteratively, for each $t = 0, 1, \ldots,$ the problem in just two variables c_t and k_{t+1}

$$\max_{\substack{c_t + k_{t+1} = f(k_t) \\ 0 \le c_t, k_{t+1} \\ (k_t \text{ given})}} \big\{ u(c_t) + \beta V(k_{t+1}) \big\}.$$

Therefore, the important question now is: how do we find the solution V to the Bellman equation above? Lets us start making a guess V_0. If we solve

$$\max_{\substack{c_0 + k_1 = f(k_0) \\ 0 \le c_0, k_1 \\ (k_0 \text{ given})}} \big\{ u(c_0) + \beta V_0(k_1) \big\}$$

we will obtain a maximum value that depends on k_0 according to some function V_1 depending. Which is this function V_1 depends of course of the choice of V_0. Thus there is another function ϕ that relates each V_0 to the corresponding V_1, i.e. such that $V_1 = \phi(V_0)$. If we now solve

$$\max_{\substack{c_0 + k_1 = f(k_0) \\ 0 \le c_0, k_1 \\ (k_0 \text{ given})}} \big\{ u(c_0) + \beta V_1(k_1) \big\}$$

we will obtain another function V_2 of k_0, which is of course $\phi(V_1)$, and so on. Proceeding in this way, we can generate a *sequence of functions*. If this sequence "converges", in some sense, to the solution of the Bellman equation V, then the construction of this sequence takes us as close as we may need for practical purposes to that solution. Thus one of the main questions to clarify then is whether such "convergence" can occur or not and, if yes, under what conditions.

Note that in order to address this question we have to give some sense to the notion of convergence of a sequence of functions or, equivalently, to the notion of distance between two functions. Thus, quite naturally, a simple economic problem like the consumption-savings decision has led us to deal with *spaces of functions*.

As a concluding remark, this example only intends to illustrate the fact that as soon as we address economic questions only slightly more complicated than the most basic ones, we are necessarily led to use more advanced mathematics. Some of this new mathematical tools (e.g. infinite-dimensional spaces, functional equations, functional spaces,...) are the subject matter of this course.